Appendix A
Fundamentals of Dynamical Systems Theory

A.1 Introduction

In this appendix, we present several mathematical tools that are useful to better comprehend the material discussed in this brief and we provide references to the proofs of the most relevant theorems, which are omitted for brevity. Readers with basic knowledge of linear dynamical systems theory as well as more experienced practitioners are invited to read this appendix, since several advanced aspects of linear algebra and theory of linear ordinary differential equations are addressed in a systematic manner. Lastly, the applicability of several results to the study of the dynamics and control of aerial vehicles is discussed at length.

A.2 Nonlinear dynamical systems, Taylor Formula, and Linearization

In this section, we review several fundamental results on the linearization of nonlinear dynamical systems.

Definition A.1 (*Linear and nonlinear functions*) Let $f : \mathcal{D} \to \mathbb{R}^l$, where $\mathcal{D} \subseteq \mathbb{R}^p$. If

$$f(\alpha\chi_1 + \beta\chi_2) = \alpha f(\chi_1) + \beta f(\chi_2), \tag{A.1}$$

for all $\alpha, \beta \in \mathbb{R}$ and $\chi_1, \chi_2 \in \mathcal{D}$, then $f(\cdot)$ is *linear in* χ. Otherwise, $f(\cdot)$ is *nonlinear in* χ.

For the statement of the next definition, consider the nonlinear dynamical system

$$\dot{\chi}(t) = f(\chi(t)), \qquad \chi(0) = \chi_0, \qquad t \geq 0, \tag{A.2}$$

where for all $t \geq 0$, $\chi(t) \in \mathcal{D}$, and $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$.

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Definition A.2 (*Linear and nonlinear dynamical systems*) Consider the dynamical system (A.2). If $f(\cdot)$ is linear in χ, then (A.2) is a *linear dynamical system*. Otherwise, (A.2) is a *nonlinear dynamical system*.

Exercise A.1 Prove that the dynamical system (3.1) is linear in χ and η. *Hint:* Define $z \triangleq [\chi^{\mathrm{T}}, \eta^{\mathrm{T}}]^{\mathrm{T}}$ and rewrite (3.1) as function of z. △

For the statement of the next result, which we refer to as *Taylor's theorem*, let $f(\chi) = [f_1(\chi), \ldots, f_l(\chi)]^{\mathrm{T}}$, $\chi = [\chi_1, \ldots, \chi_n]^{\mathrm{T}} \in \mathcal{D} \subseteq \mathbb{R}^n$, and $\xi = [\xi_1, \ldots, \xi_n]^{\mathrm{T}} \in \mathbb{R}^n$. Recall that

$$f'(\chi) = \begin{bmatrix} \frac{\partial f_1(\chi)}{\partial \chi_1} & \cdots & \frac{\partial f_1(\chi)}{\partial \chi_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_l(\chi)}{\partial \chi_1} & \cdots & \frac{\partial f_l(\chi)}{\partial \chi_n} \end{bmatrix} \in \mathbb{R}^{l \times n} \tag{A.3}$$

and

$$f'(\chi_{\mathrm{e}})\xi = \begin{bmatrix} \left.\frac{\partial f_1(\chi)}{\partial \chi_1}\right|_{\chi=\chi_{\mathrm{e}}} \xi_1 + \cdots + \left.\frac{\partial f_1(\chi)}{\partial \chi_n}\right|_{\chi=\chi_{\mathrm{e}}} \xi_n \\ \vdots \\ \left.\frac{\partial f_l(\chi)}{\partial \chi_1}\right|_{\chi=\chi_{\mathrm{e}}} \xi_1 + \cdots + \left.\frac{\partial f_l(\chi)}{\partial \chi_n}\right|_{\chi=\chi_{\mathrm{e}}} \xi_n \end{bmatrix} \in \mathbb{R}^l. \tag{A.4}$$

Theorem A.1 *If $f : \mathcal{D} \to \mathbb{R}^l$ is continuously differentiable at $\chi_{\mathrm{e}} \in \mathcal{D} \subseteq \mathbb{R}^n$, then*

$$f(\chi_{\mathrm{e}} + \xi) = f(\chi_{\mathrm{e}}) + f'(\chi_{\mathrm{e}})\xi + r_2(\xi), \tag{A.5}$$

for all $\xi \in \mathbb{R}^n$ such that $\chi_{\mathrm{e}} + \xi \in \mathcal{D}$, where

$$\lim_{\xi \to 0} \frac{r_2(\xi)}{\|\xi\|^2} = 0. \tag{A.6}$$

The term $r_2(\xi)$ in (A.5) is referred to as the *remainder* and, *per definition*, is the error for approximating $f(\chi_{\mathrm{e}}+\xi)$ with $f(\chi_{\mathrm{e}})+f'(\chi_{\mathrm{e}})\xi$. Equation (A.6) implies that if χ is sufficiently close to χ_{e}, then the $r_2(\xi)$ is negligible. Hence, Taylor's theorem allows us to *approximate* the nonlinear dynamical system (A.2) with

$$\dot{\chi}(t) = f(\chi_{\mathrm{e}}) + f'(\chi_{\mathrm{e}})\xi(t), \qquad \chi(0) = \chi_0, \qquad t \geq 0, \tag{A.7}$$

in a neighborhood of the point $\chi = \chi_{\mathrm{e}}$, where $\xi(t) = \chi(t) - \chi_{\mathrm{e}}$, $t \geq 0$.

Definition A.3 (*Equilibrium point*) The point $\chi_{\mathrm{e}} \in \mathcal{D}$ is an *equilibrium point* (or *fixed point*) for the nonlinear dynamical system (A.2) if

$$0 = f(\chi_{\mathrm{e}}). \tag{A.8}$$

Remark A.1 Consider the nonlinear dynamical system (A.2). If χ_e is an equilibrium point of (A.2) and $f(\cdot)$ is continuously differentiable at $\chi = \chi_e$, then (A.2) can be approximated by (A.7) in sufficiently small neighborhoods of the equilibrium point. Since

$$\frac{d}{dt}\chi(t) = \frac{d}{dt}[\chi(t) - \chi_e] = \frac{d}{dt}\xi(t), \qquad t \geq 0, \tag{A.9}$$

the nonlinear dynamical system (A.2) can be approximated by

$$\dot{\xi}(t) = f'(\chi_e)\,\xi(t), \qquad \xi(0) = \chi_0 - \chi_e, \qquad t \geq 0, \tag{A.10}$$

in a neighborhood of $\chi = \chi_e$.

Exercise A.2 Consider the *controlled* nonlinear dynamical system

$$\dot{\chi}(t) = F(\chi(t), \eta(t)), \qquad \chi(0) = \chi_0, \qquad t \geq 0, \tag{A.11}$$

$$\gamma(t) = H(\chi(t), \eta(t)), \tag{A.12}$$

where for all $t \geq 0$, $\chi(t) \in \mathcal{D}$, $\eta(t) \in \mathcal{U}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $\mathcal{U} \subseteq \mathbb{R}^m$ is an open set with $0 \in \mathcal{U}$, $F(\cdot,\cdot)$ and $H(\cdot,\cdot)$ are continuously differentiable on $\mathcal{D} \times \mathcal{U}$, $0 = F(0, 0)$, and $0 = H(0, 0)$. Apply Theorem A.1 to prove that

$$\dot{\chi}(t) = A\chi(t) + B\eta(t) + r_{2,1}(\xi), \qquad \chi(0) = \chi_0, \qquad t \geq 0, \tag{A.13}$$

$$\gamma(t) = C\chi(t) + D\eta(t) + r_{2,2}(\xi), \tag{A.14}$$

where

$$A \triangleq \left.\frac{\partial F(\chi, \eta)}{\partial \chi}\right|_{[\chi^T, \eta^T]^T = 0}, \qquad B \triangleq \left.\frac{\partial F(\chi, \eta)}{\partial \eta}\right|_{[\chi^T, \eta^T]^T = 0}, \tag{A.15}$$

$$C \triangleq \left.\frac{\partial H(\chi, \eta)}{\partial \chi}\right|_{[\chi^T, \eta^T]^T = 0}, \qquad D \triangleq \left.\frac{\partial H(\chi, \eta)}{\partial \eta}\right|_{[\chi^T, \eta^T]^T = 0}, \tag{A.16}$$

$\xi \in \mathbb{R}^{n+m}$, and $r_2 = [r_{2,1}^T, r_{2,2}^T]^T$. △

Remark A.2 It follows from Exercise A.2 that we can *approximate* the controlled *nonlinear* dynamical system (A.11) with the controlled *linear* dynamical system (3.1). The error for this approximation is given by $r_1\left(\left[(\chi - \chi_e)^T, (\eta - \eta_e)^T\right]^T\right)$.

A.3 The Matrix Exponential and the Solution of Linear Differential Equations

In Sect. A.2, we showed that nonlinear dynamical systems can be approximated by linear dynamical systems and the error due to this approximation is given by the remainder. In this section, we define the exponential of a real matrix and we find the solution of the linear differential equation (3.1).

Definition A.4 (*Exponential function*) Given $s \in \mathbb{C}$, we define the exponential function as

$$e^s \triangleq \sum_{k=0}^{\infty} \frac{s^k}{k!} = \lim_{n\to\infty} \left[1 + s + \cdots + \frac{s^n}{n!}\right]. \tag{A.17}$$

The following theorem is fundamental in control systems theory

Exercise A.3 (*Euler's theorem*) Prove that

$$e^{\jmath\omega} = \cos\omega + \jmath \sin\omega, \tag{A.18}$$

where $\omega \in \mathbb{R}$. *Hint*: Recall the Taylor formula of $\cos\omega$ and $\sin\omega$. △

Definition A.5 (*Exponential of a matrix*) Given the *square real* matrix $A \in \mathbb{R}^{n\times n}$, we define the *exponential of* A as

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \tag{A.19}$$

where A^k denotes the product of A by itself k times and $A^0 \triangleq I_n$.

The next result guarantees that the exponential of any square matrix is well-defined.

Proposition A.1 ([5, Proposition 11.1.2]) *Let $A \in \mathbb{R}^{n\times n}$. Then, there exists $M \in \mathbb{R}^{n\times n}$ such that $M = e^A$.*

The next result shows that matrix exponentials are key to characterize solutions of linear differential equations.

Theorem A.2 ([1, p. 30], [21, Ex. 2.37]) *Consider the linear dynamical system*

$$\dot{\chi}(t) = A\chi(t), \qquad \chi(0) = \chi_0, \qquad t \geq 0, \tag{A.20}$$

where for all $t \geq 0$, $\chi(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $\chi_0 \in \mathbb{R}^n$. Then,

$$\chi(t) = e^{At}\chi_0, \qquad t \geq 0, \tag{A.21}$$

is the unique solution of (A.20).

Next, we compute the unique solution of the controlled linear dynamical system (3.1).

Theorem A.3 ([1, p. 49], [21, Theorem 12.1]) *Consider the linear dynamical system (3.1). Then,*

$$\chi(t) = e^{At}\chi_0 + \int_0^t e^{A(t-\tau)} B\eta(\tau)\mathrm{d}\tau, \qquad t \geq 0, \tag{A.22}$$

is the unique solution of (3.1).

Theorem A.3 provides us with the *unique analytic closed-form solution* of the linear controlled differential equation (3.1). At this point, we are left with the difficult task of computing *both* e^{At} *and* $\int_0^t e^{A(t-\tau)} B\eta(\tau)\mathrm{d}\tau$ for all $t \geq 0$. Laplace transforms, which are discussed in Sect. A.11 below, will play a major role to this regard.

A.4 Row Space, Column Space, and Nullspace of a Matrix

A matrix $A \in \mathbb{R}^{n\times m}$ allows transforming vectors in $\mathbb{R}^m$ into vectors in $\mathbb{R}^n$. In this section, we provide necessary and sufficient conditions for A to transform any vector in $\mathbb{R}^m$ into *any* vector in $\mathbb{R}^n$.

Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} \in \mathbb{R}^{n\times m} \tag{A.23}$$

and recall that the *row vectors* of A are the n vectors $[a_{11}, \dots, a_{1m}]^{\mathrm{T}}, \dots, [a_{n1}, \dots, a_{nm}]^{\mathrm{T}} \in \mathbb{R}^m$, whereas the *column vectors* of A are the m vectors $[a_{11}, \dots, a_{n1}]^{\mathrm{T}}, \dots, [a_{1m}, \dots, a_{nm}]^{\mathrm{T}} \in \mathbb{R}^n$.

Definition A.6 (*Row space and column space*) Let $A \in \mathbb{R}^{n\times m}$. The *row space* of A is the subspace of $\mathbb{R}^m$ spanned by the row vectors of A. The *column space* of A is the subspace of $\mathbb{R}^n$ spanned by the column vectors of A.

The following result relates the dimension of the row and column spaces of A.

Theorem A.4 ([29, Theorem 4.15]) *Let $A \in \mathbb{R}^{n\times m}$. The row space of A and the columns space of A have the same dimension.*

Definition A.7 (*Rank of a matrix*) Let $A \in \mathbb{R}^{n\times m}$. The *rank* of A is the dimension of the row space of A and is denoted by $\mathrm{rank}(A)$.

It follows from Definition A.6 that the dimension of the row space of $A \in \mathbb{R}^{n\times m}$ is the largest number of linearly independent row vectors of A, whereas the dimension of the column space of $A \in \mathbb{R}^{n\times m}$ is the largest number of linearly independent column vectors of A. Hence, it follows from Theorem A.4 that the rank of A is the largest number of linearly independent row (or equivalently column) vectors of A.

Definition A.8 (*Nullspace of a matrix*) Let $A \in \mathbb{R}^{n \times m}$. The set

$$\mathcal{N}(A) \triangleq \{x \in \mathbb{R}^m : Ax = 0\} \tag{A.24}$$

is the *nullspace* of A.

It follows from the linearity of matrices that the nullspace of $A \in \mathbb{R}^{n \times m}$ is a subspace of $\mathbb{R}^n$ and the following definition is therefore justified.

Definition A.9 (*Nullity of a matrix*) Let $A \in \mathbb{R}^{n \times m}$. The dimension of $\mathcal{N}(A)$ is the *nullity* of A and is denoted by $\text{null}(A)$.

It follows from Definition A.9 that the nullity of a matrix $A \in \mathbb{R}^{n \times m}$ is maximum number of linearly independent vectors in $\mathcal{N}(A)$. The next result relates the nullity and the rank of a matrix. Specifically, the next result shows that the the sum of the rank and the nullity of a matrix A is equal to the number of columns of A.

Theorem A.5 ([29, Theorem 4.17], *Rank-nullity theorem*) Let $A \in \mathbb{R}^{n \times m}$. Then

$$m = \text{rank}(A) + \text{null}(A). \tag{A.25}$$

A.5 Determinant of a Matrix

In spite of their simple structure, matrices have an outstanding number of properties, a handful of which are presented in this Appendix. In most cases, the study of these properties often involves computing some scalar or vector functions, such as the determinant of a matrix.

Definition A.10 (*Cofactor matrix and minor of an element*) Let $A \in \mathbb{R}^{n \times n}$. The *cofactor matrix* of A is defined as

$$C_A \triangleq \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{bmatrix}, \tag{A.26}$$

where

$$c_{ij} = (-1)^{i+j} \det(M_{ij}) \tag{A.27}$$

and $M_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is obtained from A eliminating the ith row and the jth column. The determinant of M_{ij}, that is, $\det(M_{ij})$, is the *minor* of the element on the ith row and jth column of A.

Definition A.11 (*Determinant of a matrix*) Let $A \in \mathbb{R}^{n\times n}$. The *determinant of* A is defined as

$$\det(A) \triangleq \sum_{j=1}^{n} a_{1j}c_{1j}, \tag{A.28}$$

where a_{1j} is the element of A on the first row and jth column and c_{1j} is given by (A.27) with $i = 1$.

It is important to note that determinants are defined for *square* matrices only. In Sect. A.7 below, we provide alternative ways to compute the determinant of a matrix. The next result allows us to compute the determinant of the product of two matrices.

Theorem A.6 ([5, Proposition 2.7.3]) *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. Then,*

$$\det(AB) = \det(BA) = \det(A)\det(B). \tag{A.29}$$

A.6 Special Matrices

There exist several classes of matrices, whose properties are fundamental to better understand the dynamics of linear systems.

A.6.1 Invertible Matrices

For the statement of the next definition, let $\mathbb{X}^{n\times n}$ denote $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$.

Definition A.12 (*Matrix inverse and singular matrices*) The *inverse* of $A \in \mathbb{X}^{n\times n}$ is a matrix $B \in \mathbb{X}^{n\times n}$ such that $BA = AB = I_n$. If the inverse of A exists, then A is *nonsingular*, or equivalently *invertible*, and the inverse of A is denoted by A^{-1}. Otherwise, A is *singular*.

The next result allows us to compute the inverse of a matrix.

Theorem A.7 ([5, Proposition 2.7.5]) *Let $A \in \mathbb{R}^{n\times n}$ be invertible. Then,*

$$A^{-1} = \frac{1}{\det(A)} C_A^{\mathrm{T}}, \tag{A.30}$$

where $C_A \in \mathbb{R}^{n\times n}$ is the cofactor matrix of A.

Exercise A.4 Let $A \in \mathbb{R}^{n\times n}$. Prove that $\det(A^{-1}) = [\det(A)]^{-1}$. △

For the statement of the next results, let $\mathbb{X}^{n\times n}$ denote $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$.

Theorem A.8 ([29, Theorem 3.7]) *Let $A \in \mathbb{X}^{n\times n}$. Then, A is singular if and only if* $\det(A) = 0$.

Theorem A.9 ([5, Theorem 2.6.1]) *Let $A \in \mathbb{X}^{n\times n}$. Then, A is singular if and only if there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that*

$$Ax = 0. \tag{A.31}$$

A.6.2 Orthogonal, Rotation, and Reflection Matrices

As discussed in Sect. 1.4, orthogonal matrices play a key role in the description of the rotational dynamics of rigid bodies.

Definition A.13 (*Orthogonal matrix*) Let $A \in \mathbb{R}^{n\times n}$ be invertible. If $A^{-1} = A^{\mathrm{T}}$, then A is *orthogonal*.

Exercise A.5 Let $A_1 \in \mathbb{R}^{n\times n}$ and $A_2 \in \mathbb{R}^{n\times n}$ be orthogonal matrices. Prove that $A_1 A_2$ is an orthogonal matrix. △

Exercise A.6 Let $A \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Prove that $\det(A) = \pm 1$. *Hint:* Recall Theorem A.6. △

The sign of the determinant of an orthogonal matrix allows us distinguishing between two types of matrices, namely rotations and reflections.

Definition A.14 (*Rotation and reflection matrices*) Let $A \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. If $\det(A) = 1$, then A is a *rotation matrix*. Alternatively, if $\det(A) = -1$, then A is a *reflection matrix*.

A.7 Eigenvalues and Eigenvectors of Matrices

In this section, we recall some fundamental property of eigenvalues and eigenvectors of matrices, which play a key role in estimating the dynamics of linear and nonlinear systems.

Definition A.15 (*Characteristic polynomial and eigenvalues*) Let $A \in \mathbb{R}^{n\times n}$. The *characteristic polynomial* of A is the defined as

$$\chi_A(s) \triangleq \det(sI_n - A), \qquad s \in \mathbb{C}. \tag{A.32}$$

The complex number $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if it is a root of the characteristic polynomial $\chi_A(\cdot)$, that is,

$$\det(\lambda I_n - A) = 0. \tag{A.33}$$

The next result shows that the characteristic polynomial of the matrix A has n (possibly repeated) roots $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

Theorem A.10 *Let $A \in \mathbb{R}^{n\times n}$. Then,*

$$\chi_A(s) = \prod_{k=1}^{n}(s - \lambda_k), \tag{A.34}$$

where λ_k is an eigenvalue of A.

Proof The result directly follows from the fundamental theorem of algebra [23, p. 561]. ■

Definition A.16 (*Algebraic multiplicity and simple eigenvalues*) Let $A \in \mathbb{R}^{n\times n}$. If the root λ_i, $i = 1, \ldots, n$, of $\chi_A(s)$ is repeated p_i times, then we say that the eigenvalue λ_i of A has *algebraic multiplicity* p_i. If $p_i = 1$, $i = 1, \ldots, n$, then λ_i is a *simple* eigenvalue.

Definition A.17 (*Spectrum of a matrix*) Let $A \in \mathbb{R}^{n\times n}$. The *spectrum* of A is the set of all eigenvalues of A including multiplicity, that is,

$$\operatorname{spec}(A) \triangleq \{\lambda \in \mathbb{C} : \det(\lambda I_n - A) = 0\}. \tag{A.35}$$

Next, we relate the eigenvalues of a matrix to its determinant.

Theorem A.11 *Let $A \in \mathbb{R}^{n\times n}$. Then,*

$$\det(A) = \prod_{k=1}^{n}\lambda_k, \tag{A.36}$$

where $\lambda_k \in \operatorname{spec}(A)$, $k = 1, \ldots, n$.

Proof It follows from Theorem A.10 that

$$\chi_A(0) = \det(0 I_n - A) = \det(-A) = \prod_{k=1}^{n}(0 - \lambda_k). \tag{A.37}$$

Now, it follows from Theorem A.6 that

$$\det(-A) = \det(-I_n A) = \det(-I_n)\det(A) = (-1)^n \det(A) \tag{A.38}$$

and the result follows immediately. ■

The next theorem proves a fundamental property of eigenvalues.

Theorem A.12 *Let $A \in \mathbb{R}^{n\times n}$. Then, $\lambda \in \operatorname{spec}(A)$ if and only if there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that*

$$Ax = \lambda x. \tag{A.39}$$

Proof It follows from Definition A.15 that $\lambda \in \operatorname{spec}(A)$ if and only if $\det(\lambda I_n - A) = 0$. Hence, it follows from Theorem A.8 that $\lambda \in \operatorname{spec}(A)$ if and only if $\lambda I_n - A$ is singular, and it follows from Theorem A.9 that $\lambda I_n - A$ is singular if and only if there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that $(\lambda I_n - A)x = 0$, which concludes the proof. ■

Exercise A.7 Prove that $A \in \mathbb{R}^{n\times n}$ is singular if and only if $0 \in \operatorname{spec}(A)$. △

Theorem A.13 ([29, p. 246]) *Let $A \in \mathbb{R}^{n\times n}$. Then, A is nonsingular if and only if the column vectors of A are linearly independent.*

Exercise A.8 Let $A \in \mathbb{R}^{n\times n}$. Prove that $\operatorname{rank}(A) = n$ if and only if A is invertible, that is, if and only if $\det(A) \neq 0$, that is, if and only if $0 \notin \operatorname{spec}(A)$. △

The next proposition allows us to *immediately* find the eigenvalues of a lower or upper triangular matrix.

Proposition A.2 ([2, pp. 86–87]) *Let $A \in \mathbb{R}^{n\times n}$ be an upper (lower) triangular matrix. The elements on the diagonal of A are the eigenvalues of A, that is,* $\operatorname{spec}(A) = \operatorname{diag}(A)$.

The next results are fundamental to analyze the properties of rotation matrices.

Exercise A.9 Let $R \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix, where

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{A.40}$$

Prove that

$$r_{11} = \det(R)\,(r_{22}r_{33} - r_{23}r_{32}), \tag{A.41}$$

$$r_{22} = \det(R)\,(r_{11}r_{33} - r_{13}r_{31}), \tag{A.42}$$

$$r_{33} = \det(R)\,(r_{11}r_{22} - r_{12}r_{21}). \tag{A.43}$$

△

Theorem A.14 *Consider the rotation matrix R given by* (A.40). *Then,*

$$\operatorname{spec}(R) = \{1, e^{\imath\phi}, e^{-\imath\phi}\}, \tag{A.44}$$

where $\phi \in \mathbb{R}$ is such that

$$\cos\phi = \frac{1}{2}\left(r_{11} + r_{22} + r_{33} - 1\right). \tag{A.45}$$

Proof It follows from Definition A.15 and Theorem A.11 that

$$
\begin{aligned}
0 = -\lambda^3 + (r_{11} + r_{22} + r_{33})\,\lambda^2 - [(r_{22}r_{33} - r_{32}r_{23}) + (r_{11}r_{33} - r_{31}r_{13}) \\
+ (r_{22}r_{11} - r_{12}r_{21})]\lambda + \det(R), \qquad \lambda \in \mathbb{C},
\end{aligned}
\tag{A.46}
$$

and since R is orthogonal, it follows from Exercise A.9 that

$$
\begin{aligned}
0 &= \lambda^3 - (r_{11} + r_{22} + r_{33})\,\lambda^2 + (r_{11} + r_{22} + r_{33})\,\lambda - 1 \\
&= (\lambda - 1)\left[\lambda^2 - (r_{11} + r_{22} + r_{33} - 1)\,\lambda + 1\right], \qquad \lambda \in \mathbb{C}.
\end{aligned}
\tag{A.47}
$$

Therefore, the assertion is proven since (A.47) is satisfied by $\lambda = 1$, $\lambda = e^{\imath\phi}$, and $\lambda = e^{-\imath\phi}$. ■

It follows from Theorem A.12 that if λ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$, then there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that (A.39) is satisfied. This vector x plays a key role in linear systems dynamics.

Definition A.18 (*Eigenvectors and eigenpairs*) Let $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \operatorname{spec}(A)$. Then, any $x \in \mathbb{C}^n \setminus \{0\}$ such that (A.39) is satisfied is an *eigenvector* of A associated with λ. The pair (λ, x) is an *eigenpair* of A.

It is important to emphasize the fact that the zero vector cannot be an eigenvector of a matrix. The next result shows that the eigenvalues of a real matrix occur in complex conjugate pairs.

Theorem A.15 ([5, Proposition 4.4.5]) *Let* $A \in \mathbb{R}^{n\times n}$. *If* $\lambda \in \operatorname{spec}(A)$, *then* $\lambda^* \in \operatorname{spec}(A)$.

Remark A.3 Consider $A \in \mathbb{R}^{n\times n}$ and let $\lambda_i \in \operatorname{spec}(A)$, $i = 1, \ldots, n$. Assume, without loss of generality, that the first $2p$ eigenvalues of A are complex conjugate pairs, that is, $\lambda_{p+1} = \lambda_1^*$, $\lambda_{p+2} = \lambda_2^*, \ldots, \lambda_{2p} = \lambda_p^*$, and the remaining $n - 2p$ eigenvalues are real. It follows from Theorems A.10 and A.15 that

$$
\chi_A(s) = \prod_{k=1}^{2p}(s - \lambda_k) \prod_{k=2p+1}^{n}(s - \lambda_k).
\tag{A.48}
$$

Now, let $\lambda_i = \sigma_i + \jmath\omega_i$, $i = 1, \ldots, p$, and note that

$$
\chi_A(s) = \prod_{k=1}^{p}(s^2 - 2\sigma_k s + \sigma_k^2 + \omega_k^2) \prod_{k=2p+1}^{n}(s - \lambda_k).
\tag{A.49}
$$

Hence, *we can always write the characteristic polynomial of a real matrix as the product of monomials and binomials with real coefficients*.

Exercise A.10 Let $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \operatorname{spec}(A)$. Prove that if $x \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector of A associated to $\lambda \in \operatorname{spec}(A)$, then there exist *infinitely many eigenvectors* associated with λ. △

The next result proves that if a real matrix has n distinct eigenvalues, then its eigenvectors are linearly independent.

Theorem A.16 ([29, Theorem 7.6]) *Let $A \in \mathbb{R}^{n\times n}$ and (λ_i, x_i), $i = 1, \ldots, n$, be an eigenpair of A. If A has n distinct eigenvalues, that is, $\lambda_i \neq \lambda_j$ for all $i, j, = 1, \ldots, n$, then $\{x_1, \ldots, x_n\}$ are linearly independent.*

If $A \in \mathbb{R}^{n\times n}$ has p distinct eigenvalues, where $p < n$, then each eigenvalue of A may be associated to multiple linearly independent eigenvectors. For details, see [1, p. 479] and [5, Chap. 5].

Definition A.19 (*Geometric multiplicity*) Let $A \in \mathbb{R}^{n\times n}$. The *geometric multiplicity* γ_λ of $\lambda \in \operatorname{spec}(A)$ is the number of linearly independent eigenvectors associated with λ. That is,

$$\gamma_\lambda = \dim\left[\mathcal{N}(A - \lambda I)\right], \tag{A.50}$$

where $\dim(\cdot)$ denotes the dimension of a space.

Remarkably, it follows from Theorem A.5 that $\gamma_\lambda = n - \operatorname{rank}(A - \lambda I)$.

Definition A.20 (*Semisimple eigenvalue*) Let $A \in \mathbb{R}^{n\times n}$. Then, $\lambda \in \operatorname{spec}(A)$ is semisimple if the algebraic and geometric multiplicities of λ are equal.

Consider the matrix $A \in \mathbb{R}^{n\times n}$. It follows from Definition A.16 that if $\lambda \in \operatorname{spec}(A)$ has algebraic multiplicity greater than one, it is not always possible to apply Definition A.18 and find n linear independent eigenvectors of A. For this reason, the notion of generalized eigenvector has been introduced.

Definition A.21 Let $A \in \mathbb{R}^{n\times n}$, $\lambda \in \operatorname{spec}(A)$, and $x \in \mathbb{C}^n \setminus \{0\}$. If

$$(A - \lambda I)^p x = 0, \tag{A.51}$$

and

$$(A - \lambda I)^{p-1} x \neq 0, \tag{A.52}$$

then x a *generalized eigenvector associated to λ of rank p*.

It follows from Definitions A.18 and A.21 that if (λ, x) is an eigenpair of A, then x is a generalized eigenvector of rank 1. It can be proven that if $A \in \mathbb{R}^{n\times n}$ does *not* have n distinct eigenvalues, then there is a set of n linearly independent eigenvectors and generalized eigenvectors; for details, see [5, Fact 5.14.9].

A.8 Stability and Asymptotic Behavior of linear dynamical systems

In Sect. A.3, we proved that (A.21) and (A.22) are the solutions of the linear differential equations (A.20) and (3.1), respectively, and discussed the difficulty of finding an algorithm that allows us to compute (A.21) and (A.22) numerically. Now, suppose we designed an algorithm to numerically compute (A.21) and (A.22) for some $t \geq 0$. An extremely challenging task would be to apply this algorithm and compute (A.21) and (A.22) for $t \to \infty$. In this section, we discuss some technique to predict the behavior of (A.21) as $t \to \infty$. A study on the asymptotic behavior of (A.22) would require introducing notions such as *input-to-state* and *input-output stability* [27, Chap. 5], which are beyond the scopes of this brief.

The relevance of (A.20) and (A.21), however, should not be underestimated. In fact, assume (3.1) captures the dynamics of an aircraft. It is undesirable for a pilot to constantly operate the airplane controls, such as the ailerons and the rudder, during the entire cruise. Therefore, aircraft are designed so that if an external perturbation, such as a wind gust, occurs, then it will *eventually* return to its state of equilibrium without any action from the pilot.

A.8.1 Stability of Linear Dynamical Systems

In this section, we present the notion of stability of the linear dynamical system (A.20). For the statement of the next result, let $\overline{\mathbb{C}_-}$ denote the set of complex numbers with nonpositive real part and $\mathbb{C}_-$ the set of complex numbers with negative real part.

Definition A.22 (*Stability of a linear dynamical system*) Consider the linear dynamical system $\mathcal{G}$ given by (A.20).

(i) $\mathcal{G}$ is *Lyapunov stable* if $\lambda \in \text{spec}(A) \subset \overline{\mathbb{C}_-}$ and, if the real part of λ is equal to zero, then λ is semisimple.
(ii) $\mathcal{G}$ is *semistable* if $\lambda \in \text{spec}(A) \subset \mathbb{C}_- \cup \{0\}$ and, if $\lambda = 0$, then λ is semisimple.
(iii) $\mathcal{G}$ is *asymptotically semistable* if $\lambda \in \text{spec}(A) \subset \mathbb{C}_-$.
(iv) $\mathcal{G}$ is *unstable* if $\mathcal{G}$ is not Lyapunov stable.

It follows from Definition A.22 that asymptotic stability implies semistability, which implies Lyapunov stability. The converse, however, is not true. If there is one eigenvalue of A, whose real part is positive, then the linear dynamical system (A.20) is unstable. In addition, if there is one purely imaginary eigenvalue of A which is not semisimple, then the linear dynamical system (A.20) is unstable. If the the linear dynamical system (A.20) is Lyapunov stable, semistable, asymptotically stable, or unstable, then we say that the matrix A is Lyapunov stable, semistable, asymptotically stable, or unstable, respectively.

The next result is key to deduce the behavior of the uncontrolled linear dynamical system (A.20) from the spectrum of A.

Theorem A.17 *Consider the linear dynamical system $\mathcal{G}$ given by* (A.20).

(i) If $\mathcal{G}$ is Lyapunov stable, then $e^{At}\chi_0$ is bounded for all $t \geq 0$ and for all $\chi_0 \in \mathbb{R}^n$, that is, there exists $M > 0$ such that

$$\|e^{At}\chi_0\| < M, \qquad \chi_0 \in \mathbb{R}^n, \qquad t \geq 0. \tag{A.53}$$

(ii) If $\mathcal{G}$ is semistable, then

$$\lim_{t\to\infty} e^{At}\chi_0 = \chi_{\mathrm{e}} \tag{A.54}$$

for all $\chi_0 \in \mathbb{R}^n$, where χ_{e} is such that $0 = A\chi_{\mathrm{e}}$.

(iii) If $\mathcal{G}$ is asymptotically stable, then

$$\lim_{t\to\infty} e^{At}\chi_0 = 0 \tag{A.55}$$

for all $\chi_0 \in \mathbb{R}^n$.

(iv) If $\mathcal{G}$ is unstable, then

$$\lim_{t\to\infty} \|e^{At}\chi_0\| = \infty \tag{A.56}$$

for all $\chi_0 \in \mathbb{R}^n \setminus \{0\}$.

The proof of Theorem A.17 is provided in Sect. A.8.2. It follows from Definition A.22 and Theorem A.17 that if the linear dynamical system (A.20) is semistable or asymptotically stable, then $e^{At}\chi_0$ is bounded for all $t \geq 0$ and for all $\chi_0 \in \mathbb{R}^n$. Let (A.20) approximate the equations of motion of an aircraft in a neighborhood of an equilibrium point. Theorem A.17 allows predicting the aircraft response to disturbances, such as wind gusts, which induced an initial displacement χ_0 from the equilibrium point. A well-designed civil aircraft is asymptotically stable, since it is desirable that the aircraft response to any perturbation is bounded and that the aircraft eventually returns to its state of equilibrium. Some military aircraft, such as the F-16 Fighting Falcon, is designed to be unstable in order to improve its responsiveness and maneuverability. Should an aircraft not be asymptotically stable, one of the control engineer's main tasks is to design state- or output-feedback control laws, such that the controlled aircraft is asymptotically stable.

It follows from Definition A.22 that the stability properties of the linear dynamical system (A.20) depend on the eigenvalues of A, which are the roots of the characteristic polynomial. In many cases of practical interest, the degree of a characteristic polynomial is three or higher and hence it is not easy to compute the eigenvalues of a matrix without the assistance of a computer. *Routh stability criterion* provides a formidable tool to determine the sign of the real part of the roots of a real polynomial by analyzing the coefficients of the polynomial. For details, see [36, Chap. 5].

A.8.2 Analysis of a Linear System—The Jordan Decomposition

In this section, we show that a square matrix A can be written as the product of two invertible matrices and a block-diagonal matrix, known as *Jordan form of* A. Moreover, we provide a proof of Theorem A.17.

The *Jordan form* of $A \in \mathbb{R}^{n\times n}$ is a block-diagonal matrix $J = \text{block}-\text{diag}[J_1, \ldots, J_m]$, where J_i is called the i-th Jordan block. The i-th Jordan block is such that $J_i = \lambda_i I_{n_i} + N_{n_i}$, $i = 1, \ldots, m$, where $\lambda_i \in \text{spec}(A)$ and $N_{n_i} \in \mathbb{R}^{n_i\times n_i}$ has ones on the superdiagonal and zeros elsewhere; by convention $N_1 = 0$. We associate to each eigenvalue of A as many Jordan blocks as its geometric multiplicity and the sum of the number of rows of the Jordan blocks associated to an eigenvalue is equal to the eigenvalue's algebraic multiplicity.

Example A.1 Let $A = \begin{bmatrix} 2 & 4 & -8 \\ 0 & 0 & 4 \\ 0 & -1 & 4 \end{bmatrix}$. It holds that $\text{spec}(A) = \{2, 2, 2\}$ and the geometric multiplicity of $\lambda = 2$ is 2. Thus, the Jordan form of A is $J = \text{block}-\text{diag}[J_1, J_2]$, where $J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $J_2 = 2$ are its Jordan blocks. Hence, $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. △

Remarkably, if $\lambda_i \in \text{spec}(A)$ is semisimple, then there are as many Jordan blocks as the algebraic multiplicity of λ_i. Moreover, in this case the Jordan blocks associated to λ_i are scalars.

Example A.2 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 4 & -2 & 3 \end{bmatrix}$. It holds that $\text{spec}(A) = \{1, 1, 2\}$ and the geometric multiplicity of $\lambda = 1$ is 2. Thus, the Jordan form of A is $J = \text{block}-\text{diag}[J_1, J_2, J_3]$, where $J_1 = 1$, $J_2 = 1$, and $J_3 = 2$ are its Jordan blocks. Hence, $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. △

Theorem A.18 ([5, Theorem 5.3.3]) *Let $A \in \mathbb{R}^{n\times n}$. Then there exists an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$A = P^{-1} J P, \tag{A.57}$$

where J is the Jordan form of A.

The matrix P, such that (A.57) is satisfied, is the *generalized modal matrix* of A and the column vectors of P are the generalized eigenvectors of A [5, Fact 5.14.9].

Exercise A.11 Let $A \in \mathbb{R}^{n \times n}$. Prove that

$$e^{At} = P^{-1} e^{Jt} P, \qquad t \geq 0, \tag{A.58}$$

where $J \in \mathbb{C}^{n \times n}$ is the Jordan form of A. *Hint:* Recall Definition A.5. △

Proof of Theorem A.17. Consider the linear dynamical system (A.20). It follows from Theorem A.2 and (A.58) that

$$y(t) = e^{Jt} y_0, \qquad t \geq 0, \tag{A.59}$$

where $y(t) = P\chi(t)$, $y_0 = P\chi_0$, and $\chi(\cdot)$ satisfies (A.20). Since $e^{Jt} = \text{block} - \text{diag}\,[e^{J_1 t}, \ldots e^{J_m t}]$, consider the matrix function $e^{J_i t}$, $i = 1, \ldots, m$. Since $I_{n_i} N_{n_i} = N_{n_i} I_{n_i} = N_{n_i}$, it follows from Proposition 11.1.5 of [5] that $e^{J_i t} = e^{\lambda_i I_{n_i} t} e^{N_{n_i} t}$. In addition, it follows from Exercise A.3 that

$$e^{\lambda_i I_{n_i} t} = e^{\sigma_i t} \left[\cos(\omega_i t) + \jmath \sin(\omega_i t)\right] I_{n_i}, \qquad t \geq 0, \tag{A.60}$$

where $\lambda_i = \sigma_i + \jmath \omega_i$, $i = 1, \ldots, m$. Furthermore, since $N_{n_i}^{n_i} = 0_{n_i \times n_i}$, it follows from Definition A.5 that

$$e^{J_i t} = e^{\sigma_i t} \left[\cos(\omega_i t) + \jmath \sin(\omega_i t)\right] \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n_i - 1}}{(n_i - 1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i - 2}}{(n_i - 2)!} \\ \vdots & & & & \vdots \\ 0 & & \cdots\cdots & & 1 \end{bmatrix}. \tag{A.61}$$

Hence, if $\sigma_i < 0$, then $\lim_{t \to \infty} e^{J_i t} = 0$. Alternatively, if $\sigma_i > 0$, then $\lim_{t \to \infty} e^{J_i t} = \infty$. Lastly, if $\sigma_i = 0$, then $e^{J_i t} = [\cos(\omega_i t) + \jmath \sin(\omega_i t)]\, e^{N_{n_i} t}$. Consequently, if $\sigma_i = 0$, then $e^{J_i t}$ is bounded if and only if $n_i = 1$, that is, if and only if λ_i is semisimple. In fact, if $n_i > 1$, then $\lim_{t \to \infty} e^{J_i t} = \lim_{t \to \infty} e^{N_{n_i} t} = \infty$. ■

Remark A.4 Let $y_i(\cdot)$ denote the i-th component of $y(\cdot)$ in (A.59). If $\sigma_i < \sigma_j < 0$, $i, j = 1, \ldots, m$, then it follows from (A.61) that $y_i(t)$, $t \geq 0$, converges to zero "faster" than $y_j(t)$.

In general, it is difficult to give a physical meaning to the complex vector function $y(\cdot)$. However, analyzing the Jordan form of A allowed us understanding that the effect of the eigenvalues of A, whose real part is negative and larger in absolute value, fades before the effect of the eigenvalues, whose real part is negative and smaller in absolute value. This consideration is important in the study of the linearized equations of motion of an aircraft.

A.9 Controllability and Observability

A fundamental issue in the study of dynamical system is whether it is possible to steer the system state from any initial condition to any final condition in finite time. Furthermore, it is important to assess whether one can reconstruct the system state over a finite time interval given some output function. Although addressing this issue for nonlinear dynamical systems is beyond the scopes of this brief, in this section we consider the controllability and observability problems for linear dynamical systems.

A.9.1 Controllability and Reachability of Linear Dynamical Systems

The following definition introduces the notion of reachable linear dynamical systems.

Definition A.23 (*Reachable linear dynamical system*) Consider the linear dynamical system (3.1). The point $\chi_{\mathrm{f}} \in \mathbb{R}^n$ is *reachable* if there exists a continuous control $\eta : [0, t_{\mathrm{f}}] \to \mathbb{R}^m$ such that

$$\chi_{\mathrm{f}} = \int_0^{t_{\mathrm{f}}} e^{A(t_{\mathrm{f}}-t)} B\eta(t)\mathrm{d}t, \tag{A.62}$$

where $t_{\mathrm{f}} \in [0, \infty)$. The linear dynamical system (3.1) is reachable if every $\chi_{\mathrm{f}} \in \mathbb{R}^n$ is reachable.

It follows from Theorem A.3 that, according to Definition A.23, a linear dynamical system is reachable if and only if there exists some *continuous* control $\eta(\cdot)$ that steers $\chi(\cdot)$ *from the origin to* χ_{f} *in finite time*, for all $\chi_{\mathrm{f}} \in \mathbb{R}^n$. The time t_{f} needed to steer the system from $\chi = 0$ to χ_{f} is not specified. Since the linear dynamical system (3.1) is completely characterized the pair of matrices (A, B) and the initial condition χ_0, it is common to say that the pair (A, B) is reachable to mean that (3.1) is reachable.

Next, we introduce the definition of controllability.

Definition A.24 (*Controllable linear dynamical system*) Consider the linear dynamical system (3.1). The initial condition $\chi_0 \in \mathbb{R}^n$ is *controllable* if there exists a continuous control $\eta : [0, t_{\mathrm{f}}] \to \mathbb{R}^m$ such that

$$0 = e^{At_{\mathrm{f}}}\chi_0 + \int_0^{t_{\mathrm{f}}} e^{A(t_{\mathrm{f}}-t)} B\eta(t)\mathrm{d}t, \tag{A.63}$$

where $t_{\mathrm{f}} \in [0, \infty)$. The linear dynamical system (3.1) is controllable if every $\chi_0 \in \mathbb{R}^n$ is controllable. Finally, (3.1) is uncontrollable if it is not controllable.

Definition A.24, which was introduced by Kalman in 1957, states that a linear dynamical system is controllable if and only if there exists some control law $\eta(\cdot)$ that steers $\chi(\cdot)$ *from* χ_0 *to the origin* $\chi = 0$ *in finite time*, for all $\chi_0 \in \mathbb{R}^n$. The time t_f needed to steer the system from χ_0 to $\chi = 0$ is not specified. It is common to say that the pair (A, B) is controllable to mean that (3.1) is controllable.

The following result is key to relate controllability and reachability of linear dynamical systems.

Theorem A.19 ([1, Theorem 5.20]) *The linear dynamical system (3.1) is controllable if and only if (3.1) is reachable.*

Although Definitions A.23 and A.24 are intuitive, it is quite difficult to verify reachability and controllability of a linear dynamical system by finding a control $\eta(\cdot)$ that satisfies (A.62) or (A.63). However, several tests have been developed to achieve this goal and the following is rather easy to apply.

For the statement of the next result, given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, we define the *controllability matrix*

$$\mathcal{C}(A, B) \triangleq [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times mn}. \tag{A.64}$$

Theorem A.20 ([1, Cor. 5.13]) *The linear dynamical system (3.1) is reachable if and only if*

$$\operatorname{rank}(\mathcal{C}(A, B)) = n. \tag{A.65}$$

Theorem A.21 *The linear dynamical system (3.1) is controllable if and only if (A.65) is satisfied.*

Proof The result directly follows from Theorems A.19 and A.20. ■

A.9.2 Observability of Linear Dynamical Systems

Consider the linear dynamical system given by (3.1) and (3.2). It follows from Theorem A.3 that

$$y(t) = Ce^{At}\chi_0 + \int_0^t Ce^{A(t-\tau)}B\eta(\tau)\mathrm{d}\tau + D\eta(t), \qquad t \geq 0. \tag{A.66}$$

Hence, (3.1) and (3.2) is equivalent to

$$\dot{\chi}(t) = A\chi(t) + B\eta(t), \qquad \chi(0) = \chi_0, \qquad t \geq 0, \tag{A.67}$$

$$\hat{y}(t) = Ce^{At}\chi_0, \tag{A.68}$$

where

$$\hat{\gamma}(t) \triangleq \gamma(t) - \int_0^t Ce^{A(t-\tau)}B\eta(\tau)\mathrm{d}\tau - D\eta(t).$$

This simple observation greatly simplifies the study of the output properties of linear dynamical systems, as it allows us focussing our attention, without loss of generality, on linear dynamical systems with direct transmission matrix $D = 0$.

Definition A.25 Consider the linear dynamical system given by (A.67) and (A.68). The initial condition $\chi_0 \neq 0$ is observable if

$$Ce^{At}\chi_0 \not\equiv 0, \qquad t \geq 0. \tag{A.69}$$

The linear dynamical system (A.67) and (A.68) is observable if every $\chi_0 \in \mathbb{R}^n \setminus \{0\}$ is observable.

Definition A.25 states that the initial condition $\chi_0 \neq 0$ of the linear dynamical system (A.67) and (A.68) *is observable if and only if the output is not identically equal to zero*. Hence, observability is the property whereby a non-zero initial condition can be distinguished from the zero initial condition. It is important to observe that if (A.67) and (A.68) is observable, then it may occur that $\hat{\gamma}(t) = 0$ on some *finite* time interval, that is, for some $t \in [t_1, t_2] \subset [0, \infty)$. It is common to say that the pair (A, C) is observable to mean that (A.67) and (A.68) is observable.

Given $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$, for the statement of the next result, which allows us verifying observability of a linear dynamical system by mean of a rank condition, we define the *observability matrix*

$$\mathcal{O}(A, C) \triangleq \begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{ln\times n}. \tag{A.70}$$

Theorem A.22 ([1, Cor. 5.38]) *The linear dynamical system given by* (A.67) *and* (A.68) *is observable if and only if*

$$\operatorname{rank}(\mathcal{O}(A, C)) = n. \tag{A.71}$$

The analogy between Theorems A.21 and A.22 is quite stunning. Indeed, we can establish a duality principle between controllability and observability of linear dynamical systems.

Exercise A.12 (*Duality principle*) Consider the linear dynamical system given by (A.67) and (A.68) and prove that the pair (A, B) is controllable if and only if the pair $(A^{\mathrm{T}}, B^{\mathrm{T}})$ is observable. Prove also that the pair (A, C) is observable if and only if the pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is controllable. △

A.10 Differentiation of Complex Functions

In this section, we recall several properties of analytic complex functions.

Exercise A.13 Let $h : \mathbb{C} \to \mathbb{C}$ and $\xi = \sigma + \jmath\omega$. Prove that

$$\frac{h(s)}{\xi} = \frac{1}{\sigma^2 - \omega^2}\left[\sigma\Re(h(s)) + \omega\Im(h(s))\right] + \frac{\jmath}{\sigma^2 - \omega^2}\left[\sigma\Im(h(s)) - \omega\Re(h(s)))\right]. \tag{A.72}$$

△

Definition A.26 (*Derivative of a complex function*) Let $\mathcal{D} \subset \mathbb{C}$ be an open set. We define the derivative of $g : \mathcal{D} \to \mathbb{C}$ at $s \in \mathcal{D}$ along the direction $\xi \in \mathbb{C}$ as

$$\left.\frac{dg(s)}{ds}\right|_{\xi} \triangleq \lim_{t\to 0}\frac{g(s+t\xi) - g(s)}{t\xi}, \tag{A.73}$$

where $t \in \mathbb{R}$ is such that $s + t\xi \in \mathcal{D}$.

The definition of the derivative of a complex function involves the ratio of two complex quantities. In order to better understand (A.73), let $g : \mathbb{C} \to \mathbb{C}$, $\xi \in \mathbb{C}$, and $h(s) = g(s + t\xi) - g(s)$. It follows from Exercise A.13 that

$$\begin{aligned}\frac{g(s+t\xi) - g(s)}{t\xi} &= \frac{1}{t\sigma^2 - t\omega^2}\left[\sigma\Re(h(s)) + \omega\Im(h(s))\right]\\ &\quad + \frac{\jmath}{t\sigma^2 - t\omega^2}\left[\sigma\Im(h(s)) - \omega\Re(h(s)))\right],\end{aligned} \tag{A.74}$$

where $\Re(h)$ and $\Im(h)$ denote the real and imaginary parts of h, respectively. Since $\frac{1}{t\sigma^2 - t\omega^2}\left[\sigma\Re(h(s)) + \omega\Im(h(s))\right]$ and $\frac{1}{t\sigma^2 - t\omega^2}\left[\sigma\Im(h(s)) - \omega\Re(h(s)))\right]$ are real fractional functions, computing the limit of these two quantities is a simpler task.

Note that, in general, the *derivative of* $g(\cdot)$ *at* s *depends on* s *and* ξ. Next, we introduce a class of complex functions such that $\left.\frac{dg(s)}{ds}\right|_{\xi}$ does *not* depend on ξ.

Definition A.27 (*Analytic function, ordinary points, and singular points*) Let $g : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \subseteq \mathbb{C}$ is an open set. Then $g(\cdot)$ is *analytic* (or *holomorphic* or *differentiable*) *at* $s = s_0$, $s_0 \in \mathcal{D}$, if $\left.\frac{dg(s)}{ds}\right|_{\xi}$ exists at $s = s_0$, for all $\xi \in \mathbb{C}$, and

$$\left.\frac{dg(s)}{ds}\right|_{\xi_1} = \left.\frac{dg(s)}{ds}\right|_{\xi_2} \tag{A.75}$$

at $s = s_0$, for all ξ_1 and $\xi_2 \in \mathbb{C}$. Furthermore, $g(\cdot)$ is analytic on $\mathcal{D}$ if $g(\cdot)$ is analytic for all $s \in \mathcal{D}$. If $g(\cdot)$ is analytic at $s_0 \in \mathcal{D}$, then s_0 is an *ordinary point* of $g(\cdot)$. If $g(\cdot)$

is not analytic at $s_0 \in \mathcal{D}$ and every neighborhood of s_0 contains at least one point where $g(\cdot)$ is analytic, then s_0 is a *singular point* of $g(\cdot)$. If $g(\cdot)$ is not analytic at $s_0 \in \mathcal{D}$, and there exists a neighborhood $\mathcal{B} \subseteq \mathcal{D}$ of s_0 such that $g(\cdot)$ is analytic *for all* $s \in \mathcal{B} \setminus \{s_0\}$, then s_0 is an *isolated singular point* of $g(\cdot)$.

If $g(\cdot)$ is analytic on $\mathcal{D}$, then its derivative is independent of ξ and we denote the derivative of $g(\cdot)$ at $s \in \mathcal{D}$ by $\frac{dg(s)}{ds}$. The next result provides a *necessary condition* for a complex-valued function to be analytic.

Theorem A.23 *Consider the complex-valued function $g : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \subseteq \mathbb{C}$ is an open set. If $g(\cdot)$ is analytic on $\mathcal{D}$, then*

$$\left.\frac{dg(s)}{ds}\right|_{\Re(\xi)} = \left.\frac{dg(s)}{ds}\right|_{\jmath\Im(\xi)}, \qquad s \in \mathbb{C}, \tag{A.76}$$

where $\xi \in \mathbb{C}$, and $\Re(\xi)$ and $\Im(\xi)$ denote the real and imaginary part of ξ, respectively.

Proof The result directly follows from Definition A.27 setting $\xi_1 = \Re(\xi)$ and $\xi_2 = \jmath\Im(\xi)$, where $\xi \in \mathbb{C}$. ■

The next result provides a *partial* converse of Theorem A.23.

Theorem A.24 ([3, pp. 36–38]) *Consider the complex-valued function $g : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \subseteq \mathbb{C}$ is an open set. If*

$$\left.\frac{dg(s)}{ds}\right|_{\Re(\xi)} = \left.\frac{dg(s)}{ds}\right|_{\jmath\Im(\xi)}, \qquad s \in \mathcal{D}, \tag{A.77}$$

where $\xi \in \mathbb{C}$, and $\left.\frac{dg(s)}{ds}\right|_{\Re(\xi)}$ and $\left.\frac{dg(s)}{ds}\right|_{\jmath\Im(\xi)}$ are continuous for all $s \in \mathcal{D}$, then $g(\cdot)$ is analytic.

The next result provides an alternative to Theorem A.23. For the statement of this theorem, let $\Re(s)$ and $\Im(s)$ denote the real and imaginary parts of $s \in \mathbb{C}$.

Theorem A.25 ([3, pp. 36–38], *Cauchy–Riemann condition*) *Consider the complex-valued function $g : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \subseteq \mathbb{C}$ is an open set. Let $g_x(s) = \Re(g(s))$, $s \in \mathcal{D}$, $g_y(s) = \Im(g(s))$, $\sigma = \Re(s)$, and $\omega = \Im(s)$. If $\frac{\partial g_x(s)}{\partial \sigma}$, $\frac{\partial g_y(s)}{\partial \omega}$, $\frac{\partial g_y(s)}{\partial \sigma}$, and $\frac{\partial g_x(s)}{\partial \omega}$ are continuous on $\mathcal{D}$, and*

$$\frac{\partial g_x(s)}{\partial \sigma} = \frac{\partial g_y(s)}{\partial \omega} \quad \text{and} \quad \frac{\partial g_y(s)}{\partial \sigma} = -\frac{\partial g_x(s)}{\partial \omega}, \tag{A.78}$$

for all $s \in \mathcal{D}$, then $g(\cdot)$ is analytic.

Remark A.5 If $g : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathcal{D} \subseteq \mathbb{C}$, then we can use the classic differentiation techniques known from real calculus to compute the derivative of $g(\cdot)$ on $\mathcal{D}$.

Definition A.28 Consider $\mathcal{D} \subseteq \mathbb{C}$. The complex-valued matrix function $G : \mathcal{D} \to \mathbb{C}^{l \times m}$ is analytic if all $l \times m$ entries of $G(\cdot)$ are analytic.

A.11 Laplace Transforms

In this section, we introduce Laplace and inverse Laplace transforms. These tools will allow us computing e^{At} and $\int_0^t e^{A(t-\tau)} B\eta(\tau)\mathrm{d}\tau$, which characterize the solutions of (3.1) and (A.20).

Definition A.29 (*Laplace transformable function*) The real-valued function $f : [0, \infty) \to \mathbb{R}$ is *Laplace transformable* at $s_0 \in \mathbb{C}$ if $\int_0^\infty |f(t)|e^{-s_0 t}\mathrm{d}t$ exists and is finite.

It follows from Definition A.29 that if $f(\cdot)$ is Laplace transformable for some $s_0 \in \mathbb{C}$, then $f(\cdot)$ is Laplace transformable for all $s \in \mathbb{C}$ such that $\Re(s) \geq \Re(s_0)$. The next results provide necessary and sufficient conditions for a function to be Laplace transformable.

Theorem A.26 ([51, Sect. 6.28.1]) *Let $f : [0, \infty) \to \mathbb{R}$ be Laplace transformable for all $s \in \mathbb{C}$. Then, $f(\cdot)$ is locally integrable on $[0, \infty)$, that is, $\int_0^a |f(t)|\mathrm{d}t < \infty$ for any $a > 0$.*

Theorem A.27 ([51, Sec. 6.28.1]) *Let $f : [0, \infty) \to \mathbb{R}$ be locally integrable. If there exist $M, t_0 > 0$ and $\gamma \in \mathbb{R}$, such that*

$$|f(t)| \leq Me^{\gamma t}, \qquad t \geq 0, \tag{A.79}$$

then, $f(\cdot)$ is Laplace transformable for all $s \in \mathbb{C}$, such that $\Re(s) > \gamma$.

Definition A.30 (*Laplace transform*) Let $f : [0, \infty) \to \mathbb{R}$ be Laplace transformable at $s_0 \in \mathbb{C}$. The *Laplace transform* of $f(\cdot)$ is defined as

$$\mathcal{L}[f(t)] \triangleq \int_0^\infty f(t)e^{-st}\mathrm{d}t, \qquad s \in \{s \in \mathbb{C} : \Re(s) \geq \Re(s_0)\}. \tag{A.80}$$

Note that the Laplace transform of a real-valued function is a complex-valued function. It can be proven that the Laplace transform of a function is unique.

Definition A.31 (*Extension of a complex function*) Consider $F : \mathcal{D} \to \mathbb{C}$ and $\hat{F} : \mathcal{E} \to \mathbb{C}$, where $\mathcal{D} \subseteq \mathcal{E} \subseteq \mathbb{C}$ and $\mathcal{D}$ and $\mathcal{E}$ are open sets. If $F(s) = \hat{F}(s)$, $s \in \mathcal{D}$, then $\hat{F}(\cdot)$ is an *extension* of $F(\cdot)$ from $\mathcal{D}$ to $\mathcal{E}$.

Remark A.6 It follows from Definition A.30 that the Laplace transform of the real-valued function $f(\cdot)$ is defined for all $s \in \mathbb{C}$ such that $\Re(s) \geq \Re(s_0)$, where s_0 is the complex number with smallest real part such that $f(\cdot)$ is Laplace transformable. In this brief, we define the Laplace transform of $f(\cdot)$ as the extension of (A.80) to the set of all points in $\mathbb{C}$, where the complex-valued function $\int_0^\infty f(t)e^{-st}\mathrm{d}t$ is defined.

We define the Laplace transform of $G : [0, \infty) \to \mathbb{R}^{n\times m}$ as the Laplace transform of each component of $G(\cdot)$, that is,

$$\mathcal{L}[G(t)] = \begin{bmatrix} \mathcal{L}[G_{11}(t)] & \dots & \mathcal{L}[G_{1m}(t)] \\ \vdots & \ddots & \vdots \\ \mathcal{L}[G_{n1}(t)] & \dots & \mathcal{L}[G_{nm}(t)] \end{bmatrix}, \qquad s \in \mathbb{C}, \tag{A.81}$$

where $G(\cdot) = \begin{bmatrix} G_{11}(\cdot) & \dots & G_{1m}(\cdot) \\ \vdots & \ddots & \vdots \\ G_{n1}(\cdot) & \dots & G_{nm}(\cdot) \end{bmatrix}$.

Definition A.32 (*Inverse Laplace transform*) Let $F : \mathbb{C} \to \mathbb{C}$ be analytic for all $s \in \mathbb{C}$, such that $\Re(s) > c$. If $f : [0, \infty) \to \mathbb{R}$ is Laplace transformable for all $s \in \mathbb{C}$ such that $\Re(s) > c$ and

$$\mathcal{L}[f(t)] = F(s), \qquad s \in \{s \in \mathbb{C} : \Re(s) > c\}, \tag{A.82}$$

then $f(\cdot)$ is an *inverse Laplace transform of* $F(\cdot)$.

It can be proven that if the inverse Laplace transform of an analytic complex function exists, then it is unique. In this brief, we denote the inverse Laplace transform of $F : \mathbb{C} \to \mathbb{C}$ by $\mathcal{L}^{-1}[F(s)]$. The next result provides a mathematical expression for the inverse Laplace transform of a class of analytic complex functions.

Theorem A.28 (Mellin's inverse formula) *Let $F : \mathbb{C} \to \mathbb{C}$ be analytic for all $s \in \mathbb{C}$ such that $\Re(s) > c$, $c \in \mathbb{R}$, and assume there exists $s_0 \in \mathbb{C}$, $M \in \mathbb{R}$, and $k > 1$ such that*

$$|F(s)| \leq M \left| \frac{1}{s^k} \right| \tag{A.83}$$

for all $s \in \{s \in \mathbb{C} : |s| > |s_0|\}$. The inverse Laplace transform of $F(\cdot)$ is given by

$$\mathcal{L}^{-1}[F(s)] \triangleq \frac{1}{2\pi \jmath} \int_{c-\jmath\infty}^{c+\jmath\infty} F(s)e^{st}\mathrm{d}s, \qquad t \geq 0. \tag{A.84}$$

Given $G : \mathbb{C} \to \mathbb{C}^{n\times m}$, we define the inverse Laplace transform of $G(\cdot)$ as the inverse Laplace transform of each component of $G(\cdot)$, that is,

$$\mathcal{L}^{-1}[G(s)] = \begin{bmatrix} \mathcal{L}^{-1}[G_{11}(s)] & \dots & \mathcal{L}^{-1}[G_{1m}(s)] \\ \vdots & \ddots & \vdots \\ \mathcal{L}^{-1}[G_{n1}(s)] & \dots & \mathcal{L}^{-1}[G_{nm}(s)] \end{bmatrix}, \qquad \Re(s) > c, \tag{A.85}$$

where $G(\cdot) = \begin{bmatrix} G_{11}(\cdot) & \dots & G_{1m}(\cdot) \\ \vdots & \ddots & \vdots \\ G_{n1}(\cdot) & \dots & G_{nm}(\cdot) \end{bmatrix}$ and $c \in \mathbb{R}$ is larger than the real part of the singular points of $G_{ij}(\cdot)$, for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

It follows from Definition A.32 that the inverse Laplace transform of $F(s)$ is defined for all s, whose real part is larger than the real part of all singular points of $F(s)$. In order to define the inverse Laplace transform of $F(s)$ for all $s \in \mathbb{C}$, except the singular points of $F(s)$, one needs to resort to the *analytic extension theorem*; for details, see [7].

The Laplace transform has numerous useful properties and the reader is referred to [36, Chap. 2] for a brief review of the most relevant ones. In this section, we recall those results that are useful for the purposes of this brief.

Exercise A.14 Let $f : [0, \infty) \to \mathbb{R}$. Prove that the Laplace transform of $f(\cdot)$ is a linear operator, that is,

$$\mathcal{L}[\alpha f_1(t) + \beta f_2(t)] = \alpha \mathcal{L}[f_1(t)] + \beta \mathcal{L}[f_2(t)], \tag{A.86}$$

for all $\alpha, \beta \in \mathbb{R}$ and for all $f_1, \ f_2 : [0, \infty) \to \mathbb{R}$. △

Theorem A.29 (Real differentiation theorem, [36, p. 27]) *Let $f : [0, \infty) \to \mathbb{R}^n$ be continuously differentiable and Laplace transformable with its first derivative. Then,*

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = s\mathcal{L}[f(t)] - f(0), \qquad s \in \mathbb{C}. \tag{A.87}$$

Theorem A.30 (Real integration theorem, [36, p. 31]) *Let $f : [0, \infty) \to \mathbb{R}^n$ be integrable and Laplace transformable with $\int_0^t f(\tau)\mathrm{d}\tau$. Then,*

$$\mathcal{L}\left[\int_0^t f(\tau)\mathrm{d}\tau\right] = \frac{\mathcal{L}[f(t)]}{s}, \qquad s \in \mathbb{C} \setminus \{0\}. \tag{A.88}$$

Theorem A.31 (Final value theorem, [36, p. 29]) *Let $f : [0, \infty) \to \infty$ be continuously differentiable and Laplace transformable with its first derivative. If* $\lim_{s\to 0} s\mathcal{L}[f(t)]$ *exists and is unique, then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\mathcal{L}[f(t)]. \tag{A.89}$$

The final value theorem is an extremely powerful tool, since it allows finding the limit at infinity of a real function by computing the limit at zero of a complex function, which is usually a simpler task.

Theorem A.32 ([5, Proposition 11.2.2]) *Let* $A \in \mathbb{R}^{n\times n}$. *Then,*

$$\mathcal{L}\left[e^{At}\right] = (sI_n - A)^{-1}, \tag{A.90}$$

where $s \in \mathbb{C}\backslash\mathrm{spec}(A)$.

Remark A.7 It follows from Theorems A.7 and A.32 and Definitions A.15 and A.34 that

$$\mathcal{L}[e^{At}] = \frac{1}{\chi_A(s)}C^{\mathrm{T}}_{(sI-A)}, \qquad s \in \mathbb{C}\backslash\mathrm{spec}(A), \tag{A.91}$$

where $\chi_A(s)$ is the characteristic polynomial of A and $C_{(sI-A)}$ is the cofactor matrix of $(sI - A)$.

Definition A.33 (*Convolution*) Let $f_1, f_2 : [0, \infty) \to \mathbb{R}$ be integrable functions. The convolution of $f_1(\cdot)$ and $f_2(\cdot)$ is defined as

$$(f_1(t) \star f_2(t))(t) \triangleq \int_0^t f_1(t-\tau) f_2(\tau)\mathrm{d}\tau. \tag{A.92}$$

Theorem A.33 ([36, p. 33]) *Let* $f_1, f_2 : [0, \infty) \to \mathbb{R}$ *be Laplace transformable functions. Then,*

$$\mathcal{L}\left[(f_1(t) \star f_2(t))(t)\right] = \mathcal{L}\left[f_1(t)\right]\mathcal{L}\left[f_2(t)\right], \qquad s \in \mathbb{C}. \tag{A.93}$$

Exercise A.15 Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Show that

$$\mathcal{L}\left[\left(e^{At} \star B\eta(t)\right)(t)\right] = (sI - A)^{-1}B\mathcal{L}[\eta(t)], \qquad s \in \mathbb{C} \setminus \mathrm{spec}(A). \tag{A.94}$$

△

Theorem A.33 allows us to numerically compute the solution of the linear differential equation (3.1). Specifically, the solution $\chi(t)$, $t \geq 0$, of (3.1) can be computed as

$$\chi(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}(\chi_0 + B\mathcal{L}[\eta(t)])\right], \qquad t \geq 0, \tag{A.95}$$

for all $s \in \mathbb{C} \setminus \mathrm{spec}(A)$.

A.12 Smith–McMillan Form, Poles, and Zeros

In Definition A.27, we introduced the notion of ordinary and singular points. Next, we introduce two additional classes of points for complex-valued functions, namely, poles and zeros. To this goal, we need to recall the following important result from complex analysis.

Theorem A.34 ([6, Sect. 69]) *Let $s_0 \in \mathcal{D}$ be an isolated singular point of $g : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \subseteq \mathbb{C}$ is an open set. Then there exists $R > 0$, $a_0 \in \mathbb{C}$, $a_n \in \mathbb{C}$, and $b_m \in \mathbb{C}$, $n, m \in \mathbb{N}$, such that*

$$g(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(s - s_0)^m}, \qquad s \in \mathcal{B}, \tag{A.96}$$

where $\mathcal{B} \triangleq \{s \in \mathbb{C} : 0 < |s - s_0| < R\} \subset \mathcal{D}$.

Definition A.34 (*Poles and zeros of scalar-valued complex functions*) Consider $g : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \subseteq \mathbb{C}$ is an open set. If s_0 is an ordinary point of $g(\cdot)$ and $g(s_0) = 0$, $s_0 \in \mathcal{D}$, then s_0 is a *zero* of $g(\cdot)$. If s_0 is an isolated singular point of $g(\cdot)$ and there exists $k \in \mathbb{N}$ such that (A.96) specializes to

$$g(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n + \sum_{m=1}^{k} \frac{b_m}{(s - s_0)^m}, \qquad s \in \mathcal{B}, \tag{A.97}$$

then s_0 is a *pole* of $g(\cdot)$.

It is important to note that Definition A.34 applies to scalar-valued complex functions. In the following, we define the poles and the zeros of matrix-valued complex functions. Specifically, we consider rational matrices only, that is, matrices, whose entries are the ratios of complex polynomials. For the statement of the next results, if $G(\cdot)$ is a complex-valued $l \times m$ rational matrix, then we write $G : \mathbb{C} \to \mathbb{C}^{l \times m}$ and tacitly assume that $G(\cdot)$ is defined for all $s \in \mathbb{C}$ except the singular points of the entries of $G(\cdot)$.

Definition A.35 (*Normal rank*) Let $G : \mathbb{C} \to \mathbb{C}^{l \times m}$ be a rational matrix. Then $G(\cdot)$ has *normal rank* r if $\operatorname{rank} G(s) = r$ for all $s \in \mathbb{C}$ except the singular points of the entries of $G(\cdot)$.

For the statement of the next result, it is important to recall that a polynomial is *monic* if the coefficient of the highest order term is one. Furthermore, the pair of polynomials $(\varepsilon(\cdot), \psi(\cdot))$ is *coprime* if $\varepsilon(\cdot)$ and $\psi(\cdot)$ have no common factors.

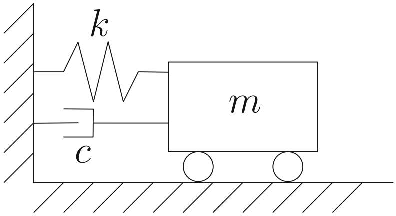

Fig. A.1 Mass-spring-damper system

Theorem A.35 (Smith–McMillan Transform) *Let* $G : \mathbb{C} \to \mathbb{C}^{l\times m}$ *be a rational matrix. Then there exist rational matrices* $T : \mathbb{C} \to \mathbb{C}^{l\times l}$*,* $S : \mathbb{C} \to \mathbb{C}^{m\times m}$*, and* $\Lambda : \mathbb{C} \to \mathbb{C}^{l\times m}$ *such that*

$$G(s) = T(s)\Lambda(s)S(s), \qquad s \in \mathbb{C}, \tag{A.98}$$

and

$$\Lambda(s) = \begin{bmatrix} \frac{\varepsilon_1(s)}{\psi_1(s)} & 0 & \dots & 0 & 0 & \dots 0 \\ 0 & \frac{\varepsilon_2(s)}{\psi_2(s)} & \dots & 0 & 0 & \dots 0 \\ \vdots & \dots & \ddots & \vdots & \dots & \dots \vdots \\ 0 & \dots & \dots & \frac{\varepsilon_r(s)}{\psi_r(s)} & 0 & \dots 0 \\ 0 & \dots & \dots & 0 & 0 & \dots 0 \\ \vdots & \dots & \ddots & \vdots & \dots & \dots \vdots \\ \\ 0 & \dots & \dots & 0 & 0 & \dots 0 \end{bmatrix}, \tag{A.99}$$

where $\varepsilon_i, \psi_i : \mathbb{C} :\to \mathbb{C}$*,* $i = 1, \dots, r$*, are a monic polynomials. Furthermore, the pairs* $(\varepsilon_i(\cdot), \psi_i(\cdot))$*,* $i = 1, \dots, r$*, are coprime for each* i*,* $\varepsilon_i(\cdot)$*,* $i = 1, \dots, r-1$*, divides* $\varepsilon_{i+1}(\cdot)$ *without remainder, and* $\psi_{i+1}(\cdot)$*,* $i = 1, \dots, r-1$*, divides* $\psi_i(\cdot)$ *without remainder.*

Definition A.36 (*Smith–McMillan form*) Let $G : \mathbb{C} \to \mathbb{C}^{l\times m}$ be a rational matrix. The matrix $\Lambda(\cdot)$ satisfying (A.98) and (A.99) is the *Smith–McMillan form* of $G(\cdot)$.

It follows from Theorem A.35 that if $G(\cdot)$ is in Smith–McMillan form, then none of the entries of $G(\cdot)$ can be further simplified.

Definition A.37 (*Zeros and poles of a matrix-valued complex function*) Let $G : \mathbb{C} \to \mathbb{C}^{l\times m}$ be a rational matrix and (A.99) be the Smith–McMillan form $G(\cdot)$. Then the roots of $\varepsilon_1(\cdot), \dots, \varepsilon_r(\cdot)$ are the *zeros* of $G(\cdot)$ and the roots of $\psi_1(\cdot), \dots, \psi_r(\cdot)$ are the *poles* of $G(\cdot)$.

Providing an algorithm to find the Smith–McMillan form of a complex-valued rational matrix is beyond the scopes of this brief. For details, see [32, Chap. 2].

Example A.3 The normal rank of

$$G(s) = \left[\tfrac{s}{s^2+2s-1} \;\; \tfrac{1}{2s^3-s^2+4s}\right], \qquad s \in \mathbb{C} \setminus \{-1-\sqrt{2}, \sqrt{2}-1, 0, 0.25 \pm 1.3919\jmath\}, \tag{A.100}$$

is one and the Smith–McMillan form of $G(\cdot)$ given by

$$\Lambda = \left[\tfrac{s}{2s^3+s^2-4s+7} \;\; 0\right], \qquad s \in \mathbb{C} \setminus \{-2.1676, 0.8338 \pm 0.9589\jmath\}. \tag{A.101}$$

Hence, $s = 0$ is a zeros of $G(\cdot)$ and $s = -2.1676$, $s = 0.8338 + 0.9589\jmath$, and $s = 0.8338 - 0.9589\jmath$ are poles of $G(\cdot)$. △

A.13 Second-order Linear Differential Equations

The second-order linear dynamical system

$$m\frac{d^2x(t)}{dt^2} = -c\frac{dx(t)}{dt} - kx(t), \qquad x(0) = x_0, \qquad \dot{x}(0) = v_0, \qquad t \geq 0, \tag{A.102}$$

where c, $k \geq 0$ and $m > 0$, captures, for instance, the equations of motion of an unforced mechanical system given by a mass that is free to translate in the x direction and is connected to a fixed surface by a linear spring of stiffness k and a linear damper of damping coefficient c. This dynamical model also plays a key role in the study of the linearized equations of motion of an aircraft; for details, see Sects. 2.7 and 2.8.

The system (A.102) is equivalent to the first-order linear dynamical system

$$\dot{\chi}(t) = A\chi(t), \qquad \chi(0) = \chi_0, \qquad t \geq 0, \tag{A.103}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \tag{A.104}$$

$\chi(t) = [x(t), \dot{x}(t)]^{\mathrm{T}}$, and $\chi_0 = [x_0, v_0]^{\mathrm{T}}$.

Exercise A.16 Prove that the linear dynamical system

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & a \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \qquad \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}, \qquad t \geq 0, \tag{A.105}$$

where $a \in \mathbb{R} \setminus \{0\}$, is equivalent to

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{ka}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \frac{z_{10}}{a} \\ z_{20} \end{bmatrix}, \qquad t \geq 0, \tag{A.106}$$

where $y_1(t) = \frac{z_1(t)}{a}$ and $y_2(t) = z_2(t)$. △

Consider the matrix A given by (A.104). The characteristic polynomial of A is

$$\chi_A(s) = s^2 + \frac{c}{m}s + \frac{k}{m}, \qquad s \in \mathbb{C}, \tag{A.107}$$

and

$$\mathrm{spec}(A) = \left\{ -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}, -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \right\}. \tag{A.108}$$

It follows from Definition A.22 that if $c = 0$ and $k > 0$, then (A.20) is Lyapunov stable, if $k = 0$ and $c > 0$, then (A.20) is semistable, and if $c > 0$ and $k > 0$, then (A.20) is asymptotically stable. Furthermore, if $\left(\frac{c}{2m}\right)^2 \geq \frac{k}{m}$, then the eigenvalues of A are real and if $\left(\frac{c}{2m}\right)^2 < \frac{k}{m}$, then the eigenvalues of A are complex conjugate.

It is customary to rewrite (A.107) as

$$\chi_A(s) = s^2 + 2\zeta\omega_\mathrm{n}s + \omega_\mathrm{n}^2, \qquad s \in \mathbb{C}, \tag{A.109}$$

where

$$\omega_\mathrm{n} \triangleq \sqrt{\frac{k}{m}} \tag{A.110}$$

is the system's *natural frequency* and

$$\zeta \triangleq \frac{c}{2\sqrt{km}} \tag{A.111}$$

is the system's *damping ratio*.

Second-order linear dynamical systems can be classified according to the value of the damping ratio.

Definition A.38 Consider the linear dynamical system (A.103). If $\zeta = 0$, then (A.103) is *undamped*; if $0 < \zeta < 1$, then (A.103) is *underdamped*; if $\zeta = 1$, then (A.103) is *critically damped*; if $\zeta > 1$, then (A.103) is *overdamped*.

Per definition, $\omega_\mathrm{n} \geq 0$ and the roots of (A.109) are given by

$$\mathrm{spec}(A) = \left\{ -\zeta\omega_\mathrm{n} - \omega_\mathrm{n}\sqrt{\zeta^2 - 1},\ -\zeta\omega_\mathrm{n} + \omega_\mathrm{n}\sqrt{\zeta^2 - 1} \right\}. \tag{A.112}$$

If (A.103) is underdamped, then

$$\mathrm{spec}(A) = \left\{ -\zeta\omega_\mathrm{n} - \jmath\omega_\mathrm{n}\sqrt{1 - \zeta^2},\ -\zeta\omega_\mathrm{n} + \jmath\omega_\mathrm{n}\sqrt{1 - \zeta^2} \right\}. \tag{A.113}$$

and (A.109) is equivalent to

$$\chi_A(s) = [s - (-\zeta\omega_\mathrm{n} + \jmath\omega_\mathrm{d})]\,[s - (-\zeta\omega_\mathrm{n} - \jmath\omega_\mathrm{d})]\,, \qquad s \in \mathbb{C}, \tag{A.114}$$

where $\omega_\mathrm{d} \triangleq \omega_\mathrm{n}\sqrt{1-\zeta^2}$ is the system's *damped natural frequency*.

The dynamics of an underdamped second-order linear dynamical system is characterized by several parameters, namely

(i) the *rise time*

$$t_r \triangleq \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_\mathrm{d}}, \tag{A.115}$$

(ii) the *peak time*

$$t_\mathrm{p} \triangleq \frac{\pi}{\omega_\mathrm{d}}, \tag{A.116}$$

(iii) the *maximum overshoot*

$$M_\mathrm{p} \triangleq e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}, \tag{A.117}$$

The reason for introducing these parameters is the following. Consider the controlled linear dynamical system (3.1) and (3.2), where A is given by (A.104), $B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$, $C = [1,\ 0]$, and $D = 0$. It follows from (3.3) that

$$\mathcal{L}\left[y(t)\right] = C(sI - A)^{-1}B\mathcal{L}[\eta(t)], \qquad s \in \mathbb{C}\backslash\mathrm{spec}(A), \tag{A.118}$$

and if $\eta(t) = 1$, $t \geq 0$, then

$$\mathcal{L}\left[y(t)\right] = \frac{1}{\left(ms^2 + cs + k\right)s}, \qquad s \in \mathbb{C}\backslash\mathrm{spec}(A). \tag{A.119}$$

It can be proven that if $\zeta \in (0, 1)$, then the rise time t_r captures the time required for the response to rise from 0% to 100% of its final value, the peak time t_p captures the time required for the response to reach the first peak of its oscillatory response, and the maximum overshoot M_p captures the maximum peak value of the response curve from unit; for details, see [36, Chap. 5].

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