Appendix A
Additional Classical Results on Glasses

A.1 Instability of the Replica Symmetric Solution

A question that naturally arises about the replica symmetric solution for the SK-model is if it is *stable* respect to fluctuations. From a mathematical point of view, the solution is stable if the Hessian matrix $\mathsf{H}(\mathsf{q})$ in Eq. (3.2.29) is *positive definite* in the replica symmetric solution in Eq. (3.2.33), $Q_{\alpha\beta} = q$. The simplest way to verify the stability is therefore by solving the eigenvalue problem

$$\mathsf{H}(\mathsf{Q})\boldsymbol{\eta} = \lambda\boldsymbol{\eta}, \tag{A.1.1}$$

and verify that all eigenvalues are positive. This analysis was performed for the first time by Almeida and Thouless [1]. We will not reproduce here all the steps of their computation, but we will present only the main results. By direct computation we can obtain the following expression for the Hessian matrix

$$H_{(\alpha\beta)(\gamma\delta)} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \beta^2\Gamma^2\left(\left\langle\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta\right\rangle_z - \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z\left\langle\sigma^\gamma\sigma^\delta\right\rangle_z\right). \tag{A.1.2}$$

In other words, the structure of the Hessian matrix is the following

$$H_{(\alpha\beta)(\alpha\beta)} = 1 - \beta^2\Gamma^2\left(1 - \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z^2\right) \equiv a, \tag{A.1.3a}$$

$$H_{(\alpha\beta)(\alpha\gamma)} = -\beta^2\Gamma^2\left(\left\langle\sigma^\beta\sigma^\gamma\right\rangle_z - \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z\langle\sigma^\alpha\sigma^\gamma\rangle_z\right) \equiv b, \tag{A.1.3b}$$

$$H_{(\alpha\beta)(\gamma\delta)} = -\beta^2\Gamma^2\left(\left\langle\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta\right\rangle_z - \left\langle\sigma^\alpha\sigma^\beta\right\rangle_z\left\langle\sigma^\gamma\sigma^\delta\right\rangle_z\right) \equiv c, \tag{A.1.3c}$$

The Hessiam matrix has therefore a particular symmetric form of the type (here $n = 4$)

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$$\mathsf{H} = \begin{pmatrix}
a & b & b & b & b & b & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c \\
b & a & b & b & b & b & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c \\
b & b & a & b & b & b & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c \\
b & b & b & a & b & b & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c \\
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b & b & b & b & b & a & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c \\
c & c & c & c & c & c & a & b & b & b & b & b & c & c & c & c & c & c & c & c & c & c & c & c \\
c & c & c & c & c & c & b & a & b & b & b & b & c & c & c & c & c & c & c & c & c & c & c & c \\
c & c & c & c & c & c & b & b & a & b & b & b & c & c & c & c & c & c & c & c & c & c & c & c \\
c & c & c & c & c & c & b & b & b & a & b & b & c & c & c & c & c & c & c & c & c & c & c & c \\
c & c & c & c & c & c & b & b & b & b & a & b & c & c & c & c & c & c & c & c & c & c & c & c \\
c & c & c & c & c & c & b & b & b & b & b & a & c & c & c & c & c & c & c & c & c & c & c & c \\
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c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & b & b & b & b & a & b \\
c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & c & b & b & b & b & b & a
\end{pmatrix}. \tag{A.1.4}$$

In the *paramagnetic phase*, being $q = 0$, a direct computation shows that $b = c = 0$ and $a = 1$, so the solution is stable.

In the *ferromagnetic phase*, Almeida and Thouless [1] first assumed an eigenvector symmetric under the exchange of replica indexes, i.e., in the form

$$\eta^{\alpha\beta} = \eta, \quad \forall\alpha, \beta. \tag{A.1.5}$$

Under this assumption, they found the following corresponding eigenvalue:

$$\lambda_1 = \frac{1 - a - 4b + 3c \pm |1 - a + 4b - 3c|}{2}. \tag{A.1.6}$$

They proved also that the eigenvalue λ_2 obtained for the $n - 1$ eigenvectors in the form

$$\eta^{\alpha\beta} \equiv \eta_0(\delta_{\alpha\theta} + \delta_{\beta\theta}) + \eta_1 \tag{A.1.7}$$

for a certain θ (i.e., in each eigenvector one replica is "special"), is such that $\lambda_2 \xrightarrow{n\to 0} \lambda_1$. The third eigenvalue λ_3 is obtained assuming that there are two replicas, of indices ν, θ, such that

$$\eta^{\theta\nu} \equiv \eta_1, \quad \eta^{\theta\alpha} = \eta^{\nu\alpha} \equiv \eta_2, \quad \eta^{\alpha\beta} \equiv \eta_3, \quad \forall\alpha, \beta. \tag{A.1.8}$$

There are $\frac{n(n-3)}{2}$ eigenvectors in this form. Again, they proved that the corresponding eigenvalue is

$$\lambda_3 = a - 2b + c. \tag{A.1.9}$$

It can be seen that, for $Q_{\alpha\beta} \equiv q$, $\lambda_1 = \lim_{n\to 0} \lambda_2$ is always positive. However

$$\lambda_3 = a - 2b + c > 0 \Rightarrow \frac{1}{\beta^2\Gamma^2} > \frac{1}{\sqrt{2\pi}} \int e^{-\frac{w^2}{2}} \operatorname{sech}^2\left(\beta\Gamma\sqrt{q}w + \beta h\right) \mathrm{d}\, w. \tag{A.1.10}$$

The previous inequality defines a region in the (β, h) space in which the replica symmetric solution is *unstable*, corresponding to values (β, h) that violate the inequality. The curve obtained imposing the equality is called *Almeida–Thouless line* (AT-line).

A.2 The TAP Approach for Glasses

A different but instructive approach to the study of the SK-model was proposed in 1977 by Thouless, Anderson, and Palmer [5]. They obtained a set of equations for the local magnetization in the SK-model. Given a certain realization of the SK-model, the free energy can be written as

$$-\beta F[\beta; \mathbf{h}; \mathbf{J}] := \ln \left[\sum_{\{\sigma_i\}} \exp \left(\beta \sum_{i \neq j} J_{ij} \sigma_i \sigma_j + \beta \sum_i h_i \sigma_i \right) \right]. \tag{A.2.1}$$

In the previous equation, we have denoted by $\mathbf{h} := \{h_i\}_{i=1,\dots,N}$ a set of local magnetic fields, and by $\mathbf{J} := \{J_{ij}\}_{ij}$ the set of coupling constants. Note that here we suppose that the external magnetic field depends on the site position. As known from the general theory, the local magnetisation is given by $m_i := \langle \sigma_i \rangle = -\partial_{h_i} F$, F free energy. Let us now perform a Legendre transform, introducing

$$G[\beta; \mathbf{m}; \mathbf{J}] = \max_{h_1,\dots,h_N} \left[F[\beta; \mathbf{h}; \mathbf{J}] + \sum_i m_i h_i \right], \tag{A.2.2}$$

where $\mathbf{m} := \{m_i\}_i$. The new functional is a *constrained free energy*, in the form

$$-\beta G[\beta; \mathbf{m}; \mathbf{J}] = \ln \left[\sum_{\{\sigma_i\}} \exp \left(\beta \sum_{i \neq j} J_{ij} \sigma_i \sigma_j + \beta \sum_i h_i(m_i)(\sigma_i - m_i) \right) \right], \tag{A.2.3}$$

where $h_i(m_i)$ are such that $\langle \sigma_i \rangle = m_i$. Let us now expand the previous functional around $\beta = 0$ (infinite temperature). We obtain

$$\begin{aligned} -\beta G[\beta; \mathbf{m}; \mathbf{J}] = &\sum_{i=1}^N \left[\frac{1 - m_i}{2} \ln \left(\frac{1 - m_i}{2} \right) - \frac{1 + m_i}{2} \ln \left(\frac{1 + m_i}{2} \right) \right] \\ &+ \beta \sum_{ij} J_{ij} m_i m_j + \frac{\beta^2}{2} \sum_{ij} J_{ij}^2 (1 - m_i^2)(1 - m_j^2) + o(\beta^2). \end{aligned} \tag{A.2.4}$$

Plefka [4] showed that, for the SK-model, the additional terms in $o(\beta^2)$ can be neglected in the $N \to +\infty$. The minimum condition respect to m_i, $\partial_{m_i} G = 0$, gives the celebrated *Thouless–Anderson–Palmer equations* (TAP-equations):

$$m_i = \tanh\left[\beta\left(\sum_{j\neq i} J_{ij} m_j + h_i - m_i \sum_{j\neq i} J_{ij}^2 \beta(1-m_j)^2\right)\right] \tag{A.2.5}$$

In the previous expression an additional term appears respect to the equation for the mean field magnetization in the Ising model (3.2.14), i.e. the so called *Onsager reaction*

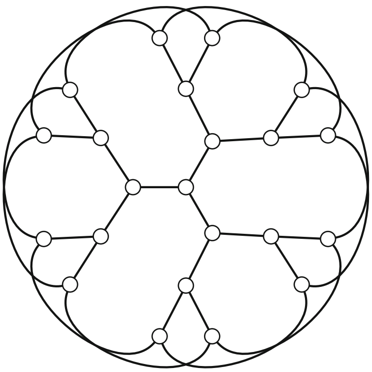
$$h_i^O := -m_i \sum_{j\neq i} J_{ij}^2 \beta(1-m_j)^2. \tag{A.2.6}$$

This additional term is not negligible in the spin glass case due to the fact that $J_{ij} \sim \frac{1}{\sqrt{N}}$ and *not* $J_{ij} \sim \frac{1}{N}$ as in the ferromagnetic case.

A.3 The Cavity Method for Spin Glasses on a Bethe Lattice

The cavity method was originally conceived for the analysis of the ground state of spin glasses on particular lattices. In this section, therefore, we start from this problem, to properly identify the physics behind the algorithm that the method inspired, following the seminal paper of Mézard and Parisi [3].

Let us first introduce the so called *Bethe lattice*. The Bethe lattice $\mathtt{B}_N^k = \text{Graph}(\mathcal{V}; \mathcal{E})$ is a random graph of $N \gg 0$ vertices with fixed connectivity $k+1$. For example, $\mathtt{B}_N^2$ has the structure

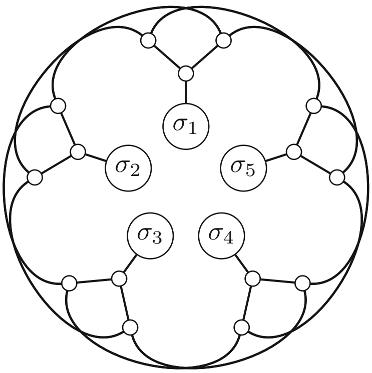


The Bethe lattice $\mathtt{B}_N^k$ has cycles[1] of typical length $\ln N$, and therefore we expect that we can consider the $(k+1)$-Cayley tree as the limiting graph of a Bethe lattice $\mathtt{B}_N^k$ having $N \to \infty$.

We define now a spin model on the Bethe lattice, associating to each vertex $v_i \in \mathcal{V}$ a spin variable $\sigma_i \in \{-1, 1\}$, and to each edge (v_i, v_j) a random coupling J_{ij}. In particular, we suppose that $\{J_{ij}\}_{ij}$ are identically and independently distributed random variables with a certain probability distribution density $\rho(J)$. The spin glass Hamiltonian is simply

$$H_{\mathtt{B}_N^k}[\boldsymbol{\sigma}; \mathbf{J}] = -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j. \tag{A.3.1}$$

To find the global ground state (GGS) energy $\mathcal{C}_N$ of the system, we benefit of the tree-like structure of the graph. Indeed, let us consider a new graph $\mathtt{G}_{N,q}^k \subseteq \mathtt{B}_N^k$ obtained from $\mathtt{B}_N^k$ decreasing the degree of $q > k$ spins $\{\sigma_1, \ldots, \sigma_q\}$ from $k+1$ to k. This spins are called *cavity spins* and surround a "cavity" in the graph. For example, $\mathtt{G}_{N,5}^2$ has the shape

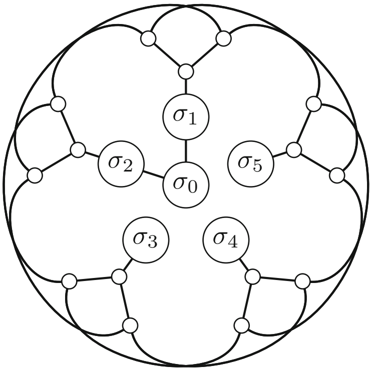


We suppose now the values of these spins $\sigma_1, \ldots, \sigma_q$ *fixed* and that the GGS energy of this new system, let us call it $\mathcal{C}_N(\sigma_1, \ldots, \sigma_q)$, is related to the old one in the following way

$$\mathcal{C}_N(\sigma_1, \ldots, \sigma_q) - \mathcal{C}_N = \sum_{i=1}^{q} h_i\sigma_i, \tag{A.3.2}$$

for a proper set $\{h_i\}_{i=1,\ldots,q}$ of *local cavity fields*. Due to the random nature of the system, we assume that $\{h_i\}_i$ are i.i.d. random variables as well, with distribution $\varrho(h)$. Now suppose that a new spin σ_0 is added to the lattice and coupled to k cavity spins $\sigma_1, \ldots, \sigma_k$ by $J_1, \ldots, J_k$ respectively,

[1]Note that in the literature it is common to identify a Bethe lattice with a Cayley tree, that is a tree with no cycles.



We fix the value of σ_0 and change the value of $\sigma_1, \dots, \sigma_k$ in such a way that the new GGS energy is minimized. The energy ϵ_i of each link (σ_i, σ_0) is minimized taking

$$\epsilon_i = \min_{\sigma_i} [(-h_i - J_i\sigma_0)\sigma_i] =: -a(J_i, h_i) - \sigma_0 b(J_i, h_i). \tag{A.3.3}$$

In this notation, it is clear that the final cavity field on σ_0 is

$$h_0 = \sum_{i=1}^{k} b(J_i, h_i). \tag{A.3.4}$$

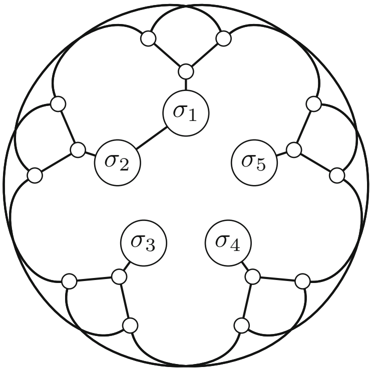
It follows that the recursion relation

$$\varrho(h) = \overline{\prod_{i=1}^{k} \left(\int \varrho(h_i)\, \mathrm{d}\, h_i \right) \delta \left(h - \sum_{i=1}^{k} b(J_i, h_i) \right)} \tag{A.3.5}$$

holds. Suppose now that we are able to solve the previous equation for ϱ. Then, denoting by $\delta\epsilon^1$ the energy shift of the GGS due to a site addition, we have

$$\delta\epsilon_0^1 = -\overline{\prod_{i=1}^{k+1} \left(\int \varrho(h_i)\, \mathrm{d}\, h_i \right) \left(\sum_{j=1}^{k+1} a(J_j, h_j) + \left| \sum_{j=1}^{k+1} b(J_j, h_j) \right| \right)}. \tag{A.3.6}$$

Let us consider back the graph $\mathsf{G}_{N,q}^k$ and let us add, this time, an edge, e.g. connecting σ_1 and σ_2,



Again, the energy shift of the GGS $\delta\epsilon^2$ due to the addition of an edge can be written as

$$\delta\epsilon^2 = -\overline{\prod_{i=1}^{2}\left(\int \varrho(h_i)\,\mathrm{d}\,h_i\right)\max_{\sigma_1,\sigma_2}(h_1\sigma_1 + h_2\sigma_2 + J_{12}\sigma_1\sigma_2)}. \tag{A.3.7}$$

We have now two expression for the shifts in energy due to local alterations of the graph, a vertex addition and an edge addition. The importance of these to quantities is due to the fact that Mézard and Parisi [2] proved that

$$\lim_{N\to\infty}\frac{\mathcal{C}_N}{N} = \delta\epsilon^1 - \frac{k+1}{2}\delta\epsilon^2. \tag{A.3.8}$$

Note that the tree-like structure of the Bethe lattice plays a fundamental role in the derivation of the equations above, allowing us to write recursive relations. The cavity method suggests that, at least on a tree-like graph, we can obtain information about the ground state evaluating the shift in energy after a local modification of the graph and writing down proper self-consistent equations for the distributions of the random quantities. The method, in the formulation presented here, works however only in the replica symmetric hypothesis. In other words, we have implicitly assumed that there are *no local ground states*, i.e., local minima in energy whose corresponding configurations can be obtained from the global minimum ones only by an infinite number of spin flips. This eventuality is not so rare and, in this case, it might be necessary to break the replica symmetry to correctly reproduce the GGS energy.

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Appendix B
The Wiener Process and the Brownian Bridge Process

In the present appendix we summarize some fundamental properties of the Wiener process and of the Brownian bridge process. Excellent books on the subject are available in the literature (see, for example, the book of Rogers and Williams [5] or the monograph of Mörters and Peres [3]). We present also some useful probability distributions for the Wiener process and the Brownian bridge process.

B.1 Wiener Process and Brownian Bridge Process

Let us suppose that Ω is a sample space and that $\mathcal{F}$ corresponds to a set of events, σ-algebra over Ω. Recall that a σ-algebra on a certain set A is a collection Σ of subsets of A such that $A \in \Sigma$, $a \in \Sigma \Rightarrow A \setminus a \in \Sigma$ and $a, b \in \Sigma \Rightarrow a \cup b \in \Sigma$. If $\mathcal{C}$ is a set of subsets of A, then $\sigma(\mathcal{C})$ is, by definition, the smallest σ-algebra on A containing $\mathcal{C}$. Let us suppose also that a probability $\mathbb{P} : \mathcal{F} \to \mathbb{R}^+$ is given on the space of events. In this space, we denote by $\mathbb{E}(\bullet)$ the expected value of a certain quantity $f : \Omega \to \mathbb{R}$,

$$\mathbb{E}(f) := \int_{\Omega} \mathbb{P}(\omega) f(\omega) \, \mathrm{d}\, \omega \,. \tag{B.1.1}$$

Similarly, if $\mathcal{G} \subset \mathcal{F}$ is a sub σ-algebra of $\mathcal{F}$, $\mathbb{E}(f|\mathcal{G}) := \int_\Omega f(\omega)\mathbb{P}(\omega\,|\mathcal{G})\,\mathrm{d}\,\omega$.

A *Wiener process* W (Fig. B.1), or *standard Brownian process* $\mathsf{W} : \mathbb{R}^+ \to \mathbb{R}$ on this probability space is such that

- $\forall\, \omega \in \Omega$ $\mathsf{W}(0) = 0$, i.e., any realization of the process starts in the origin;
- the map $t \to \mathsf{W}(t)$ is continuous $\forall t \in \mathbb{R}^+$ and $\forall\, \omega \in \Omega$;
- $\forall t, \tau \in \mathbb{R}^+$ the increment $\mathsf{W}(t+\tau) - \mathsf{W}(t)$ is independent from $\mathsf{W}(u)$, $u \in [0, t]$;
- the increment $\mathsf{W}(t + \tau) - \mathsf{W}(t)$, $\forall t, \tau \in \mathbb{R}^+$, is normally distributed with zero mean and variance τ.

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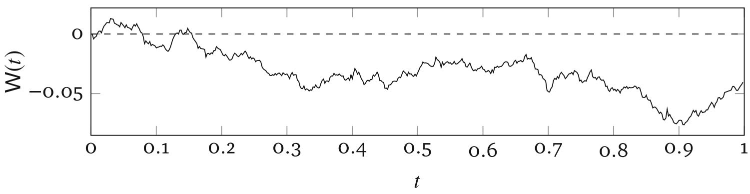

Fig. B.1 A realization of a Wiener process

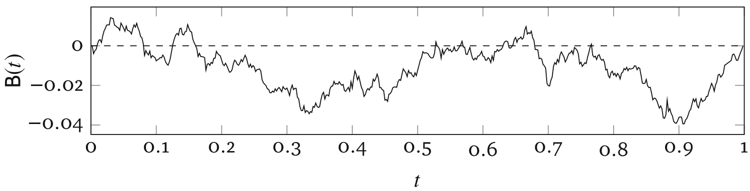

Fig. B.2 A realization of a Brownian bridge process

This set of properties uniquely identifies the Wiener process: it can be proved that such a process *exists*.

Definition B.1.1 (*Gaussian process*) A *Gaussian process* is a certain stochastic process $\mathsf{X}(t) : T \subseteq \mathbb{R} \to \mathbb{R}$ such that, given a set of n element $\{t_1, \ldots, t_n\} \in T$, the joint distribution of $(\mathsf{X}(t_1), \ldots, \mathsf{X}(t_n))$ is a multivariate Gaussian distribution. It follows that the process is characterized only by the mean $\mu(t) := \mathbb{E}(\mathsf{X}(t))$ and by the covariance $\mathbb{E}\left(\mathsf{X}(t)\mathsf{X}(t')\right)$.

The Wiener process is a Gaussian process with mean $\mathbb{E}(\mathsf{W}(t)) = 0$ and covariance $\mathbb{E}\left(\mathsf{W}(t)\mathsf{W}(t')\right) = \min\{t, t'\}$. Most importantly, if a continuous process satisfies these properties, it is a Wiener process.

Definition B.1.2 (*Martingale*) Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as above. We say that a family $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$, sub σ-algebras of $\mathcal{F}$, is a *filtration* if, for $s < t$,

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \sigma\left(\bigcup_{\tau} \mathcal{F}_\tau\right) \subseteq \mathcal{F}, \tag{B.1.2}$$

A certain process $\mathsf{X}(t)$, $t \in \mathbb{R}^+$ on our probability space is said to be *adapted* to the filtration if $\mathsf{X}(t)$ is measurable on $\mathcal{F}_t$. Given an adapted process such that $\mathbb{E}\left(\mathsf{X}(t)\right) < +\infty\ \forall t$ and $\mathbb{E}\left(\mathsf{X}(t)|\mathcal{F}_s\right) = \mathsf{X}(s)$ for $s < t$ almost surely, then the process is called *martingale*.

The Wiener process is a martingale, due to the fact that, denoting by $\mathcal{W}_s := \sigma\left(\{\mathsf{W}(\tau) : \tau \le s\}\right)$ the filtration of the probability space, $\mathbb{E}\left(\mathsf{W}(t) - \mathsf{W}(s)|\mathcal{W}_s\right) = 0 \Rightarrow \mathbb{E}\left(\mathsf{X}(t)|\mathcal{W}_s\right) = \mathsf{X}(s)$ directly from the defining properties of the Wiener process.

Definition B.1.3 (*Markov process*) An adapted process $\mathsf{X}(t)$, $t \in \mathbb{R}^+$, with filtration $\{\mathcal{F}_s\}_{s\in\mathbb{R}^+}$, is *Markov process* if there exists a Markov kernel $p_{\mathsf{X}}(\tau, \mathsf{X}(s), A)$ for A open subset of $\mathbb{R}$, such that

$$\Pr\left(\mathsf{X}(t) \in A|\mathcal{F}_s\right) = p_{\mathsf{X}}(t-s, \mathsf{X}(s), A). \tag{B.1.3}$$

The Wiener process is a Markov process, having

$$p_{\mathsf{W}}(\tau, y, (x, x+\mathrm{d}\,x)) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(x-y)^2}{2\tau}\right) \mathrm{d}\,x =: \rho(x-y;\tau)\,\mathrm{d}\,x. \tag{B.1.4}$$

It is well known that the probability density $\rho(x, \tau)$ is the fundamental solution of the heat equation

$$\partial_t \rho(x,t) = \frac{1}{2}\partial_x^2 \rho(x,t), \tag{B.1.5}$$

obtained by Einstein in his celebrated work on Brownian diffusion.

A stochastic process strictly related to the Wiener process is the *Brownian bridge process* (Fig. B.2). A Brownian bridge process $\mathsf{B}(t)$ is a Gaussian process on the unit interval $[0, 1]$ such that $\mathbb{E}(\mathsf{B}(t)) = 0$ and $\mathbb{E}(\mathsf{B}(s)\mathsf{B}(t)) = \min\{s, t\} - st$. It can be written in terms of a Wiener process on the unit interval as

$$\mathsf{B}(t) := \mathsf{W}(t) - t\mathsf{W}(1), \quad t \in [0, 1]. \tag{B.1.6}$$

It can be proved that the previous process can be obtained simply conditioning a Wiener process to be equal to zero for $t = 1$. We can introduce also the Gaussian stochastic process

$$\mathsf{B}^{(-1)}(t) := \int_0^t \mathsf{B}(s)\,\mathrm{d}\,s, \quad t \in [0, 1]. \tag{B.1.7}$$

It is easily proved that

$$\mathbb{E}\left(\mathsf{B}^{(-1)}(t)\right) = 0, \quad \mathbb{E}\left(\left[\mathsf{B}^{(-1)}(1)\right]^2\right) = \frac{1}{12}. \tag{B.1.8}$$

Finally, the *zero-area Brownian bridge* can be defined as

$$\mathsf{B}^0(t) = \mathsf{B}(t) - \mathsf{B}^{(-1)}(1). \tag{B.1.9}$$

The previous process has

$$\mathbb{E}\left(\mathsf{B}^0(t)\right) = 0, \quad \mathbb{E}\left(\mathsf{B}^0(s)\mathsf{B}^0(t)\right) = \frac{1}{2}\left(|s-t| - \frac{1}{2}\right)^2 - \frac{1}{24}. \tag{B.1.10}$$

B.2 Probability Distributions

Many mathematical results are available for the probability distributions of some interesting quantities related to the Wiener process or the Brownian bridge process on the unit interval. We collect here some of these results, without proving them (for further details and more general results, see the detailed paper of Beghin and Orsingher [1]). These results are mostly obtained through the so-called *reflection principle*. Using the reflection principle, it can be proved that

$$\Pr\left(\sup_{\tau\in(0,t)} \mathsf{W}(\tau) > W\right) = 2\int_W^\infty \frac{e^{-\frac{z^2}{2t}}}{\sqrt{2\pi t}}\,dz. \tag{B.2.1}$$

Similarly we have

$$\Pr\left(\sup_{\tau\in(0,t)} \mathsf{W}(\tau) > W|\mathsf{W}(t) = w\right) = \begin{cases}\exp\left(-\frac{2W(W-w)}{t}\right) & W > w,\\ 1 & W < w.\end{cases} \tag{B.2.2}$$

It follows that, if $w = 0$ and $t = 1$ we have the probability distribution for the sup of the Brownian bridge process,

$$\Pr\left(\sup_{\tau\in(0,1)} \mathsf{B}(\tau) > B\right) = e^{-2B^2}. \tag{B.2.3}$$

For the absolute value of the Wiener process we have that

$$\Pr\left(\sup_{\tau\in(0,t)} |\mathsf{W}(\tau)| < W|\mathsf{W}(t) = w\right) = \sum_{k\in\mathbb{Z}}(-1)^k \exp\left(-\frac{2kW(kW-w)}{t}\right). \tag{B.2.4}$$

For $w = 0$ and $t = 1$ we have

$$\Pr\left(\sup_{\tau\in(0,1)} |\mathsf{B}(\tau)| < B\right) = \sum_{k\in\mathbb{Z}}(-1)^k \exp\left(-2k^2B^2\right). \tag{B.2.5}$$

Darling [2] proved that for the zero-area Brownian bridge the following distribution holds:

$$\Pr\left(\mathsf{B}^0(t) < B\right) = \frac{4\sqrt{\pi}}{3}\sum_{n=1}^\infty \frac{1}{\alpha_n}\psi\left(\frac{\sqrt{8}B}{3\alpha_n}\right), \quad B > 0, \tag{B.2.6}$$

where $0 < \alpha_1 < \cdots < \alpha_n < \cdots$ are the zeros of the combination of Bessel functions

$$f(\alpha) := J_{\frac{1}{3}}(\alpha) + J_{-\frac{1}{3}}(\alpha), \tag{B.2.7}$$

whilst $\psi(x)$ is the solution of the following integral equation:

$$\int_0^\infty e^{-\lambda x}\, \psi(x)\, d\, x = e^{-\lambda^{\frac{2}{3}}} \,. \tag{B.2.8}$$

The theorem of Darling is not trivial at all, both in the derivation and in the final result. The explicit treatment of the probability distribution density is indeed quite involved. More implicit results on the Laplace transform of the probability distribution of $\mathsf{B}^{(-1)}$ have been obtained by Perman and Wellner [4], but we do not present them here.

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Curriculum Vitae

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Publications

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