Appendix A
Some Relevant Proofs

This appendix contains the proofs of the theorems, propositions and lemmas which were only mentioned in the main text. The expert reader may follow the main text only. The details and main arguments for the statements therein can be found here.

Lemma 2.15 Let S be a subset of $\mathbb{R}$ which is bounded above and let p be the supremum of S. If S is a closed subset of $\mathbb{R}$, then $p \in S$.

Proof Suppose $p \in \mathbb{R} - S$. As $\mathbb{R} - S$ is open there exist real numbers a and b with $a < b$ such that $p \in (a, b) \subseteq \mathbb{R} - S$. As p is the least upper bound for S and $a < p$ it is clear that there exists an $x \in S$ such that $a < x$. Also $x < p < b$ and so $x \in (a, b) \subseteq \mathbb{R} - S$. But this is a contradiction since $x \in S$. Hence our supposition is false and $p \in S$.

Proposition 2.16 Let T be a clopen subset of $\mathbb{R}$. Then either $T = \mathbb{R}$ or $\mathbb{T} = \emptyset$.

Proof Suppose $T \neq \mathbb{R}$ and $T \neq \emptyset$. Then there exists an element $x \in T$ and an element $z \in \mathbb{R} - T$. Without loss of generality, assume $x < z$. Put $S = T \cap [x, z]$. Then S, being the intersection of two closed sets, is closed. It is also bounded above, since z is obviously an upper bound. Let p be the supremum of S. By the previous Lemma, $p \in S$. Since $p \in [x, z]$, $p \leq z$. As $z \in \mathbb{R} - S$, $p \neq z$ and so $p < z$. Now T is also an open set and $p \in T$. So there exist a and b in $\mathbb{R}$ with $a < b$ such that $p \in (a, b) \subseteq T$. Let t be such that $p < t < min(b, z)$, where $min(b, z)$ denotes the smaller of b and z. So, $t \in T$ and $t \in [p, z]$. Thus $t \in T \cap [x, z] = S$. This is a contradiction since $t > p$ and p is the supremum of S. Hence out supposition is false and consequently $T = \mathbb{R}$ or $T = \emptyset$.

Lemma 2.24 Let f be a function mapping $\mathbb{R}$ into itself. Then f is continuous if and only if for each $a \in \mathbb{R}$ and each open set U containing $f(a)$, there exists an open set V containing a such that $f(V) \subseteq U$.

Proof Assume that f is continuous. Let $a \in \mathbb{R}$ and let U be any open set containing $f(a)$. Then there exist real numbers c and d such that $f(a) \in (c, d) \subseteq U$. Put ϵ equal to the smaller of the two numbers $d - f(a)$ and $f(a) - c$, so that

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$$(f(a) - \epsilon, f(a) + \epsilon) \subseteq U \tag{A.1}$$

As the mapping f is continuous, there exists a $\delta > 0$ such that $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$ for all $x \in (a - \delta, a + \delta)$. Let V be the open set $(a - \delta, a + \delta)$. Then $a \in V$ and $f(V) \subseteq U$ as required.

Reversely, assume that for each $a \in \mathbb{R}$ and each open set U containing $f(a)$ there exists an open set V containing a such that $f(V) \subseteq U$. We have to show that f is continuous. Let $a \in \mathbb{R}$ and ϵ be any positive real number. Put $U = (f(a) - \epsilon, f(a) + \epsilon)$. So U is open an open set containing $f(a)$. Therefore there exists an open set V containing a such that $f(V) \subseteq U$. As V is an open set containing a, there exist real numbers c and d such that $a \in (c, d) \subseteq V$. Put δ equal to the smaller of the two numbers $d - a$ and $a - c$, so that $(a - \delta, a + \delta) \subseteq V$. Then for all $x \in (a - \delta, a + \delta)$, $f(x) \in f(V) \subseteq U$ as required. So, f is continuous.

Lemma 2.25 Let f be a mapping of a topological space (X, τ) into a topological space (Y, τ'). Then the following two conditions are equivalent:

- for each $U \in \tau'$, $f^{-1}(U) \in \tau$
- for each $a \in X$ and each $U \in \tau'$ with $f(a) \in U$, there exists a $V \in \tau$ such that $a \in V$ and $f(V) \subseteq U$.

Proof Assume that the first condition is satisfied. Let $a \in X$ and $U \in \tau'$ with $f(a) \in U$. Then $f^{-1}(U) \in \tau$. Put $V = f^{-1}(U)$ and we have that $a \in V$, $V \in \tau$ and $f(V) \subseteq U$. So, the second condition is satisfied. Reversely, assuming that the second condition is satisfied, let $U \in \tau'$. If $f^{-1}(U) = \emptyset$ then $f^{-1} \in \tau$. If $f^{-1} \neq \emptyset$, let $a \in f^{-1}(U)$. Then $f(a) \in U$. Therefore there exists a $V \in \tau$ such that $q \in V$ and $f(V) \subseteq U$. So for each $a \in f^{-1}(U)$ there exists a $V \in \tau$ such that $a \in V \subseteq f^{-1}(U)$. This implies that $f^{-1}(U) \in \tau$. So the first condition is satisfied.

Putting these two lemmas together we can see that $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for each open subset U of $\mathbb{R}$, $f^{-1}(U)$ is an open set.

Proposition 2.29 Let (X, τ) and (Y, τ_1) be topological spaces. Then $f : (X, \tau) \to (Y, \tau_1)$ is continuous if and only if for every closed subset S of Y, $f^{-1}(S)$ is a closed subset of X.

Proof The result is direct if one observes that

$$f^{-1}(compl(S)) = compl(f^{-1}(S)) \tag{A.2}$$

where *compl* is the complement.

Proposition 2.32 Let (X, τ) and (Y, τ_1) be topological spaces and $f : (X, \tau) \to (Y, \tau_1)$ surjective and continuous. If (X, τ) is connected then (Y, τ_1) is connected.

Proof Suppose (Y, τ_1) is not connected. Then it has a clopen subset U such that $U \neq \emptyset$ and $U \neq Y$. Then $f^{-1}(U)$ is an open set, since f is continuous, and also a closed set, that is, $f^{-1}(U)$ is a clopen subset of X. Now, $f^{-1}(U) \neq \emptyset$ as f is surjective and $U \neq \emptyset$. Also $f^{-1}(U) \neq X$ since if it were, U would equal Y by the surjectivity of f. Thus (X, τ) is not connected. This is a contradiction. Therefore (Y, τ_1) is connected.

Theorem 2.34 Let $f : [a, b] \to \mathbb{R}$ be continuous and let $f(a) \neq f(b)$. Then for every number p between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = p$.

Proof As $[a, b]$ is connected and f is continuous we have that $f([a, b])$ is connected. This implies that $f([a, b])$ is an interval. now $f(a)$ and $f(b)$ are in $f([a, b])$. So, if p is between $f(a)$ and $f(b)$ then $p \in f([a, b])$, that is, $p = f(c)$ for some $c \in [a, b]$.

Corollary 2.36 (The fixed point theorem) Let f be a continuous mapping of [0, 1] into [0, 1]. Then there exists a $z \in [0, 1]$ such that $f(z) = z$. The point is called a fixed point.

Proof If $f(0) = 0$ or $f(1) = 1$ the result is obviously true. Thus it suffices to consider the case when $f(0) > 0$ and $f(1) < 1$. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = x - f(x)$. Clearly g is continuous $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$ Consequently there exists a $z \in [0, 1]$ such that $g(z) = 0$ that is, $z - f(z) = 0$ or $f(z) = z$.

Proposition 2.42 Let (X, d) be a metric space and τ the topology induced on X by the metric d. Then a subset U of X is open in (X, τ) if and only if for each $a \in U$ there exists an $\epsilon > 0$ such that the open ball $B_\epsilon(a) \subseteq U$.

Proof Assume $U \in \tau$. Then for any $a \in U$ there exists a point $b \in X$ and a $\delta > 0$ such that

$$a \in B_\delta(b) \subseteq U \tag{A.3}$$

Let then $\epsilon = \delta - d(a, b)$. Then

$$a \in B_\epsilon(a) \subseteq U \tag{A.4}$$

Reversely, assume that U is a subset of X with the property that for each $a \in U$ there exists an ϵ_a such that $B_{\epsilon_a}(a) \subseteq U$. Then U is an open set.

Proposition 2.44 Let (X, d) be any metric space and τ the topology induced on the X by d. Then (X, τ) is Hausdorff.

Proof Let a and b be any points of X with $a \neq b$. Then $d(a, b) > 0$. Put $\epsilon = d(a, b)$. Consider the open balls $B_{\epsilon/2}(a)$ and $B_{\epsilon/2}(b)$. Then these are open sets in (X, τ) with $a \in B_{\epsilon/2}(a)$ and $b \in B_{\epsilon/2}(b)$. So, to show τ is Hausdorff we have to prove that $B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b) = \emptyset$.

Suppose $x \in B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b)$. Then $d(x, a) < \frac{\epsilon}{2}$ and $d(x, b) < \frac{\epsilon}{2}$. Hence

$$d(a, b \leq d(a, x) + d(x, b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{A.5}$$

This says $d(a, b) < \epsilon$ which is false. Consequently there exists no $x \in B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b)$; that is, $B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b) = \emptyset$, as required.

Proposition 2.48 Let (X, d) be a metric space. A subset A of X is closed in (X, d) if and only if every convergent sequence of points in A converges to a point in A. This means that A is closed in (X, d) if and only if $a_n \to x$ where $x \in X$ and $a_n \in A$ for all n, implies $x \in A$.

Proof Assume that A is closed in (X, d) and let $a_n \to x$ where $a_n \in A$ for all positive integers n. Suppose that $x \in X - A$. Then, as $X - A$ is an open set containing x, there exists an open ball $B_\epsilon(x)$ such that $x \in B_\epsilon(x) \subseteq X - A$. Noting that each $a_n \in A$, this implies that $d(x, a_n) > \epsilon$ for each n. Hence the sequence $a_1, a_2, \ldots, a_n \ldots$ does not converge to x. This is a contradiction. So, $x \in A$, as required. Conversely, assume that every convergent sequence of points in A converges to a point of A. Suppose that $X - A$ is not open. Then there exists a point $y \in X - A$ such that for each $\epsilon > 0$, $B_\epsilon(y) \cap Q \neq \emptyset$. For each positive integer n, let x_n be any point in $B_{1/n}(y) \cap A$. Then we claim that $x_n \to y$. To see this let ϵ be any positive real number, and n_0 any integer greater than $\frac{1}{\epsilon}$. Then for each $n \geq n_0$

$$x_n \in B_{1/n}(y) \subseteq B_{1/n_0}(y) \subseteq B_\epsilon(y) \tag{A.6}$$

So $x_n \to y$ and by our assumption, $y \in A$. This is a contradiction and so $X - A$ is open and thus A is closed in (X, d).

Lemma 3.2 Any point x in the simplex can be written as $x = \sum_i x_i v_i$ with $x_i \geq 0$ and $\sum_i x_i = 1$. The x_i are called barycentric coordinates and they are unique.

Proof In general we can always translate a simplex into another one with $v_0 = 0$. Now, if $v_0 = 0$ then $v_1, \ldots, v_i$ have to be independent. So, $x = \sum_{p=0}^{i} x_p v_p = \sum_{p=1}^{i} x_p v_p$ hence x_p, $p > 0$ are determined by x and then so is $x_0 = 1 - \sum_{p=1}^{i} x_p$.

Lemma 3.8 The above formula for ∂_i gives a well defined $\mathbb{K}$-map $\partial_i : C_i(X, \mathcal{T}; \mathbb{K}) \to C_{i-1}(X, \mathcal{T}; \mathbb{K})$.

Proof

- First it is verified that the formula only depends on the orientation. For instance for two orderings xyz and zxy which give the same orientation one has $\partial\sigma_{zxy} = \sigma_{xy} - \sigma_{zy} + \sigma_{zx}$ and $\partial\sigma_{xyz} = \sigma_{yz} - \sigma_{xz} + \sigma_{xy}$ coincide.
- Second, it is also verified that the map descends to $C_i \to C_{i-1}$ i.e. that the opposite orientations produce opposite results.

The two verified requirements signify that for any permutation one has the action of the boundary operator expressed in terms of the sign of the permutation.

Definition 4.16 (Quasi-isomorphisms) We say that a map of complexes $f : A^* \to B^*$ is a quasi-isomorphism if the induced maps of cohomology groups $H^n(f) : H^n(A^*) \to H^n(B^*)$, $n \in Z$ are all isomorphisms.

Lemma 4.17 A left resolution of M is the same as a quasi-isomorphism of complexes $P^* \to M^\#$ such that $P^i = 0$ for $i > 0$.

Proof If (P^*, q) is a resolution of M then the only non-zero cohomology group of P^* is $H^0(P^*) \cong M$, the same being true for $M^\#$. Moreover the morphism of complexes $P^* \to M^\#$ is given by $q : P^0 \to M$ which induces isomorphisms of $H^0(P^*) = P^0/dP^{-1}$ onto $M = H^0(M^\#)$.

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Curriculum Vitae

- Name: Patrascu, Andrei-Tudor
- Work experience, date starting
  - 2001: License in physics with honors, Babes-Bolyai university, Romania
  - 2005: Master in theoretical physics, Babes-Bolyai university, Romania
  - 2006: Erasmus mundus master courses in theoretical physics at RWTH-Aachen
  - 2006: Master in theoretical physics from university of Paris, Orsay/Ecole Normale Superieure, Paris
  - 2008: Teaching assistant at Rensselaer polytechnic institute

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- 2010: University of Paris Diderot, 7th framework programme, FET open grant number 250056
- 2011: University college London, "ExoMol" ERC advanced investigator project 267219
- Self initiated research project (final PhD research project at UCL): "Universal coefficient theorem and quantum field theory",
- 2011: Invited visiting scholar at Rice university (TX, USA)

- Publications
  - Study of the electronic and rovibronic structure of the X $2\Sigma^+$, A 2Π, and B $2\Sigma^+$ states of AlO, published in J. Chem. Phys. 141, 144312, 2014
  - ExoMol molecular line lists: IX The spectrum of AlO, published in the monthly notices of the royal astronomical society, ref. MN-14-3616-MJ.R1, 2015
  - Quantization, holography, and the universal coefficient theorem, published in Phys. Rev. D 90, 045018, 2014
  - Black coles, Information and the universal coefficient cheorem, arXiv:1410.5291
  - On SU(2) anomaly and Majorana fermions, arXiv:1402.7283
  - On quantization and general relativity, arXiv:1406.5769
  - A quantum field theoretical approach to the P vs. NP problem via the sign of the fermionic quantum monte carlo, arXiv:1403.2067
  - The quantization of topology, from quantum Hall effect to quantum gravity, arXiv:1411.4475
  - The mapping of non-planar diagrams into planar diagrams and the exact solution of QCD
  - Entanglement, spacetime, and the Mayer-Vietoris theorem

- Conference Talks and Invited Talks
  - International society of infrared, millimeter and terahertz waves, Rome (Italy 2010)
  - Electronic structures theory for strongly correlated systems, Palermo (Italy 2012)
  - Invited research scholar at Rice university, Houston (USA 2012)
  - Cumberland lodge conference, characterizing exoplanets (UK 2013)
  - High resolution molecular spectroscopy conference, Budapest (Hungary 2013)
  - Mathematics days in Sofia, invited talk (Bulgaria 2014)
  - 30th International colloquium on group theoretical methods in physics, talk, Ghent (Belgium 2014)
  - Institut d'ete du LPTENS Paris, invited (France 2014)
  - Max Planck institute for gravitational physics, invited seminar, Potsdam (Germany 2015)