# Appendix A
# Solutions of Exercises

**Exercise** 1.1
$P^{\rho}_{M}(1) = \frac{5}{16}$, $P^{\rho}_{M}(2) = \frac{5}{16}$, and $P^{\rho}_{M}(3) = \frac{3}{8}$.

**Exercise** 1.2
$A = E_1 - E_2$ and $B = F_1 - F_2$, where $E_1 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, $E_2 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, $F_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $F_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

**Exercise** 1.3
The expectation can be calculated in two ways. (1) $1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$. (2) $\operatorname{Tr} \rho A = 0$.

**Exercise** 1.4
The variance $\Delta_\rho A$ can be calculated in two ways. (1) $1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$. (2) $\operatorname{Tr} \rho A^2 = 1$.

**Exercise** 1.5
Since $\operatorname{Tr} B\rho = 0$, the value $\Delta_\rho A \circ B$ is $\operatorname{Tr} \frac{1}{2}(AB + BA)\rho = \operatorname{Tr} \frac{1}{2}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right)\rho = 0$.

**Exercise** 1.6
The state $|X\rangle\rangle_{A,B}\ {}_{A,B}\langle\langle X|$ is entangled if and only if the vector $|X\rangle\rangle_{A,B}$ is in a tensor product form. This condition is equivalent to $\det X \neq 0$.

**Exercise** 1.7
The orthogonality can be shown by using (1.16).



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**Exercise** 1.8
The spectral decomposition of $\rho^{\otimes n}$ is $\rho^{\otimes n} = \sum_{k=0}^{n} \frac{2^k}{3^n} F_k$, where $F_k := \sum_{i_1,i_2,\ldots,i_n : i_1+i_2+\cdots+i_n=k} E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_n}$.

**Exercise** 1.9
Since $\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{it(h_j-h_{j'})}dt = 0$ for $h_j \neq h_{j'}$, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{itH}\rho e^{-itH}dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{it(h_j-h_{j'})}dt E_j\rho E_{j'} = \sum_j E_j\rho E_j.$$

**Exercise** 1.10
The PVM $\{E_i \otimes F_j\}_{i,j}$ gives the simultaneous diagonalization of $A \otimes I$ and $I \otimes B$, where $E_i$ and $F_j$ are given in Exercise 1.2.

**Exercise** 1.11
For simplicity, we show the case with $d = 1$. Given two non-negative integers $\alpha, \beta \in \mathbb{Z}_+$, we assume that $\beta \geq 1$. Then, we have

$$\begin{aligned} &\sup_x \left|x^\alpha \frac{d^\beta}{dx^\beta} x f(x)\right| = \sup_x \left|x^\alpha \frac{d^{\beta-1}}{dx^{\beta-1}} f(x) + x^{\alpha+1}\frac{d^\beta}{dx^\beta} f(x)\right| \\ \leq &\sup_x \left|x^\alpha \frac{d^{\beta-1}}{dx^{\beta-1}} f(x)\right| + \sup_x \left|x^{\alpha+1}\frac{d^\beta}{dx^\beta} f(x)\right| < \infty. \end{aligned}$$

When $\beta = 0$,

$$\sup_x \left|x^\alpha \frac{d^\beta}{dx^\beta} x f(x)\right| = \sup_x \left|x^{\alpha+1} f(x)\right| < \infty.$$

So, he function $xf(x)$ belongs to $\mathscr{S}(\mathbb{R}^d)$.

For a given $\alpha, \beta \in \mathbb{Z}_+$, we assume that $\beta \geq 1$. Then, we have $\sup_x \left|x^\alpha \frac{d^\beta}{dx^\beta}\frac{d}{dx} f(x)\right| = \sup_x \left|x^\alpha \frac{d^{\beta+1}}{dx^{\beta+1}} f(x)\right| < \infty$. So, the function $\frac{d}{dx} f(x)$ belongs to $\mathscr{S}(\mathbb{R}^d)$.

**Exercise** 1.12
For simplicity, we show the case with $d = 1$. Given two non-negative integers $\alpha, \beta \in \mathbb{Z}_+$, we assume that $\beta \geq 1$. Then, we have

$$\begin{aligned} &\sup_x \left|x^\alpha \frac{d^\beta}{dx^\beta} g(x) f(x)\right| = \sup_x \left|x^\alpha \sum_{k=0}^{\beta} \binom{\beta}{k} \frac{d^k}{dx^k} g(x) \frac{d^{\beta-k}}{dx^{\beta-k}} f(x)\right| \\ \leq &\sum_{k=0}^{\beta} \binom{\beta}{k} \sup_x \left|x^\alpha \frac{d^k}{dx^k} g(x)\right| \cdot \left|\frac{d^{\beta-k}}{dx^{\beta-k}} f(x)\right| < \infty. \end{aligned}$$

So, the product $f(x)g(x)$ belongs to $\mathscr{S}(\mathbb{R}^d)$.

**Exercise** 2.1
Let $(a, b, c)$ be the cyclic permutation among $a, b, c$. Then, the permutation group $S_3$ is composed of $(1, 2, 3)$, $(3, 1, 2)$, $(2, 3, 1)$, $(1, 2)$, $(2, 3)$, and $(3, 1)$. The orders of $(1, 2, 3)$, $(3, 1, 2)$, $(2, 3, 1)$ are 3. The orders of $(1, 2)$, $(2, 3)$, $(3, 1)$ are 2.

**Exercise** 2.2
This statement can be shown by the following relation. $(1, 2)(3, 4)(1, 3)(2, 4) = (1, 3)(2, 4)(1, 2)(3, 4) = (1, 4)(2, 3)$, $(1, 3)(2, 4)(1, 4)(2, 3) = (1, 4)(2, 3)(1, 3)(2, 4) = (1, 2)(3, 4)$, $(1, 4)(2, 3)(1, 2)(3, 4) = (1, 2)(3, 4)(1, 4)(2, 3) = (1, 3)(2, 4)$, $((1, 2)(3, 4))^2 = ((1, 3)(2, 4))^2 = ((1, 4)(2, 3))^2 = e$.

**Exercise** 2.3
The following map gives the isometric relation. $e \mapsto e$, $(1, 2)(3, 4) \mapsto i$, $(1, 3)(2, 4) \mapsto j$, $(1, 4)(2, 3) \mapsto k$.

**Exercise** 2.4
It is enough to show that $(a, b)\{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}(a, b) = \{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$ because any permutation is given as the products of transpositions. Consider the case of $(a, b) = (1, 2)$ because other cases can be shown in the same way. $(1, 2)(1, 2)(3, 4)(1, 2) = (1, 2)(1, 2)(1, 2)(3, 4) = (1, 2)(3, 4)$. $(1, 2)(1, 3)(2, 4)(1, 2) = (2, 3)(1, 2)(1, 2)(1, 4) = (2, 3)(1, 4) = (1, 4)(2, 3)$. $(1, 2)(1, 4)(2, 3)(1, 2) = (2, 4)(1, 2)(1, 2)(1, 3) = (2, 4)(1, 3) = (1, 3)(2, 4)$.

**Exercise** 2.5
Consider the subgroup $S_3$ of $S_4$, which permutes the letters $\{1, 2, 3\}$. For any different element $g \neq e \in S_3$, the set $g(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is different from $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, for any different elements $g_1 \neq g_2 \in S_3$, the set $g_1(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is different from $g_2(\mathbb{Z}_2 \times \mathbb{Z}_2)$. So, each element of $S_3$ gives different residue class. Since the order of $S_3$ is 6 and the order of $S_4/\mathbb{Z}_2 \times \mathbb{Z}_2$ is $24/4 = 6$, all of elements of $S_4/\mathbb{Z}_2 \times \mathbb{Z}_2$ can be written by $S_3$. Since they have the same products, the quotient group $S_4/\mathbb{Z}_2 \times \mathbb{Z}_2$ is isometric to $S_3$.

**Exercise** 2.6
It is trivial that the subset $\{g \in G | T(g)\theta_0 = \theta_0\}$ contains the identity element. When $g_1, g_2 \in \{g \in G | T(g)\theta_0 = \theta_0\}$, we have $T(g_1 g_2)\theta_0 = T(g_1)T(g_2)\theta_0 T(g_1)\theta_0 = \theta_0$. Hence, $\{g \in G | T(g)\theta_0 = \theta_0\}$ is a subgroup.

**Exercise** 2.7
Firstly, we fix a point $\theta_0$ in $\Theta$. For an element $\theta \in \Theta$, we choose $g_\theta \in G$ such that $g_\theta \cdot \theta_0 = \theta$. Then, $\theta \mapsto [g_\theta] \in G/H$ is an isomorphism for homogeneous spaces.

**Exercise** 2.8
Let $X$ be the set $\{g^{-1}h^{-1}gh|g, h \in G\}$ of generators of the commutator subgroup $[G, G]$. For elements $z \in G$ and $x, y \in G$, we have $z(xyx^{-1}y^{-1})z^{-1} = (zxz^{-1})(zyz^{-1})(zxz^{-1})^{-1}(zyz^{-1})^{-1} \in X$. That is, $zXz^{-1} \subset X$. Since $X$ generates $[G, G]$, we have $z[G, G]z^{-1} \subset [G, G]$. Especially, considering the case when $z$ is an element of $[G, G]$, we find that $[G, G]$ is a normal subgroup.

**Exercise** 2.9
The desired statement follows from the calculation given in Exercise 2.4.

**Exercise** 2.10
Since

$$\begin{aligned}
&((h, k)(h', k'))(h'', k'') = (hT(k)(h')\phi(k, k'), kk')(h'', k'') \\
=&(hT(k)(h')\phi(k, k')T(kk')(h'')\phi(kk', k''), kk'k''), \\
&(h, k)((h', k')(h'', k'')) = (h, k)(h'T(k')(h'')\phi(k', k''), k'k'') \\
=&(hT(k)(h'T(k')(h'')\phi(k', k''))\phi(k, k'k''), kk'k''),
\end{aligned}$$

it is sufficient to show that

$$\begin{aligned}
&hT(k)(h')\phi(k, k')T(kk')(h'')\phi(kk', k'') \\
=&hT(k)(h'T(k')(h'')\phi(k', k''))\phi(k, k'k''). \qquad \text{(A.1)}
\end{aligned}$$

This equation can be shown as follows.

$$\begin{aligned}
&T(k)(h'T(k')(h'')\phi(k', k''))\phi(k, k'k'') \\
=&T(k)(h')T(k)(T(k')(h''))T(k)(\phi(k', k''))\phi(k, k'k'') \\
\overset{(a)}{=}&T(k)(h')\phi(k, k')T(kk')(h'')\phi(k, k')^{-1}\phi(k, k')\phi(kk', k'')\phi(k, k'k'')^{-1} \\
&\cdot \phi(k, k'k'') \\
=&T(k)(h')\phi(k, k')T(kk')(h'')\phi(kk', k''), \qquad \text{(A.2)}
\end{aligned}$$

where $(a)$ follows from (2.7) and (2.8).

**Exercise** 2.11
$hk = (h, e)(e, k) = (h, k)$. $khk^{-1} = (e, k)(h, e)(e, k^{-1}) = (e, k)(h, k^{-1}) = (T(k)h, kk^{-1}) = (T(k)h, e) = T(k)(h)$.

**Exercise** 2.12
The desired argument can be checked by the relations $(h, k)(h', e)(h, k)^{-1} = (h, k)(h', e)(h^{-1}, k^{-1}) = (h, k)(h'h^{-1}, k^{-1}) = (h, T(k)h'h^{-1}, kk^{-1}) = (h, T(k)h' h^{-1}, e) \in H$.

**Exercise** 2.13
When $T(k)$ is the identity map, the discussions in Sect. 2.2.2 give the central extension.

Conversely, we assume that the given extension is a central extension. Since any factor system can be replaced by a normalized factor system, we treat only the case with a normalized factor system. We consider the case with $(h', e) = (e, k)(h', e)(e, k)^{-1} = (e, k)(h', k^{-1}) = (T(k)h'\phi(k, k^{-}), kk^{-1}) = (T(k)h'\phi(k, k^{-}), e)$ $h' = e$. So, we obtain $\phi(k, k^{-}) = e$. Hence, we have $T(k)h' = h'$, which implies that $T(k)$ is the identity map.

**Exercise** 2.14
The desired argument can be shown by focusing on the simultaneous diagonalization.

**Exercise** 2.15
When $\mathsf{f}_\lambda$ is a self-conjugate representation, we have $\overline{\mathsf{f}_\lambda(g)} = \mathsf{f}_{\lambda^*}(g) = \mathsf{f}_\lambda(g)$. So, the character takes real numbers.

**Exercise** 2.16
The representation is complexifiable if and only if it is commutative with the multiplication of the imaginary number $i$, which is equivalent to (2.44).

**Exercise** 2.17
It is enough to consider the case when $j = 2$ and $i = 1$ in (2.72). So, (2.72) implies that

$$d_{S_n,(\frac{n+k}{2},\frac{n-k}{2})} = \frac{n!}{(\frac{n+k}{2}+1)!\frac{n-k}{2}!}\Big(\frac{n+k}{2} - \frac{n-k}{2} + 2 - 1\Big)$$
$$= \frac{(n+1)!}{(n+1)(\frac{n+k}{2}+1)!\frac{n-k}{2}!}(k+1) = \frac{k+1}{n+1}\binom{n+1}{\frac{n+k}{2}+1}.$$

**Exercise** 3.1
Any element $g$ of $\mathrm{SO}(2, \mathbb{R})$ is written as $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$. So, any element connected to the connected components including $I$. Thus, $\mathrm{SO}(2, \mathbb{R})$ is connected.

**Exercise** 3.2
Since $gg^\dagger = g^\dagger g$, there are $d$ eigenvectors $v_1, \ldots, v_d$ of $g$. Since $g$ preserves the inner product, the eigenvalue $\lambda_j$ corresponding to $v_j$ has the absolute value 1. Taking the complex conjugate of the eigenvalue equation $\lambda_j v_j = g v_j$, we have $\overline{\lambda_j} v_j^* = g v_j^*$. This fact shows that the number of non-real eigenvectors is even, which is denoted by $2r$. We sort the eigenvectors $v_i$ such that initial $2r$ eigenvectors are non-real, and $v_{2j}$ is the complex conjugate of $v_{2j-1}$. Now, we define two vectors $w_{2j-1} := \frac{1}{\sqrt{2}}(v_{2j-1} + v_{2j})$ and $w_{2j} := \frac{i}{\sqrt{2}}(v_{2j-1} - v_{2j})$ for $j = 1, \ldots, r$. For

$j \geq 2r+1$, we define $w_j := v_j$. Since $|\lambda_j| = 1$, $\lambda_j$ is written as $\cos\theta + i\sin\theta$. So, we have $gw_{2j-1} = \cos\theta w_{2j-1} + \sin\theta w_{2j}$ and $gw_{2j} = -\sin\theta w_{2j-1} + \cos\theta w_{2j}$ for $j = 1, \ldots, r$. Arranging the $d$ vectors $w_1, \ldots, w_d$, we define orthogonal matrix $g'$. Then, $g'gg'^{-1}$ satisfies the required condition.

**Exercise** 3.3
Due to Exercises 3.1 and 3.2, Any element $g \in \mathrm{SO}(d, \mathbb{R})$ is connected to the connected components including $I$. So, $\mathrm{SO}(d, \mathbb{R})$ is connected.

**Exercise** 3.4
We choose one-dimensional differentiable subsets $g_i(t)$ of $G_i$. Then, $g_1(t)$ and $g_2(t)$ are commutative with each other as elements of $G_1 \times G_2$. This fact shows that $\mathfrak{g}(G_1) \oplus \mathfrak{g}(G_2) = \mathfrak{g}(G_1 \times G_2)$.

**Exercise** 3.5
The desired argument can be checked by expanding both sides of (3.20) and (3.21) up to the order $t^2$.

**Exercise** 3.6
It is sufficient to show that (3.41) satisfies Jacobi law, which can be shown as follows.

$$
\begin{aligned}
&[(X_1, X_2), [(Y_1, Y_2), (Z_1, Z_2)]] + [(Y_1, Y_2), [(Z_1, Z_2), (X_1, X_2)]] \\
=&[(X_1, X_2), (T(Y_2)Z_1 - T(Z_2)Y_1 + [Y_1, Z_1], [Y_2, Z_2])] \\
&+ [(Y_1, Y_2), (T(Z_2)X_1 - T(X_2)Z_1 + [Z_1, X_1], [Z_2, X_2])] \\
=&(T(X_2)(T(Y_2)Z_1 - T(Z_2)Y_1 + [Y_1, Z_1]) - T([Y_2, Z_2])X_1 \\
&+ [X_1, (T(Y_2)Z_1 - T(Z_2)Y_1 + [Y_1, Z_1]], [X_2, [Y_2, Z_2]]) \\
&+ (T(Y_2)(T(Z_2)X_1 - T(X_2)Z_1 + [Z_1, X_1]) - T([Z_2, X_2])Y_1 \\
&+ [Y_1, (T(Z_2)X_1 - T(X_2)Z_1 + [Z_1, X_1]], [Y_2, [Z_2, X_2]]) \\
=&(T(X_2)(T(Y_2)Z_1 - T(Z_2)Y_1 + [Y_1, Z_1]) - T([Y_2, Z_2])X_1 \\
&+ T(Y_2)(T(Z_2)X_1 - T(X_2)Z_1 + [Z_1, X_1]) - T([Z_2, X_2])Y_1 \\
&+ [X_1, (T(Y_2)Z_1 - T(Z_2)Y_1 + [Y_1, Z_1]] \\
&+ [Y_1, (T(Z_2)X_1 - T(X_2)Z_1 + [Z_1, X_1]], \\
&[X_2, [Y_2, Z_2]] + [Y_2, [Z_2, X_2]]) \\
=&([T(X_2), T(Y_2)]Z_1 + T(Z_2)T(Y_2)X_1 - T(Z_2)T(X_2)Y_1 \\
&+ T(Z_2)[X_1, Y_1] + [Z_1, (T(X_2)Y_1 - T(Y_2)X_1] + [Z_1, [X_1, Y_1]], \\
&[Z_2, [X_2, Y_2]]) \\
=&[(Z_1, Z_2), (T(X_2)Y_1 - T(Y_2)X_1 + [X_1, Y_1], [X_2, Y_2])] \\
=&[(Z_1, Z_2), [(X_1, X_2), (Y_1, Y_2)]].
\end{aligned}
$$

**Exercise 3.7**

$$\begin{aligned}&[\mathsf{f}((X_1, X_2)), \mathsf{f}((Y_1, Y_2))]\\ =&(\mathsf{f}_1(X_1) + \mathsf{f}_2(Y_1))(\mathsf{f}_1(X_2) + \mathsf{f}_2(Y_2)) - (\mathsf{f}_1(X_2) + \mathsf{f}_2(Y_2))(\mathsf{f}_1(X_1) + \mathsf{f}_2(Y_1))\\ =&[\mathsf{f}_1(X_1), \mathsf{f}_1(Y_1)] + [\mathsf{f}_2(X_2), \mathsf{f}_2(Y_2)] + [\mathsf{f}_1(X_1), \mathsf{f}_2(Y_2)] + [\mathsf{f}_2(X_2), \mathsf{f}_1(Y_1)]\\ =&\mathsf{f}_1([X_1, Y_1]) + \mathsf{f}_2([X_2, Y_2]) - \mathsf{f}_1(T(Y_2)X_1) + \mathsf{f}_1(T(X_2)Y_1)\\ =&\mathsf{f}([(X_1, X_2), (Y_2, Y_2)]).\end{aligned}$$

**Exercise 3.8**
Property (3.44) shows that the matrix $\mathsf{ad}(Y)$ is represented as an alternative matrix. Hence, $\exp(\mathsf{ad}(Y))$ is an orthogonal matrix. When we choose $Y$ as $g = \exp(Y)$, we have $(\mathsf{ad}(g)X, \mathsf{ad}(g)Z)_{\mathfrak{g}} = (X, Z)_{\mathfrak{g}}$.

**Exercise 3.9**
Since

$$\mathcal{F}_{\mathrm{U}(1)} \circ \mathcal{F}_{\mathrm{U}(1)}^{-1}[\{a_n\}](n') = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{Z}} a_n e^{i\theta n} e^{-i\theta n'} d\theta = \sum_{n\in\mathbb{Z}} a_n \delta_{n,n'} = a_n,$$

the map $\mathcal{F}_{\mathrm{U}(1)} \circ \mathcal{F}_{\mathrm{U}(1)}^{-1}$ is the identity map.

For any function $\phi(\neq 0) \in Ł^2(\mathrm{U}(1))$, $\mathcal{F}_{\mathrm{U}(1)}[\phi]$ is not zero. So, the kernel of $\mathcal{F}_{\mathrm{U}(1)}$ is $\{0\}$. Thus, the map $\mathcal{F}_{\mathrm{U}(1)}^{-1} \circ \mathcal{F}_{\mathrm{U}(1)}$ is also the identity map.

**Exercise 3.10**

$$\begin{aligned}\|\mathcal{F}_{\mathrm{U}(1)}^{-1}[\{a_n\}]\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n,m} \overline{a_n} a_m e^{-in\theta} e^{im\theta} d\theta\\ =\sum_{n,m} \overline{a_n} a_m \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta &= \sum_{n,m} \overline{a_n} a_m \delta_{n,m} = \|\{a_n\}\|^2.\end{aligned}$$

Thus, $\mathcal{F}_{\mathrm{U}(1)}^{-1}$ is a unitary map. Since $\mathcal{F}_{\mathrm{U}(1)}^{-1} \circ \mathcal{F}_{\mathrm{U}(1)}$ is the identity map, $\mathcal{F}_{\mathrm{U}(1)}$ is a unitary map.

**Exercise** 3.11

$$
\begin{aligned}
&-i\mathcal{F}_d[\frac{df}{dx_j}](\lambda) = -i\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\frac{df}{dx_j}\phi(x)e^{-ix\cdot\lambda}dx\\
=&-\frac{i}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^{d-1}}[\phi(x)e^{-ix\cdot\lambda}]_{x_j=-\infty}^{\infty}dx_1\dots dx_{j-1}dx_{j+1}\dots dx_d\\
&+\frac{i}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^{d-1}}\phi(x)\frac{df}{dx_j}e^{-ix\cdot\lambda}dx\\
=&\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^{d-1}}\phi(x)\lambda_j e^{-ix\cdot\lambda}dx = \lambda_j\mathcal{F}_d[f](\lambda).
\end{aligned}
$$

**Exercise** 3.12

$$
\begin{aligned}
&\mathcal{F}_d^{-1}[\phi]^*(\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}}\left(\int_{\mathbb{R}^d}\phi(x)e^{ix\cdot\lambda}dx\right)^*\\
=&\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\phi(x)^*e^{-ix\cdot\lambda}dx = \mathcal{F}_d[\phi^*](\lambda).
\end{aligned}
$$

**Exercise** 3.13

$$
\begin{aligned}
&\mathcal{F}_d[\phi](-\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}}\left(\int_{\mathbb{R}^d}\phi(x)e^{-ix\cdot(-\lambda)}dx\right)^*\\
=&\frac{1}{(2\pi)^{\frac{d}{2}}}\left(\int_{\mathbb{R}^d}\phi(x)e^{ix\cdot\lambda}dx\right)^* = \mathcal{F}_d^{-1}[\phi].
\end{aligned}
$$

**Exercise** 3.14

For simplicity, we show the case with $d = 1$. Using Exercise 3.11, we have

$$
\begin{aligned}
&\sup_\lambda\left|\lambda^\alpha\frac{d^\beta}{d\lambda^\beta}\mathcal{F}[f](\lambda)\right| = \sup_\lambda\left|\lambda^\alpha\frac{d^\beta}{d\lambda^\beta}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\phi(x)e^{-ix\lambda}dx\right|\\
=&\sup_\lambda\left|\lambda^\alpha\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{d^\beta}{d\lambda^\beta}\phi(x)e^{-ix\lambda}dx\right|\\
=&\sup_\lambda\left|\lambda^\alpha\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(-ix)^\beta\phi(x)e^{-ix\lambda}dx\right|\\
=&\sup_\lambda\left|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{d^\alpha}{dx^\alpha}(-ix)^\beta\phi(x)e^{-ix\lambda}dx\right|.
\end{aligned}
$$

Exercise 1.11 guarantees that $\frac{d^\alpha}{dx^\alpha}(-ix)^\beta \phi(x)$ belongs to $\mathscr{S}(\mathbb{R})$.

For a function $f$ in $\mathscr{S}(\mathbb{R}^d)$, the values $\sup_x |x|^2|f(x)|$ and $\sup_x |f(x)|$ have finite values $C_1$ and $C_2$, we have $|\phi(x)| \le \frac{C_1}{|x^2|}$ and $|\phi(x)| \le C_2$. We apply this fact to $\frac{d^\alpha}{dx^\alpha}(-ix)^\beta \phi(x)$. So, we have

$$\sup_\lambda \Big|\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{d^\alpha}{dx^\alpha}(-ix)^\beta \phi(x) e^{-ix\lambda} dx\Big|$$
$$\le \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Big|\frac{d^\alpha}{dx^\alpha}(-ix)^\beta \phi(x)\Big| dx \le \frac{1}{\sqrt{2\pi}} \left(\int_{|x|>1} \frac{C_1}{|x^2|} dx + \int_{|x|\le 1} C_2 dx\right) < \infty.$$

So, $\mathcal{F}_d[\phi]$ belongs to $\mathscr{S}(\mathbb{R}^d)$.

**Exercise 3.15**

$$\langle \phi_1 | \mathcal{F}_d[\phi_2]\rangle = \int_{\mathbb{R}^d} \overline{\phi_1(x)} \mathcal{F}_d[\phi_2](x) dx$$
$$= \int_{\mathbb{R}^d} \overline{\phi_1(x)} \left(\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \phi_2(y) e^{-iy\cdot x} dy\right) dx$$
$$= \int_{\mathbb{R}^d} \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-iy\cdot x} \overline{\phi_1(x)} dx\right) \phi_2(y) dy$$
$$= \int_{\mathbb{R}^d} \overline{\left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{iy\cdot x} \phi_1(x) dx\right)} \phi_2(y) dy = \langle \mathcal{F}_d^{-1}[\phi_1] | \phi_2\rangle.$$

**Exercise 3.16**

**a** Firstly, considering the complex integral in the path $-R + i\frac{y}{\sqrt{\epsilon}} \to R + i\frac{y}{\sqrt{\epsilon}} \to R \to -R \to -R + i\frac{y}{\sqrt{\epsilon}}$. Then, we have

$$\frac{\sqrt{\epsilon}}{\sqrt{2\pi}} \int_{-R}^{R} e^{-\frac{\epsilon}{2}(x - i\frac{y}{\sqrt{\epsilon}})^2} dy = \frac{\sqrt{\epsilon}}{\sqrt{2\pi}} \int_{-R+i\frac{y}{\sqrt{\epsilon}}}^{R+i\frac{y}{\sqrt{\epsilon}}} e^{-\frac{\epsilon}{2}x^2} dy = \frac{\sqrt{\epsilon}}{\sqrt{2\pi}} \int_{-R}^{R} e^{-\frac{\epsilon}{2}x^2} dy.$$

Taking the limit $R \to \infty$, we have

$$\frac{\sqrt{\epsilon}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\epsilon}{2}(x - i\frac{y}{\sqrt{\epsilon}})^2} dy = \frac{\sqrt{\epsilon}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\epsilon}{2}x^2} dy = 1.$$

Thus,

$$\mathcal{F}[\varphi_{1/\epsilon}](y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{4}} e^{-\frac{\epsilon x^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\frac{\epsilon x^2}{2} - ixy} dy$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\frac{\epsilon}{2}(x - i\frac{y}{\sqrt{\epsilon}})^2 - \frac{x^2}{2\epsilon}} dy = \left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2\epsilon}} = \varphi_\epsilon(x).$$

**b**

$$\begin{aligned}
&\left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty}\phi(x-y)\varphi_\epsilon(y)dy = (\frac{1}{2\pi\epsilon})^{\frac{1}{4}}\langle\overline{\phi(x-\cdot)}|\mathcal{F}[\varphi_{1/\epsilon}]\rangle\\
=&\left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{4}}\langle\mathcal{F}^{-1}[\overline{\phi(x-\cdot)}]|\varphi_{1/\epsilon}\rangle = \left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{4}}\langle\mathcal{F}[\overline{\phi(x+\cdot)}]|\varphi_{1/\epsilon}\rangle\\
=&\left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{4}}\langle\overline{\mathcal{F}^{-1}[\phi(x+\cdot)]}|\varphi_{1/\epsilon}\rangle = \left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty}\mathcal{F}^{-1}[\phi(x+\cdot)](y)\varphi_{1/\epsilon}(y)dy\\
=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}[\phi](y)e^{ixy}e^{-\epsilon y^2/2}dy.
\end{aligned}$$

**c** Since $(\frac{1}{2\pi\epsilon})^{\frac{1}{4}}\varphi_\epsilon(y)$ is the Gaussian distribution with variance $\epsilon$, taking the limit $\epsilon \to 0$, we have

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}[\phi](y)e^{ixy}dy = \mathcal{F}^{-1}[\mathcal{F}[\phi]].$$

So, $\mathcal{F}^{-1}\circ\mathcal{F}$ is the identity map on $\mathcal{S}(\mathbb{R})$. Since

$$\mathcal{F}^{-1}[\mathcal{F}[\phi]] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}[\phi](y)e^{ixy}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{F}^{-1}[\phi](-z)e^{-ixz}dz = \mathcal{F}[\mathcal{F}^{-1}[\phi]],$$

$\mathcal{F}\circ\mathcal{F}^{-1}$ is the identity map on $\mathcal{S}(\mathbb{R})$.
**d** Since $\mathcal{S}(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R})$, $\mathcal{F}^{-1}\circ\mathcal{F}$ and $\mathcal{F}\circ\mathcal{F}^{-1}$ are the identity map on $L^2(\mathbb{R})$.
**e** Since $\mathcal{F}_d = \mathcal{F}^{\otimes d}$ and $\mathcal{F}_d^{-1} = (\mathcal{F}^{-1})^{\otimes d}$, $\mathcal{F}_d^{-1}\circ\mathcal{F}_d$ and $\mathcal{F}_d\circ\mathcal{F}_d^{-1}$ are the identity map on $L^2(\mathbb{R}^d)$.

**Exercise** 3.17
Substituting $\mathcal{F}_d[\phi_2]$ into $\phi_1$ in Exercise 3.15, we have

$$\langle\mathcal{F}_d[\phi_2]|\mathcal{F}_d[\phi_2]\rangle = \langle\mathcal{F}_d^{-1}[\mathcal{F}_d[\phi_2]]|\phi_2\rangle. = \langle\phi_2]|\phi_2\rangle.$$

So, $\mathcal{F}_d$ preserves the norm, i.e., $\mathcal{F}_d$ is a unitary map.

**Exercise** 3.18
Since $\mathcal{S}(\mathbb{R}^d)$ is a dense subset of $L^2(\mathbb{R}^d)$, Exercise 3.11 implies (3.70), which contains (3.61) as a special case.

**Exercise** 3.19

$$\mathcal{F}[e^{ip\mathsf{Q}}\phi](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} e^{ipx} \phi(x) dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(\lambda - p)x} = \mathcal{F}[\phi](\lambda - p).$$

**Exercise** 3.20
When we replace $\lambda$ by $-\lambda$ in (3.61), we have

$$\mathcal{F}^{-1}[\mathsf{P}\psi](\lambda) = -\lambda \mathcal{F}^{-1}[\psi](\lambda),$$

which implies that $\mathcal{F}^{-1}\mathsf{P} = -\mathsf{Q}\mathcal{F}^{-1}$, i.e., $\mathsf{P} = -\mathcal{F}^{-1}\mathsf{Q}\mathcal{F}^{-1}$. Substitute $\mathcal{F}^{-1}[\phi]$ into $\phi$ in (3.64), we have

$$e^{-ip\mathsf{P}}[\phi](\lambda) e^{ip\mathcal{F}\mathsf{Q}\mathcal{F}^{-1}}[\phi](\lambda) = \mathcal{F} e^{ip\mathsf{Q}} \mathcal{F}^{-1}[\phi](\lambda)$$
$$= \mathcal{F}[e^{ip\mathsf{Q}} \mathcal{F}^{-1}[\phi]](\lambda) = \mathcal{F}[\mathcal{F}^{-1}[\phi]](\lambda - p) = \phi(\lambda - p).$$

**Exercise** 3.21
Consider the inverse Fourier transform of the given function. Exercise 3.22 shows that the product $\mathcal{F}_d^{-1}[\phi_1] \cdot \mathcal{F}_d^{-1}[\phi_2]$ of the inverse Fourier transforms of $\phi_1, \phi_2$. Exercise 3.14 shows that these inverse Fourier transforms $\mathcal{F}_d^{-1}[\phi_1]$ and $\mathcal{F}_d^{-1}[\phi_2]$ belong to $\mathscr{S}(\mathbb{R}^d)$. So, Exercise 1.12 shows that their product $\mathcal{F}_d^{-1}[\phi_1] \cdot \mathcal{F}_d^{-1}[\phi_2]$ also belongs to $\mathscr{S}(\mathbb{R}^d)$. Since Exercise 3.14 shows that the Fourier transform $\mathcal{F}_d[\mathcal{F}_d^{-1}[\phi_1] \cdot \mathcal{F}_d^{-1}[\phi_2]]$ belongs to $\mathscr{S}(\mathbb{R}^d)$, the given function belongs to $\mathscr{S}(\mathbb{R}^d)$.

**Exercise** 3.22
We can show this relation as follows.

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathcal{F}_d[\phi_1](\lambda') \mathcal{F}_d[\phi_2](\lambda - \lambda') d\lambda'$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}_d[\phi_1](\lambda') \phi_2(x) e^{-ix\cdot(\lambda - \lambda')} dx d\lambda'$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}_d[\phi_1](\lambda') e^{ix\cdot\lambda'} d\lambda' \phi_2(x) e^{-ix\cdot\lambda} dx$$
$$= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathcal{F}_d^{-1} \circ \mathcal{F}_d[\phi_1](x) \phi_2(x) e^{-ix\cdot\lambda} dx$$
$$= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \phi_1(x) \phi_2(x) e^{-ix\cdot\lambda} dx = \mathcal{F}_d[\phi_1 \cdot \phi_2](\lambda).$$

**Exercise** 4.1

**a** The set $\mathcal{M}_{1,1}$ is the set of Hermitian matrices $X$ satisfying the condition $\operatorname{Tr} X \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$. Since $\operatorname{Tr} gXg^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \operatorname{Tr} Xg^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \operatorname{Tr} X \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the image of $\iota$ is limited to the connected component, we obtain the desired statement.

**b** $\det gXg^\dagger = \det g \det X \det g^\dagger = \det X$.

**c** Since $\det \begin{pmatrix} z & x - yi \\ x + yi & z \end{pmatrix} = z^2 - x^2 - y^2$, any element $g \in \mathrm{SU}(1,1)$ gives a $3 \times 3$ matrix $\iota(g) \in \mathrm{SO}(1,2)$ based on the coordinate $(x, y, z) \in \mathbb{R}^3$ and the above relation. The map $\iota$ is a homomorphism $\iota$ from $\mathrm{SU}(2)$ to $\mathrm{SO}(3, \mathbb{R})$, and is not an isomorphism because the matrices $I, -I \in \mathrm{SU}(1,1)$ are mapped to the unit matrix on $\mathbb{R}^3$. In particular, we find that $\iota^{-1}(e) = \{I, -I\}$. So, we obtain the relation $\mathrm{SO}(1,2)_0 \cong \mathrm{SU}(1,1)/\mathbb{Z}_2$.

**Exercise** 4.2

Using (4.13), we have

$$
\begin{aligned}
&\mathsf{K}_{+,s}\mathsf{K}_{-,s}v_{n-1} = (\mathsf{K}_{-,s}^n \mathsf{K}_{+,s} + [\mathsf{K}_{+,s}, \mathsf{K}_{-,s}^n])v_0 \\
=&(\mathsf{K}_{-,s}^n \mathsf{K}_{+,s} + \mathsf{K}_{-,s}^{n-1} ns(2\mathsf{E}_{0,s} - (n-1)I))v_0 \\
=&(\mathsf{K}_{-,s}^{n-1} ns(2\lambda - (n-1)))v_0 = ns(2\lambda - (n-1))v_{n-1}.
\end{aligned}
$$

**Exercise** 4.3

Since

$$
[\mathsf{E}_{0,s}, \mathsf{K}_{-,s}^n] = \sum_{j=0}^{n-1} \mathsf{K}_{-,s}^j [\mathsf{E}_{0,s}, \mathsf{K}_{-,s}] \mathsf{K}_{-,s}^{n-1-j} = -n\mathsf{K}_{-,s}^n,
$$

we obtain the first equation in (4.14). The second equation in (4.14) follows from the relations

$$
\begin{aligned}
&[\mathsf{K}_{+,s}, \mathsf{K}_{-,s}^n] = \sum_{j=0}^{n-1} \mathsf{K}_{-,s}^j [\mathsf{K}_{+,s}, \mathsf{K}_{-,s}] \mathsf{K}_{-,s}^{n-1-j} = \sum_{j=0}^{n-1} \mathsf{K}_{-,s}^j 2s\mathsf{E}_{0,s} \mathsf{K}_{-,s}^{n-1-j} \\
=&\sum_{j=0}^{n-1} (\mathsf{K}_{-,s}^{n-1} 2s\mathsf{E}_{0,s} + \mathsf{K}_{-,s}^j [2s\mathsf{E}_{0,s}, \mathsf{K}_{-,s}^{n-1-j}]) \\
=&\sum_{j=0}^{n-1} (\mathsf{K}_{-,s}^{n-1} 2s\mathsf{E}_{0,s} + \mathsf{K}_{-,s}^j \cdot -2s(n-1-j)\mathsf{K}_{-,s}^{n-1-j}) \\
=&\mathsf{K}_{-,s}^{n-1} ns(2\mathsf{E}_{0,s} - (n-1)I).
\end{aligned}
$$

**Exercise 4.4**
Equation (4.25) is changed to

$$m(\alpha^2-\beta^2)+2\sqrt{(\lambda_1+\frac{1}{2})^2-m^2\alpha\beta}=(\lambda-\lambda_1)(\lambda+\lambda_1+1)+\frac{1}{4}$$
$$=\begin{cases}\lambda+\frac{1}{2} & \text{when } \lambda=\lambda_1+\frac{1}{2}\\ -\lambda-\frac{1}{2} & \text{when } \lambda=\lambda_1-\frac{1}{2},\end{cases}$$

which implies 3.28.

**Exercise 4.5**
Due to the invariance of the LHS of (4.32), it is sufficient to consider the case with the state $|\lambda;\lambda\rangle$. Since $|\langle\lambda;\lambda|\lambda:\zeta\rangle|^2=(1+|\zeta|^2)^{-2\lambda}=(1+(x^2+y^2)/2)^{-2\lambda}$, using $r=\sqrt{x^2+y^2}$ and $t=r^2/2$, we have

$$\begin{aligned}&\langle\lambda;\lambda|\int_{\hat{\mathbb{C}}}|\lambda:\zeta\rangle\langle\lambda:\zeta|\frac{dxdy}{2\pi(1+|\zeta|^2)^2}|\lambda;\lambda\rangle\\
=&\int_{\hat{\mathbb{C}}}|\langle\lambda;\lambda|\lambda:\zeta\rangle|^2\frac{dxdy}{2\pi(1+|\zeta|^2)^2}\\
=&\int_{\hat{\mathbb{C}}}(1+(x^2+y^2)/2)^{-2\lambda}\frac{dxdy}{2\pi(1+(x^2+y^2)/2)^2}\\
=&\int_0^\infty\int_0^{2\pi}(1+(x^2+y^2)/2)^{-2\lambda-2}\frac{d\theta rdr}{2\pi}\\
=&\int_0^\infty(1+t)^{-2\lambda-2}dt\\
=&\frac{1}{-2\lambda-1}[(1+t)^{-2\lambda-1}]_0^\infty=\frac{1}{2\lambda+1}.\end{aligned}$$

**Exercise 4.6**
Due to the invariance of the LHS of (4.33), it is sufficient to consider the case with the state $|\lambda;\lambda\rangle$. Since $|\langle\lambda;\lambda|\lambda:\zeta\rangle|^2=(1-|\zeta|^2)^{-2\lambda}=(1-(x^2+y^2)/2)^{-2\lambda}$, using $r=\sqrt{x^2+y^2}$ and $t=r^2/2$, we have

$$\begin{aligned}&\langle\lambda;\lambda|\int_D|\lambda:\zeta\rangle\langle\lambda:\zeta|\frac{dxdy}{2\pi(1-|\zeta|^2)^2}|\lambda;\lambda\rangle\\
=&\int_D|\langle\lambda;\lambda|\lambda:\zeta\rangle|^2\frac{dxdy}{2\pi(1-|\zeta|^2)^2}\\
=&\int_D(1-(x^2+y^2)/2)^{-2\lambda}\frac{dxdy}{2\pi(1-(x^2+y^2)/2)^2}\\
=&\int_0^{\sqrt{2}}\int_0^{2\pi}(1-(x^2+y^2)/2)^{-2\lambda-2}\frac{d\theta rdr}{2\pi}\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (1-t)^{-2\lambda-2} dt \\
&= -\frac{1}{-2\lambda-1}[(1-t)^{-2\lambda-1}]_0^1 = -\frac{1}{2\lambda+1}.
\end{aligned}
$$

**Exercise** 4.7

$$
|\langle \lambda; \lambda | \lambda : \zeta \rangle|^2 \frac{dxdy}{2\pi(1+|\zeta|^2)^2} = (1+(x^2+y^2)/2)^{-2\lambda-2} \frac{dxdy}{2\pi}.
$$

**Exercise** 4.8

$$
|\langle \lambda; \lambda | \lambda : \zeta \rangle|^2 \frac{dxdy}{2\pi(1-|\zeta|^2)^2} = (1-(x^2+y^2)/2)^{-2\lambda-2} \frac{dxdy}{2\pi}.
$$

**Exercise** 4.9
Since the basis of $\mathcal{U}_{[n,0,\ldots,0]}(\mathrm{U}(r))$ are identified with $n$-combination with repetition of $r$ things, which is $\frac{(n+r-1)!}{(r-1)!n!}$.

**Exercise** 4.10
Substituting 1 into both of $\lambda_1$ and $\lambda_2$ in (4.48), we obtain the dimension $\frac{(r+1)r(r-1)}{3}$.

**Exercise** 4.11
Counting the dimension of both sides in (4.11), we have $r^2 = 1 + \dim \mathcal{U}_{[1,0,\ldots,0,1]}(\mathrm{U}(r))$.

**Exercise** 4.12
Taking the complex conjugate in (4.65), we obtain (4.68).

**Exercise** 5.1
We show only the case with $t = 1, l = 2$. Since

$$
\begin{aligned}
&(x_2\partial_3 - x_3\partial_2)(x_3\partial_1 - x_1\partial_3) \\
=&x_2\partial_1 + (x_2x_3\partial_3\partial_1 - x_2x_1\partial_3\partial_3 - x_3^2\partial_2\partial_1 + x_3x_1\partial_2\partial_3), \\
&(x_3\partial_1 - x_1\partial_3)(x_2\partial_3 - x_3\partial_2) \\
=&x_1\partial_2 + (x_2x_3\partial_3\partial_1 - x_2x_1\partial_3\partial_3 - x_3^2\partial_2\partial_1 + x_3x_1\partial_2\partial_3),
\end{aligned}
$$

we have

$$
[L_1, L_2] = -\hbar^2[(x_2\partial_3 - x_3\partial_2), (x_3\partial_1 - x_1\partial_3)] = \hbar^2(x_1\partial_2 - x_2\partial_1) = i\hbar L_3.
$$

**Exercise** 5.2

For $k > l$, we have

$$F_{k,l}^2 = Q_k^2 \partial_l^2 + Q_l^2 \partial_k^2 - 2Q_k Q_l \partial_k \partial_l - Q_k \partial_k - Q_l \partial_l.$$

Taking the sum for $k \neq l$, we have

$$\begin{aligned}\sum_{k>l} F_{k,l}^2 &= \sum_{k>l} Q_k^2 \partial_l^2 + Q_l^2 \partial_k^2 - 2Q_k Q_l \partial_k \partial_l - Q_k \partial_k - Q_l \partial_l \\ &= \sum_{k\neq l} Q_k^2 \partial_l^2 - \sum_{k\neq l} Q_k Q_l \partial_k \partial_l - \sum_{k\neq l} Q_k \partial_k \\ &= \sum_{k\neq l} Q_k^2 \partial_l^2 - \sum_{k\neq l} Q_k Q_l \partial_k \partial_l - (d-1) \sum_k Q_k \partial_k,\end{aligned}$$

which implies (5.40). Also, we have

$$\begin{aligned}\left(\sum_k Q_k \partial_k\right)^2 &= \sum_k Q_k^2 \partial_k^2 + \sum_{k\neq l} Q_k \partial_k Q_l \partial_l \\ &= \sum_k Q_k^2 \partial_k^2 + \sum_k Q_k \partial_k + \sum_{k\neq l} Q_k Q_l \partial_k \partial_l,\end{aligned}$$

which implies (5.41).

**Exercise** 5.3

For $k = 1, \ldots, d-1$, $j > k$, we have

$$\frac{\partial r}{\partial x_k} = \frac{\frac{1}{2} \cdot 2x_k}{\sqrt{\sum_{k'=1}^d x_{k'}^2}} = \frac{w_k}{r},$$

which implies (5.50). Using the formula $\frac{d}{dx} \cos^{-1} f(x) = \frac{-1}{\sqrt{1-f(x)^2}} \frac{df}{dx}$, we have

$$\begin{aligned}\frac{\partial \theta_k}{\partial x_k} &= -\sqrt{\frac{\sum_{k'=k}^d x_{k'}^2}{\sum_{k'=k+1}^d x_{k'}^2}} \frac{\sum_{k'=k+1}^d x_{k'}^2}{(\sum_{k'=k}^d x_{k'}^2)^{3/2}} = \frac{\sqrt{\sum_{k'=k+1}^d x_{k'}^2}}{\sum_{k'=k}^d x_{k'}^2} \\ &= \frac{\sqrt{\sum_{k'=k+1}^d w_{k'}^2}}{r \sum_{k'=k}^d w_{k'}^2},\end{aligned}$$

which implies (5.51). Using the same formula, we have

$$\frac{\partial\theta_k}{\partial x_j} = \sqrt{\frac{\sum_{k'=k}^d x_{k'}^2}{\sum_{k'=k+1}^d x_{k'}^2}} \frac{x_k x_j}{(\sum_{k'=k}^d x_{k'}^2)^{3/2}} = \frac{x_k x_j}{(\sum_{k'=k}^d x_{k'}^2)\sqrt{\sum_{k'=k+1}^d x_{k'}^2}}$$
$$= \frac{w_k w_j}{r(\sum_{k'=k}^d w_{k'}^2)\sqrt{\sum_{k'=k+1}^d w_{k'}^2}},$$

which implies (5.52).

Using (5.53), for $j > l$, we have

$$\begin{aligned} F_{j,l} =& w_j \Bigg( \sum_{k=1}^{l-1} \frac{w_k w_l}{\left(\sum_{k'=k}^d w_{k'}^2\right)\sqrt{\sum_{k'=k+1}^d w_{k'}^2}} \frac{\partial}{\partial\theta_k} \\ & + \frac{\sqrt{\sum_{k'=l+1}^d w_{k'}^2}}{\sum_{k'=l}^d w_{k'}^2} \frac{\partial}{\partial\theta_l} + w_l \frac{\partial}{\partial r} \Bigg) \\ & - w_l \Bigg( \sum_{k=1}^{j-1} \frac{w_k w_j}{\left(\sum_{k'=k}^d w_{k'}^2\right)\sqrt{\sum_{k'=k+1}^d w_{k'}^2}} \frac{\partial}{\partial\theta_k} \\ & + \frac{\sqrt{\sum_{k'=j+1}^d w_{k'}^2}}{\sum_{k'=j}^d w_{k'}^2} \frac{\partial}{\partial\theta_j} + w_j \frac{\partial}{\partial r} \Bigg) \\ =& \sum_{k=j}^{l-1} \frac{w_j w_l w_k}{\left(\sum_{k'=k}^d w_{k'}^2\right)\sqrt{\sum_{k'=k+1}^d w_{k'}^2}} \frac{\partial}{\partial\theta_k} \\ & + w_j \frac{\sqrt{\sum_{k'=l+1}^d w_{k'}^2}}{\sum_{k'=l}^d w_{k'}^2} \frac{\partial}{\partial\theta_l} - w_l \frac{\sqrt{\sum_{k'=j+1}^d w_{k'}^2}}{\sum_{k'=j}^d w_{k'}^2} \frac{\partial}{\partial\theta_j}, \end{aligned}$$

which implies (5.55).

Using (5.53) and (5.54), we have

$$\begin{aligned}
F_{j,d} =& w_j \Bigg( \sum_{k=1}^{d-2} \frac{w_k w_d}{\left(\sum_{k'=k}^{d} w_{k'}^2\right)\sqrt{\sum_{k'=k+1}^{d} w_{k'}^2}} \frac{\partial}{\partial \theta_k} \\
&+ \frac{w_{d-1}}{w_{d-1}^2 + w_d^2} \frac{\partial}{\partial \theta_{d-1}} + w_d \frac{\partial}{\partial r} \Bigg) \\
&- w_d \Bigg( \sum_{k=1}^{j-1} \frac{w_k w_j}{\left(\sum_{k'=k}^{d} w_{k'}^2\right)\sqrt{\sum_{k'=k+1}^{d} w_{k'}^2}} \frac{\partial}{\partial \theta_k} \\
&+ \frac{\sqrt{\sum_{k'=j+1}^{d} w_{k'}^2}}{\sum_{k'=j}^{d} w_{k'}^2} \frac{\partial}{\partial \theta_j} + w_j \frac{\partial}{\partial r} \Bigg) \\
=& \sum_{k=j}^{d-2} \frac{w_k w_j w_d}{\left(\sum_{k'=k}^{d} w_{k'}^2\right)\sqrt{\sum_{k'=k+1}^{d} w_{k'}^2}} \frac{\partial}{\partial \theta_k} \\
&+ w_j \frac{w_{d-1}}{w_{d-1}^2 + w_d^2} \frac{\partial}{\partial \theta_{d-1}} - w_d \frac{\sqrt{\sum_{k'=j+1}^{d} w_{k'}^2}}{\sum_{k'=j}^{d} w_{k'}^2} \frac{\partial}{\partial \theta_j},
\end{aligned}$$

which implies (5.56).

**Exercise 5.4**
We have

$$\begin{aligned}
&\binom{n+d-1}{n} - \binom{n+d-3}{n-2} \\
&= \frac{(n+d-1)\ldots(n+1)}{(d-1)!} - \frac{(n+d-3)\ldots(n-1)}{(d-1)!} \\
&= \frac{[(n+d-1)(n+d-2) - n(n-1)](n+d-3)\ldots(n+1)}{(d-1)!} \\
&= \frac{(d-1)(2n+d-2)(n+d-3)\ldots(n+1)}{(d-1)!} \\
&= \frac{(2n+d-2)((n+d-3)\ldots(n+1))}{(d-2)!},
\end{aligned}$$

which implies (5.60).

**Exercise 5.5**
Since

$$
\begin{aligned}
&(x_k\partial_l - x_l\partial_k)(x_{k'}\partial_{l'} - x_{l'}\partial_{k'})\\
=&x_k\partial_{l'}\delta_{l,k'} - x_l\partial_{l'}\delta_{k,k'} - x_k\partial_{k'}\delta_{l,l'} + x_l\partial_{k'}\delta_{k,l'}\\
&+ (x_kx_{k'}\partial_l\partial_{l'} - x_kx_{l'}\partial_l\partial_{k'} - x_lx_{k'}\partial_k\partial_{l'} + x_lx_{l'}\partial_k\partial_{k'}),\\
&(x_{k'}\partial_{l'} - x_{l'}\partial_{k'})(x_k\partial_l - x_l\partial_k)\\
=&x_{k'}\partial_l\delta_{l',k} - x_{l'}\partial_l\delta_{k',k} - x_{k'}\partial_k\delta_{l',l} + x_{l'}\partial_k\delta_{k',l}\\
&+ (x_kx_{k'}\partial_l\partial_{l'} - x_kx_{l'}\partial_l\partial_{k'} - x_lx_{k'}\partial_k\partial_{l'} + x_lx_{l'}\partial_k\partial_{k'}),
\end{aligned}
$$

we have

$$[L_{k,l}, L_{k',l'}] = L_{k,l'}\delta_{l,k'} - L_{l,l'}\delta_{k,k'} - L_{k,k'}\delta_{l,l'} + L_{l,k'}\delta_{k,l'},$$

which implies (5.62).

**Exercise 5.6**
Since

$$
\begin{aligned}
\frac{\partial^2(x_2+ix_1)^n}{\partial x_1^2} &= -n(n-1)(x_2+ix_1)^{n-2}\\
\frac{\partial^2(x_2+ix_1)^n}{\partial x_2^2} &= n(n-1)(x_2+ix_1)^{n-2}\\
\frac{\partial^2(x_2+ix_1)^n}{\partial x_j^2} &= 0,
\end{aligned}
$$

for $j \geq 3$, we have $\Delta_{\mathbb{R}^d}(x_2+ix_1)^n = 0$.

**Exercise 5.7**

$$
\begin{aligned}
&\Delta_{\mathbb{R}^d}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k\binom{n}{2k}x_1^{n-2k}x_2^{2k}\\
=&\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor-1}(-1)^k(n-2k)(n-2k-1)\binom{n}{2k}x_1^{n-2k-2}x_2^{2k}\\
&+\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}(-1)^k 2k(2k-1)\binom{n}{2k}x_1^{n-2k}x_2^{2k-2}\\
=&\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}x_1^{n-2k}x_2^{2k-2}\\
&\cdot\left((-1)^{k-1}(n-2k+2)(n-2k+1)\binom{n}{2k-2}+(-1)^k 2k(2k-1)\binom{n}{2k}\right)\\
=&0.
\end{aligned}
$$

**Exercise** 5.8
Let $x = \sin\theta_1 \cos\theta_2$. We have

$$\int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \int_0^{\pi} d\theta_3 \frac{(\sin\theta_1 \cos\theta_2 + i\cos\theta_1)^n}{2(1-\cos\theta_1)} \sin^2\theta_1 \sin\theta_2$$
$$= \pi \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \frac{(\sin\theta_1 \cos\theta_2 + i\cos\theta_1)^n}{2(1-\cos\theta_1)} \sin^2\theta_1 \sin\theta_2$$
$$= -\pi \int_0^{2\pi} d\theta_1 \int_{\sin\theta_1}^{-\sin\theta_1} dx \frac{(x + i\cos\theta_1)^n}{2(1-\cos\theta_1)} \sin\theta_1$$
$$= \pi \int_0^{2\pi} d\theta_1 \int_{-\sin\theta_1}^{\sin\theta_1} dx \frac{(x + i\cos\theta_1)^n}{2(1-\cos\theta_1)} \sin\theta_1$$
$$= \pi \int_0^{2\pi} d\theta_1 \Big( \frac{1}{n+1} \frac{(\sin\theta_1 + i\cos\theta_1)^{n+1}}{2(1-\cos\theta_1)} \sin\theta_1 - \frac{1}{n+1} \frac{(-\sin\theta_1 + i\cos\theta_1)^{n+1}}{2(1-\cos\theta_1)} \sin\theta_1 \Big)$$
$$= \pi \int_0^{2\pi} d\theta_1 \Big( \frac{1}{n+1} \frac{(\sin(n+1)\theta_1 + i\cos(n+1)\theta_1)}{2(1-\cos\theta_1)} \sin\theta_1 - \frac{1}{n+1} \frac{(-\sin(n+1)\theta_1 + i\cos(n+1)\theta_1)}{2(1-\cos\theta_1)} \sin\theta_1 \Big)$$
$$= \frac{2\pi i^n}{n+1} \int_0^{2\pi} d\theta_1 \frac{\sin\theta_1}{2(1-\cos\theta_1)} \sin(n\theta_1).$$

**Exercise** 5.9
**c** We have

$$\int_0^{2\pi} \frac{\sin\theta \sin 2\theta}{1-\cos\theta} d\theta = \int_0^{2\pi} \frac{\sin\theta(1+\cos\theta)\sin 2\theta}{1-\cos^2\theta} d\theta$$
$$= \int_0^{2\pi} \frac{(1+\cos\theta)\sin 2\theta}{\sin\theta} d\theta = \int_0^{2\pi} 2(1+\cos\theta)\cos\theta d\theta$$
$$= \int_0^{2\pi} 2\cos^2\theta d\theta = \int_0^{2\pi} 1 + \cos 2\theta d\theta = 2\pi.$$

**d** Since $n \geq 2$, we have

$$\int_0^{2\pi} \frac{\sin\theta \sin(n+1)\theta}{1-\cos\theta} d\theta = \int_0^{2\pi} \frac{\sin\theta(1+\cos\theta)\sin(n+1)\theta}{1-\cos^2\theta} d\theta$$
$$= \int_0^{2\pi} \frac{(1+\cos\theta)\sin(n+1)\theta}{\sin\theta} d\theta$$
$$= \int_0^{2\pi} \frac{(1+\cos\theta)(\sin n\theta \cos\theta + \cos n\theta \sin\theta)}{\sin\theta} d\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi} \frac{(\cos\theta + \cos^2\theta)\sin n\theta}{\sin\theta} d\theta + \int_0^{2\pi} (1+\cos\theta)\cos n\theta d\theta \\
&= \int_0^{2\pi} \frac{(\cos\theta + 1 - \sin^2\theta)\sin n\theta}{\sin\theta} d\theta + 0 \\
&= \int_0^{2\pi} \frac{(\cos\theta + 1)\sin n\theta}{\sin\theta} d\theta + \int_0^{2\pi} -\sin\theta \sin n\theta d\theta \\
&= \int_0^{2\pi} \frac{(-\cos^2\theta + 1)\sin n\theta}{\sin\theta(1-\cos\theta)} d\theta + \int_0^{2\pi} -\sin\theta \sin n\theta d\theta \\
&= \int_0^{2\pi} \frac{\sin^2\theta \sin n\theta}{\sin\theta(1-\cos\theta)} d\theta + 0 = \int_0^{2\pi} \frac{\sin\theta \sin n\theta}{1-\cos\theta} d\theta.
\end{aligned}
$$

**Exercise** 5.10

$$
\begin{aligned}
T(f)(w) &= \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \int_0^{\pi} d\theta_3 \frac{\sin^2\theta_1 \sin\theta_2}{2(1-\cos\theta_1)} \\
&= \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \int_0^{\pi} d\theta_3 \frac{(1+\cos\theta_1)\sin\theta_2}{2} \\
&= 2\pi \int_0^{\pi} d\theta_2 \int_0^{\pi} d\theta_3 \frac{\sin\theta_2}{2} = 2\pi^2 \int_0^{\pi} d\theta_2 \frac{\sin\theta_2}{2} = 2\pi^2.
\end{aligned}
$$

**Exercise** 5.11
Using the commutation relation (5.2), we have

$$
\Big[\sum_{j=1}^{3} P_j^2, Q_k P_k\Big] = -2i\hbar P_k^2, \quad \Big[\sum_{j=1}^{3} P_j^2, P_k Q_k\Big] = -2i\hbar P_k^2.
$$

So, we have

$$
\Big[\sum_{j=1}^{3} P_j^2, A\Big] = \frac{-i}{\mu} \sum_{k=1}^{3} P_k^2.
$$

Also,

$$
\Big[\frac{1}{r}, Q_k P_k\Big] = -i\hbar \frac{x_k^2}{r^3}, \quad \Big[\frac{1}{r}, P_k Q_k\Big] = -i\hbar \frac{x_k^2}{r^3}.
$$

Since $\sum_{k=1}^{3} \frac{x_k^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$, we have

$$
\Big[\sum_{j=1}^{3} P_j^2, A\Big] = i \frac{Ze^2}{4\pi\epsilon_0 r}.
$$

Thus, we have the desired commutation relation

$$i[H, A] = \frac{1}{2\mu}\sum_{j=1}^{3} P_j^2 + H.$$

**Exercise 5.12**
We show (5.100) only for $k = 3$.

$$\sum_{j=1}^{4}\left(\frac{\partial w_j}{\partial p_k}\right)^2$$
$$= \frac{\left(\sum_{j=1}^{2}(4p_0 p_j p_3)^2\right) + (2p_0)^2 \cdot (p^2 - 2p_3^2 + p_0^2)^2 + (4p_0^2 p_3)^2}{(p^2 + p_0^2)^4}$$
$$= \frac{4p_0^2(4p_0^2(p^2 - p_3^2) + (p^2 - 2p_3^2 + p_0^2)^2 + 4p_0^2 p_3^2)}{(p^2 + p_0^2)^4}$$
$$= \frac{4p_0^2(p^2 + p_0^2)^2}{(p^2 + p_0^2)^4} = (\frac{2p_0}{p^2 + p_0^2})^2.$$

Next, we show (5.88) only for $k = 2, l = 3$.

$$\sum_{j=1}^{4}\frac{\partial w_j}{\partial p_2}\frac{\partial w_j}{\partial p_3}$$
$$= \frac{16p_0^2 p_1^2 p_2 p_3 - 8p_0^2(p^2 - 2p_2^2 + p_0^2)p_2 p_3 - 8p_0^2(p^2 - 2p_3^2 + p_0^2)p_2 p_3 16p_0^4 p_2 p_3}{(p^2 + p_0^2)^4}$$
$$= \frac{8p_0^2(2(p_0^2 + p_1^2)p_2 p_3 - p_2 p_3(2p^2 - 2p_2^2 - 2p_3^2 + 2p_0^2)}{(p^2 + p_0^2)^4}$$
$$= \frac{8p_0^2 p_2 p_3(2(p_0^2 + p_1^2) - (2p_1^2 + 2p_0^2)}{(p^2 + p_0^2)^4} = 0.$$

**Exercise 5.13**
(5.107) can be shown as

$$\langle \phi_1 | \phi_4 * \mathcal{F}_3[\phi_2] \rangle = \int_{\mathbb{R}^3} \phi_1(x)^* \phi_4 * \mathcal{F}_3[\phi_2](x) d^3x$$
$$= \int_{\mathbb{R}^3} \phi_1(x)^* \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \phi_4(y) \mathcal{F}_3[\phi_2](x-y) d^3y d^3x$$
$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \phi_1(x)^* \mathcal{F}_3[\phi_2](x-y) \phi_4(y) d^3y d^3x$$
$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \phi_1(x)^* \mathcal{F}_3^{-1}[\phi_2](y-x) \phi_4(y) d^3y d^3x$$
$$\stackrel{(a)}{=} \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{\frac{3}{2}}} \left( \int_{\mathbb{R}^3} \phi_1(x) \mathcal{F}_3^{-1}[\phi_2](y-x)^* d^3x \right)^* \phi_4(y) d^3y$$
$$= \int_{\mathbb{R}^3} (\phi_1 * \mathcal{F}_3^{-1}[\phi_2]^*)(y)^* \phi_4(y) d^3y = \langle \phi_1 * \mathcal{F}_3^{-1}[\phi_2]^* | \phi_4 \rangle,$$

where $(a)$ follows from Exercise 3.13.

**Exercise** 5.14

**a** We choose the spherical coordinate $(r, \theta_1, \theta_2)$ given in (5.7). We assume that $p = (0, 0, |p|)$. Then, choosing $t := \cos\theta_1$, we have

$$\int_{\mathbb{R}^3} e^{-\delta|x|} \frac{e^{-i(xp_1+yp_2+zp_3)}}{\sqrt{x^2+y^2+z^2}} dxdydz$$
$$= \int_0^\infty dr \int_0^{2\pi} d\theta_1 \int_0^\pi d\theta_2 e^{-\delta r} \frac{e^{-i|p|r\cos\theta_1}}{r} r^2 \sin\theta_1$$
$$= 2\pi \int_0^\infty dr \int_0^\pi d\theta_1 e^{-\delta r} \frac{e^{-i|p|r\cos\theta_1}}{r} r^2 \sin\theta_1$$
$$= 2\pi \int_0^\infty dr \int_{-1}^1 dt r e^{-\delta r} e^{-i|p|rt} = e^{-\delta r} \frac{4\pi}{|p|} \int_0^\infty \sin(|p|r).$$

**b**

$$\lim_{\delta\to 0} \frac{4\pi}{|p|} \int_0^\infty e^{-\delta r} \sin(|p|r) dr = \lim_{\delta\to 0} \frac{4\pi}{i|p|} \int_0^\infty e^{-\delta r} (e^{i|p|r} - e^{-i|p|r}) dr$$
$$= \lim_{\delta\to 0} \frac{2\pi}{i|p|} \left(\frac{0-1}{i|p|-\delta} + \frac{0-1}{i|p|+\delta}\right) = \frac{4\pi}{|p|^2}.$$

**c** The combination of **a** and **b** implies (5.109).

**Exercise** 5.15

Now, we consider the case when the spin is $\uparrow$. Since the flavor system has the permutation symmetry, it is enough to show the orthogonality the vectors in the spin system when the vector in the flavor system is $|udd\rangle$. We have $\langle udd|n\rangle = |\uparrow'\rangle$, $\langle udd|\Delta^0\rangle = |\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle$, and $\langle \uparrow' |(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) = 0$.

**Exercise** 5.16
Now, we consider the case when the spin is $\uparrow$. It is enough to show the orthogonality the vectors in the spin system when the vector in the flavor system is $|uds\rangle$. Since $\langle uds|\Lambda^0\rangle = |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle$, $\langle uds|\Sigma^0\rangle = |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle)$, and $\langle uds|\Sigma^{*0}\rangle = |\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle$, they are orthogonal to each other.

**Exercise** 5.17
It is enough to show that $(a^*, b^*)J_\rho \begin{pmatrix} a \\ b \end{pmatrix} \geq 0$ for two complex numbers $a$ and $b$.

$$(a, b)J_\rho \begin{pmatrix} a \\ b \end{pmatrix} = \mathrm{Tr}(a\mathsf{Q}_0 + a\mathsf{P}_0)^\dagger (a\mathsf{Q}_0 + a\mathsf{P}_0)\rho \geq 0.$$

**Exercise** 6.1
Let $\theta$ is the angle between $\alpha$ and $\beta$. Then, $\|\alpha\|\|\beta\| \cos\theta = (\alpha, \beta)$. So, $\langle\beta, \alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Thus, $\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle = 4\cos^2\theta$, which should be an integer. Hence, $\cos^2\theta$ is limited to 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, i.e., $\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle$ is limited to 0, 1, 2, 3, 4. When $\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle \neq 4$, since $\|\alpha\| \leq \|\beta\|$, the pair $(\langle\alpha, \beta\rangle, \langle\beta, \alpha\rangle)$ is limited to $(0, 0)$, $(\pm 1, \pm 1)$, $(\pm 1, \pm 2)$, and $(\pm 1, \pm 3)$. When $\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle = 4$, $\cos^2\theta = 1$, i.e., $\alpha = \pm\beta$. So, the pair $(\langle\alpha, \beta\rangle, \langle\beta, \alpha\rangle)$ is $(\pm 2, \pm 2)$.

**Exercise** 6.2
**a** Since $(\alpha, \beta) > 0$, $\langle\alpha, \beta\rangle$ and $\langle\beta, \alpha\rangle$ are positive. So, Exercise 6.1 shows that one of two is one, at least. When $\langle\alpha, \beta\rangle = 1$, $\alpha - \beta = \alpha - \langle\alpha, \beta\rangle\beta = \mathcal{W}_\beta(\alpha) \in \Phi$. When $\langle\beta\alpha = 1$, $\beta - \alpha \in \Phi$. So, $\alpha - \beta = -(\beta - \alpha) \in \Phi$.
**b** Substituting $\beta$ into $-\beta$ in **a**, we obtain the desired statement.

**Exercise** 6.3
Consider liner transform that maps $\gamma_j$ to $e_j$, where $e_j$ is the vector that has non-zero component 1 only $j$-th entry. Since $\cap_{j=1}^r V^+(e_j)$ is not empty, $\cap_{j=1}^r V^+(\gamma_j)$ is also not empty.

**Exercise** 6.4
**a** Assume that there exists an element $\alpha \in \Phi^+(\gamma)$ such that it is not written as non-negative $\mathbb{Z}$-linear combination of $\Delta(\gamma)$. We choose $\alpha' := \mathrm{argmin}_\alpha(\alpha, \gamma)$, where the minimum is taken among elements $\alpha \in \Phi^+(\gamma)$ satisfying the above condition. Since $\alpha'$ is decomposable, there exists $\beta_1, \beta_2 \in \Phi^+(\gamma)$ such that $\alpha = \beta_1 + \beta_2$. We find $(\alpha, \gamma) = (\beta_1, \gamma) + (\beta_2, \gamma)$ and $(\beta_1, \gamma), (\beta_2, \gamma) > 0$. Due to the above minimum choice, $\beta_1$ and $\beta_2$ are written as non-negative $\mathbb{Z}$-linear combinations of $\Delta(\gamma)$. So, $\alpha$ is also written as a non-negative $\mathbb{Z}$-linear combination of $\Delta(\gamma)$, which is the contradiction of the assumption.
**b** Assume that $(\alpha, \beta) > 0$ and $\alpha \neq \beta$. So, $\beta \neq -\alpha$. Exercise 6.2 guarantees that $\alpha - \beta \in \Phi$. Thus, $\alpha - \beta$ or $\beta - \alpha$ belongs to $\Phi^+(\gamma)$. When $\alpha - \beta \in \Phi^+(\gamma)$,

$\alpha = (\alpha-\beta)+\beta$, which contradicts the assumption $\alpha \in \Delta(\gamma)$. When $\beta-\alpha \in \Phi^+(\gamma)$, $\beta = (\beta-\alpha)+\alpha$, which contradicts the assumption $\beta \in \Delta(\gamma)$.
**c** Assume that there exist real numbers $r_\alpha$ such that $\sum_{\alpha\in\Delta(\gamma)} r_\alpha\alpha = 0$. Define subsets $\Delta(\gamma)_1 := \{\alpha \in \Delta(\gamma)|r_\alpha > 0\}$ and $\Delta(\gamma)_2 := \{\alpha \in \Delta(\gamma)|r_\alpha < 0\}$. So, we have $a := \sum_{\alpha\in\Delta(\gamma)_1} r_\alpha\alpha = \sum_{\alpha'\in\Delta(\gamma)_2} -r_{\alpha'}\alpha'$. Thus, **b** guarantees that $(a, a) = \sum_{\alpha\in\Delta(\gamma)_1,\alpha'\in\Delta(\gamma)_2} r_\alpha(-r_{\alpha'})(\alpha, \alpha') \le 0$. So, $a = 0$. Thus, since $\alpha \in \Phi^+(\gamma)$, $0 = (a, \gamma) = \sum_{\alpha\in\Delta(\gamma)_1} r_\alpha(\alpha, \gamma) > 0$, which contradicts the assumption.
**d** **a** and **c** guarantee the conditions **(B2)** and **(B1)**, respectively.
**e** Due to Exercise 6.3, we can choose $\gamma \in \cup_{\alpha\in\Delta} V^+(\alpha)$. Due to condition **(B2)** and **a**, we have $\Phi^+ \subset \Phi^+(\gamma')$ and $\Phi^- \subset \Phi^+(-\gamma') = -\Phi^+(\gamma')$. So, $\Phi^+ = \Phi^+(\gamma')$. Thus, $\Delta \subset \Delta(\gamma')$. Since the number of elements of both sets are dimension, we have $\Delta = \Delta(\gamma')$.

**Exercise 6.5**
Firstly, we assume that $\alpha-\beta \in \Phi$. However, it contradicts **(B2)**. So, $\alpha-\beta$ is not a root. Next, we assume that $(\alpha, \beta) > 0$, which implies$\alpha \neq -\alpha$. Since $\alpha \neq \beta$, Exercise 6.2 guarantees that $\alpha - \beta \in \Phi$, which contradicts **(B2)**. So, we have $(\alpha, \beta) \le 0$.

**Exercise 6.6**
**a** Assume that $\beta \in \Phi^-$. $\beta = -\sum_{\alpha\in\Delta} r_\alpha\alpha$ with non-negative integer $r_\alpha$. Choose $\alpha' \in \Delta$ such that $r_{\alpha'} \neq 0$. So, $\beta + \alpha' \in \Phi^-$, which contradicts the assumption.
**b** Assume that there exists an element $\alpha \in \Delta$ such that $(\beta, \alpha) < 0$, which implies $\beta \neq \alpha$. Since $\beta \in \Phi^+$, $\beta \neq -\alpha$. Exercise 6.2 guarantees that $\alpha + \beta \in \Phi$, which contradicts the assumption. So, we have $(\beta, \alpha) \ge 0$ for $\alpha \in \Delta$. Since $\beta$ is not 0 and $\Delta$ spans the vector space $V$, there exists an element $\alpha \in \Delta$ such that $(\alpha, \beta) > 0$.
**c** Due to **a** and **(B2)**, we $\beta$ is written as $\sum_{\alpha\in\Delta} r_\alpha\alpha$ with non-negative integer $r_\alpha$. Define two subsets $\Delta_1 := \{\alpha \in \Delta|r_\alpha > 0\}$ and $\Delta_2 := \{\alpha \in \Delta|r_\alpha = 0\}$. Assume that $\Delta_2$ is not empty. Exercise 6.5 guarantees that $(\beta, \alpha') = \sum_{\alpha\in\Delta_1} r_\alpha(\alpha, \alpha') \le 0$ for an element $\alpha' \in \Delta_2$. Since $\Phi$ is connected, there exist $\alpha \in \Delta_1$ and $\alpha \in \Delta_2$ such that $(\alpha, \alpha') \neq 0$. Exercise 6.5 guarantees that $(\alpha, \alpha') < 0$. Thus, $(\beta, \alpha') < 0$, which **b**.
**d** Due to **c**, we have $\beta = \sum_{\alpha\in\Delta} r_\alpha\alpha$ and $\beta = \sum_{\alpha'\in\Delta} r'_{\alpha'}\alpha'$ with $r_\alpha, r'_{\alpha'} > 0$. $(\beta, \beta') = \sum_{\alpha,\alpha'\in\Delta} r_\alpha r'_{\alpha'}(\alpha, \alpha')$, which is strictly positive due to **b**. When $\beta \neq \beta'$, Exercise 6.2 guarantees that $\beta - \beta' \in \Phi$, which implies $\beta \ge \beta'$ or $\beta' \ge \beta$, which contradicts the assumption.

**Exercise 6.7**
$X \in \mathfrak{h}$ satisfies

$$(X, [F^x_{\boldsymbol{\alpha}}, F^y_{\boldsymbol{\alpha}}])_{\mathfrak{g}} = ([X, F^x_{\boldsymbol{\alpha}}], F^y_{\boldsymbol{\alpha}})_{\mathfrak{g}} = (-\boldsymbol{\alpha}(X)F^y_{\boldsymbol{\alpha}}, F^y_{\boldsymbol{\alpha}})_{\mathfrak{g}} = -(X, Z_{\boldsymbol{\alpha}})(F^y_{\boldsymbol{\alpha}}, F^y_{\boldsymbol{\alpha}})_{\mathfrak{g}}.$$

So, as Killing form is non-degenerate, we have $[F^x_{\boldsymbol{\alpha}}, F^y_{\boldsymbol{\alpha}}] = -(F^y_{\boldsymbol{\alpha}}, F^y_{\boldsymbol{\alpha}})_{\mathfrak{g}} Z_{\boldsymbol{\alpha}}$. Similarly, we have $[F^x_{\boldsymbol{\alpha}}, F^y_{\boldsymbol{\alpha}}] = -(F^x_{\boldsymbol{\alpha}}, F^x_{\boldsymbol{\alpha}})_{\mathfrak{g}} Z_{\boldsymbol{\alpha}}$.

**Exercise 6.8**
The assumption yields that $\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j \rangle = 0$. Then, we have $\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j \rangle = \frac{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j, \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j)} = \frac{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i) + (\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j)} < 1$. Since the above value is not a integer, $\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j$ does not belong to $\Phi$.

**Exercise 6.9**
The set $\Phi^+$ of positive roots is given as $\{\boldsymbol{\alpha}_{j,l}\}_{1 \le j < l \le r}$, where $\boldsymbol{\alpha}_{j,l} = \boldsymbol{\alpha}_j + \cdots + \boldsymbol{\alpha}_{l-1}$. Hence, $\boldsymbol{\delta} = \frac{1}{2} \sum_{1 \le j < l \le r} (\boldsymbol{\alpha}_j + \cdots + \boldsymbol{\alpha}_{l-1}) = \frac{1}{2} \sum_{j=1}^{r-1} j(r-j)\boldsymbol{\alpha}_j$.

**Exercise 6.10**
It is sufficient to show that $\frac{l-j+\sum_{t=j}^{l-1} \lambda_t}{l-j} = \frac{\langle \boldsymbol{\lambda} + \boldsymbol{\delta}, \boldsymbol{\alpha}_{j,l} \rangle}{\langle \boldsymbol{\delta}, \boldsymbol{\alpha}_{j,l} \rangle}$, for each root $\boldsymbol{\alpha}_{j,l} \in \Phi^+$. Exercise 6.9 guarantees that $\langle \delta, \boldsymbol{\alpha}_j \rangle = \frac{1}{2}(2j(r-j) - (j+1)(r-j-1) - (j-1)(r-j+1)) = 1$ for $j = 2, \ldots, r-2$, and $\langle \delta, \boldsymbol{\alpha}_1 \rangle = \langle \delta, \boldsymbol{\alpha}_r \rangle = \frac{1}{2}(2(r-1) - 2(r-2)) = 1$ for $j = 1, r-1$. So, $\langle \delta, \boldsymbol{\alpha}_{j,l} \rangle = j - l$. Also, $\langle \boldsymbol{\lambda}, \boldsymbol{\alpha}_{j,l} \rangle = \sum_{t=j}^{l-1} \lambda_t$.

**Exercise 7.1**
The desired statement follows from (7.9) and (3.13).

**Exercise 7.2**
When $\zeta' = \frac{q'+ip'}{\sqrt{2}}$, the distribution of the outcome is $|\langle \zeta | \zeta' \rangle|^2 \frac{dqdp}{2\pi} = e^{-|\zeta - \zeta'|^2} \frac{dqdp}{2\pi} = e^{-\frac{(p-p')^2 + (q-q')^2}{2}} \frac{dqdp}{2\pi}$.

**Exercise 7.3**

$$
\begin{aligned}
& W_{|\alpha\rangle\langle\alpha|}(\zeta) = \langle \alpha | \mathsf{W}(\zeta) | \alpha \rangle = \langle 0 | \mathsf{W}(\alpha)^\dagger \mathsf{W}(\zeta) \mathsf{W}(\alpha) | 0 \rangle \\
= & \langle 0 | \mathsf{W}(-\alpha) \mathsf{W}(\zeta) \mathsf{W}(\alpha) | 0 \rangle = e^{\frac{1}{2}(\zeta\overline{\alpha} - \alpha\overline{\zeta})} \langle 0 | \mathsf{W}(-\alpha) \mathsf{W}(\zeta + \alpha) | 0 \rangle \\
= & e^{\frac{1}{2}(\zeta\overline{\alpha} - \alpha\overline{\zeta})} e^{\frac{1}{2}(-\alpha\overline{\zeta+\alpha} + (\zeta+\alpha)\overline{\alpha})} \langle 0 | \mathsf{W}(\zeta) | 0 \rangle = e^{\zeta\overline{\alpha} - \alpha\overline{\zeta}} e^{-\frac{|\zeta|^2}{2}}.
\end{aligned}
$$

**Exercise 7.4**
The distribution of the outcome is $|\langle \zeta | n \rangle|^2 \frac{dqdp}{2\pi} = e^{-|\zeta|^2} \frac{|\zeta|^{2n}}{n!} \frac{dqdp}{2\pi} = e^{-\frac{p^2+q^2}{2}} \frac{(\frac{p^2+q^2}{2})^{2n}}{n!} \frac{dqdp}{2\pi}$.

**Exercise 7.5**
The distribution of the outcome $n$ is $|\langle \zeta | n \rangle|^2 = e^{-|\zeta'|^2} \frac{|\zeta'|^{2n}}{n!}$.

**Exercise 7.6**

$$
[Q \otimes I + I \otimes Q, P \otimes I - I \otimes P] = [Q \otimes I, P \otimes I] + [I \otimes Q, -I \otimes P] = iI - iI = 0.
$$

**Exercise** 7.7
Assume that $\alpha = \frac{x+iy}{\sqrt{2}}$ and $\beta = \frac{x'+iy'}{\sqrt{2}} = \zeta\sqrt{1-\eta} + \alpha\sqrt{\eta}$. Then, we have

$$\begin{aligned} &\mathrm{Tr}_2\, \mathsf{U}(b_\eta)|\zeta\rangle\langle\zeta| \otimes \rho_{N/\eta} \mathsf{U}(b_\eta)^\dagger \\ =& \int_{\mathbb{C}} |\zeta\sqrt{1-\eta} + \alpha\sqrt{\eta}\rangle\langle\zeta\sqrt{1-\eta} + \alpha\sqrt{\eta}| e^{-\frac{|\alpha|^2\eta}{N}} \eta \frac{dxdy}{2\pi N} \\ =& \int_{\mathbb{C}} |\beta\rangle\langle\beta| e^{-\frac{|\beta-\zeta\sqrt{1-\eta}|^2}{N}} \frac{dx'dy'}{2\pi N} \to \Lambda_N(|\zeta\rangle\langle\zeta|). \end{aligned}$$

**Exercise** 7.8
It is enough to show that $(a_1^*, \ldots, a_4^*) J_\rho \begin{pmatrix} a_1 \\ \vdots \\ a_4 \end{pmatrix} \geq 0$ for four complex numbers $a_1, \ldots, a_4$.

$$(a_1^*, \ldots, a_4^*) J_\rho \begin{pmatrix} a_1 \\ \vdots \\ a_4 \end{pmatrix} = \mathrm{Tr}(\sum_{i=1}^4 a_i \mathsf{A}_i)^\dagger (\sum_{i=1}^4 a_i \mathsf{A}_i)\rho \geq 0.$$

**Exercise** 7.9
The relation (7.66) follows as

$$\begin{aligned} &\left(-i\frac{\partial}{\partial p} - \frac{q}{2}\right) e^{-\frac{|\zeta|^2}{2} + i(ap-qb)} = \left(-i\frac{\partial}{\partial p} - \frac{q}{2}\right) e^{-\frac{p^2+q^2}{4} + i(ap-qb)} \\ =& \left(a + \frac{-q+ip}{2}\right) e^{-\frac{p^2+q^2}{4} + i(ap-qb)} = \frac{1}{\sqrt{2}}(\alpha + \overline{\alpha} - \overline{\zeta}) e^{-\frac{p^2+q^2}{4} + i(ap-qb)} \\ =& \mathcal{F}_{\mathsf{W}}^{-1}[\frac{\mathsf{a} + \mathsf{a}^\dagger}{\sqrt{2}} |\alpha\rangle\langle\alpha|](\zeta). \end{aligned}$$

Similarly, the relation (7.67) follows as

$$\begin{aligned} &\left(i\frac{\partial}{\partial q} - \frac{p}{2}\right) e^{-\frac{|\zeta|^2}{2} + i(ap-qb)} = \left(i\frac{\partial}{\partial q} - \frac{p}{2}\right) e^{-\frac{p^2+q^2}{4} + i(ap-qb)} \\ =& \left(b - i\frac{q+ip}{2}\right) e^{-\frac{p^2+q^2}{4} + i(ap-qb)} = -\frac{i}{\sqrt{2}}(\alpha - \overline{\alpha} + \overline{\zeta}) e^{-\frac{p^2+q^2}{4} + i(ap-qb)} \\ =& \mathcal{F}_{\mathsf{W}}^{-1}[-i\frac{\mathsf{a} - \mathsf{a}^\dagger}{\sqrt{2}} |\alpha\rangle\langle\alpha|](\zeta). \end{aligned}$$

**Exercise** 7.10
Choose $\theta$ such that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ \gamma & \alpha \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \beta' & 0 \\ 0 & \alpha' \end{pmatrix}$ with suitable two real numbers $\alpha'$ and $\beta'$. So, we have $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \beta & -\gamma \\ -\gamma & \alpha \end{pmatrix}$ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \beta' & 0 \\ 0 & \alpha' \end{pmatrix}$.

Since the matrix $g = \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix}$ satisfies

$$g^T \begin{pmatrix} & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \end{pmatrix} g = \begin{pmatrix} & 0 & \frac{i}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \end{pmatrix},$$

the matrix $g$ belongs to $\mathrm{Sp}(4,\mathbb{R})$. Thus, we have $g^T J_{\alpha,\beta,\gamma} g = J_{\alpha',\beta',0}$.

**Exercise** 7.11
Using (7.65), we have

$$\begin{aligned} Q_X(\zeta) &= \langle\zeta|\Big(\int_{\mathbb{C}} W_X(\alpha)\mathsf{W}(-\alpha)\frac{dadb}{2\pi}\Big)|\zeta\rangle \\ &= \int_{\mathbb{C}} W_X(\alpha)\langle\zeta|\mathsf{W}(-\alpha)|\zeta\rangle\frac{dadb}{2\pi} = \int_{\mathbb{C}} W_X(\alpha)e^{-\zeta\overline{\alpha}+\alpha\overline{\zeta}}e^{-\frac{|\zeta|^2}{2}}\frac{dadb}{2\pi}. \end{aligned}$$

**Exercise** 8.1

$$\mathsf{W}_{\mathbb{Z}}(0,0) = I, \quad \mathsf{W}_{\mathbb{Z}}(1,0) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathsf{W}_{\mathbb{Z}}(2,0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathsf{W}_{\mathbb{Z}}(0,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix},$$

$$\mathsf{W}_{\mathbb{Z}}(1,1) = e^{\pi i/3}\begin{pmatrix} 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \end{pmatrix}, \quad \mathsf{W}_{\mathbb{Z}}(2,1) = e^{2\pi i/3}\begin{pmatrix} 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathsf{W}_{\mathbb{Z}}(0,2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix}, \quad \mathsf{W}_{\mathbb{Z}}(1,2) = e^{2\pi i/3}\begin{pmatrix} 0 & 0 & e^{2\pi i/3} \\ 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \end{pmatrix},$$

$$\mathsf{W}_{\mathbb{Z}}(2,2) = e^{4\pi i/3}\begin{pmatrix} 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \\ 1 & 0 & 0 \end{pmatrix}.$$

**Exercise** 8.2
We have $\langle\langle \frac{1}{\sqrt{d}} \mathsf{W}_{\mathbb{Z}}(s', t') | \frac{1}{\sqrt{d}} \mathsf{W}_{\mathbb{Z}}(s, t) \rangle\rangle = \frac{1}{d} \operatorname{Tr} \omega_{\mathbb{Z}}^{s't-st'} \mathsf{W}_{\mathbb{Z}}(s-s', t-t') = \delta_{s,s'}\delta_{t,t'}$. So, $\{|\frac{1}{\sqrt{d}} \mathsf{W}_{\mathbb{Z}}(s, t)\rangle\rangle\}_{(s,t)\in\mathbb{Z}_d^2}$ forms a CONS on $\mathbb{C}^d \otimes \mathbb{C}^d$.

**Exercise** 8.3
It is sufficient to show that $(x, y) = (F(x), F(y))_{\mathbb{F}_4}$ for $x, y \in \mathbb{F}_2^2$. This relation can be shown as follows.

$$(1, 1)_{\mathbb{F}_4} = 0,\ (1, \alpha)_{\mathbb{F}_4} = 1,\ (\alpha, \alpha)_{\mathbb{F}_4} = 1,\ (\alpha, \alpha+1)_{\mathbb{F}_4} = 0,$$
$$(\alpha+1, \alpha+1)_{\mathbb{F}_4} = 1,\ (\alpha+1, 1)_{\mathbb{F}_4} = 1.$$

**Exercise** 8.4
$\{(0, 0, 0, 0), (1, 1, 0, 0), (0, 0, 1, 1), (1, 1, 1, 1)\}$. It is easy to show that this subgroup is a self-orthogonal subgroup. The dimension of this subgroup is 2. So, its orthogonal group also has dimension 2. Thus, it is a strictly self-orthogonal group.

**Exercise** 8.5
Consider a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle = (ad - bc)(x'y - xy'), \tag{A.3}$$

it belongs to $\mathrm{Sp}(2, \mathbb{F}_q)$ if and only if $ad - bc = 1$.

**Exercise** 8.6

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

**Exercise** 8.7
Firstly, notice that the elements of $\mathrm{GL}(2, \mathbb{F}_2)$ corresponding to $\mathbb{F}_4$ via the $\mathbb{F}_2$-morphism $F$ are limited to the following matrices.

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \ \alpha+1 \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.4}$$

So, the matrix $g_0 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_2)$ is not contained in $\mathbb{F}_4$ in this sense.

**Exercise** 8.8

Use the element $g_0$ given in Exercise 8.7. The matrix $\begin{pmatrix} g_0 & 0 \\ 0 & g_0 \end{pmatrix}$ belongs to $\mathrm{Sp}(4, \mathbb{F}_2)$ but does not belong to $\mathrm{Sp}(2, \mathbb{F}_4)$ via the $\mathbb{F}_2$-morphism $F$.

**Exercise** 8.9

The following is an example of a generating matrix of a strictly self-orthogonal subgroup given in Exercise 8.4.

$$A(g) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

**Exercise** 8.10

The space $\mathcal{H}^{\otimes 1:\mathrm{even}}$ is spanned by $\{|0\rangle, |1\rangle + |2\rangle\}$. The space $\mathcal{H}^{\otimes 1:\mathrm{odd}}$ is spanned by $\{|1\rangle - |2\rangle\}$.

**Exercise** 8.11

$$\left(\sqrt{\frac{2}{3}}\langle 0| + \sqrt{\frac{1}{3}}\langle 1|\right)\left(\sqrt{\frac{2}{3}}|0\rangle + e^{2\pi i/3}\sqrt{\frac{1}{3}}|1\rangle\right) = \frac{2}{3} + \frac{1}{3}e^{2\pi i/3} = \frac{1}{2} + \frac{1}{2\sqrt{3}}i.$$

So,

$$\left|\left(\sqrt{\frac{2}{3}}\langle 0| + \sqrt{\frac{1}{3}}\langle 1|\right)\left(\sqrt{\frac{2}{3}}|0\rangle + e^{2\pi i/3}\sqrt{\frac{1}{3}}|1\rangle\right)\right| = \sqrt{\frac{1}{3}}.$$

Similarly, we have

$$\begin{aligned}&\left|\left(\sqrt{\frac{2}{3}}\langle 0| + \sqrt{\frac{1}{3}}\langle 1|\right)\left(\sqrt{\frac{2}{3}}|0\rangle + e^{4\pi i/3}\sqrt{\frac{1}{3}}|1\rangle\right)\right| \\ =&\left|\left(\sqrt{\frac{2}{3}}\langle 0| + e^{-2\pi i/3}\sqrt{\frac{1}{3}}\langle 1|\right)\left(\sqrt{\frac{2}{3}}|0\rangle + e^{4\pi i/3}\sqrt{\frac{1}{3}}|1\rangle\right)\right| = \sqrt{\frac{1}{3}}.\end{aligned}$$

Also, we have

$$\left|\left(\sqrt{\frac{2}{3}}\langle 0| + \sqrt{\frac{1}{3}}\langle 1|\right)|1\rangle\right| = \left|\left(\sqrt{\frac{2}{3}}\langle 0| + e^{-2\pi i/3}\sqrt{\frac{1}{3}}\langle 1|\right)|1\rangle\right|$$
$$= \left|\left(\sqrt{\frac{2}{3}}\langle 0| + e^{-4\pi i/3}\sqrt{\frac{1}{3}}\langle 1|\right)|1\rangle\right| = \sqrt{\frac{1}{3}}.$$

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110. G. Zauner, *Quantendesigns - Grundzuge einer nicht kommutativen Designtheorie*, Ph.D. thesis, University of Vienna (1999)
111. G. Zauner, Quantum designs: Foundations of a noncommutative design theory. Int. J. Quant. Inf. **9**, 445–507 (2011)
112. H. Zhu, SIC POVMs and Clifford groups in prime dimensions. J. Phys. A: Math. Theor. **43**, 305305 (2010)
113. H. Zhu, Quantum State Estimation and Symmetric Informationally Complete POMs. Ph.D. thesis, National University of Singapore (2012). http://scholarbank.nus.edu.sg/bitstream/handle/10635/35247/ZhuHJthesis.pdf
114. H. Zhu, Tomographic and Lie algebraic significance of generalized symmetric informationally complete measurements. Phys. Rev. A **90**, 032309 (2014)
115. H. Zhu, Quantum state estimation with informationally overcomplete measurements. Phys. Rev. A **90**, 012115 (2014)
116. H. Zhu, Multiqubit Clifford groups are unitary 3-designs (2015). http://arxiv.org/abs/1510.02619
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119. H. Zhu, Super-symmetric informationally complete measurements. Ann. Phys. **362**, 311–326 (2015)
120. H. Zhu, B.-G. Englert, Quantum state tomography with fully symmetric measurements and product measurements. Phys. Rev. A **84**, 022327 (2011)

# Curriculum Vitae

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**Personal**

Born on December 28, 1971.
Born in Tokushima, Japan.
Japanese citizen.

**Education**

Ph.D. Depart of Mathematics, Graduate School of Science, Kyoto University, 1999.
M.A. Depart of Mathematics, Graduate School of Science, Kyoto University, 1996.
B.S. Physics & Mathematics, Faculty of Science, Kyoto University, 1994

**Employment (Full time)**

Research Fellow, the Japan Society for the Promotion of Science, 1998–2000
Researcher, Laboratory for Mathematical Neuroscience (Amari's Laboratory), Brain Science Institute, RIKEN, 2000–2003
Research Manager, ERATO, Quantum Computation and Information Project, Japan Science and Technology Agency (JST), 2003–2006
Group Leader, Quantum Information Theory Group, ERATO-SORST Quantum Computation and Information Project, Japan Science and Technology Agency (JST), 2006–2007
Associate Professor, Graduate School of Information Sciences, Tohoku University, 2007–2012


M. Hayashi, *Group Representation for Quantum Theory*,
DOI 10.1007/978-3-319-44906-7

Full Professor, Graduate School of Mathematics, Nagoya University, 2012–current.

**Employment (Part time)**

Part-time Lecturer (Quantum Information), SOKENDAI (The Graduate University for Advanced Studies), 2002
Adjunct Associate Professor, Superrobust Computation Project, Information Science and Technology
Strategic Core (21st Century COE by MEXT), Graduate School of Information Science and Technology, The University of Tokyo, 2004–2007
Visiting Associate Professor, Graduate School of Information Sciences, Tohoku University, 2007
Visiting Research Associate Professor, Centre for Quantum Technologies, National University of Singapore, 2009–2012
Visiting Research Professor, Centre for Quantum Technologies, National University of Singapore, 2012–current.
Visiting Professor, Faculty of Engineering and Information Technology, University of Technology, Sydney, 2013

**Honors, Awards, and Fellowships**

1. ***12th Japan Academy Medal***: "Information Theory and Quantum Information Theory for Finite-Coding-Length" This prize is awarded by The Japan Academy to up to 6 prestigious Japanese researchers under 45 years old among all scienti_c areas. Awardees are selected from among the annual recipients of the JSPS Prize.
2. ***FY2015 JSPS PRIZE***: "Information Theory and Quantum Information Theory for Finite-Coding-Length" The JSPS PRIZE is meant to recognize at an early stage in their careers young researchers with fresh ideas who have the potential to become world leaders in their _elds, while helping to enhance their opportunities to advance their research and make breakthroughs. Twenty-_ve researchers under 45 years old were selected for this year's PRIZE. Their _elds of research run the spectrum from the humanities and social sciences to the natural sciences.
3. ***2011 IEEE Information Theory Society Paper Award***: "Information spectrum approach to second-order coding rate in channel coding" *IEEE Transactions on Information Theory*, Vol. 55, No. 11, 4947–4966 (2009). This prize is the most distinguished paper award in the information theory community.
4. *Senior Member of IEEE*, 2013.
5. *Japan IBM Prize in the Computer Science Section 2010*: "Universal protocol in quantum information and its application to quantum key distribution" This prize is one of the most distinguished prizes in Japan in information science for researchers under 45 across Japan.
6. *Funai Foundation for Information Technology Award in the Computer Science Category 2010*: "Universal quantum information protocol and its application to quantum cryptography".
7. *Research Fellowship for Young Scientists*, the Japan Society for the Promotion of Science, 1998.

8. *2001 SITA Encouragement Award* (by The Society of Information Theory and its Applications (SITA)): "Variable length universal entanglement concentration by local operations".
9. *16th TEPIA Video Award, 2006* (by Association for Technological Excellence Promoting Innovative Advances (TEPIA)): This prize is awarded to Public Video for ERATO Quantum computation and information project. I managed the production process as Research Manager.

## Professional Activities

### Editorial Board Member

1. Editorial Board Member, International Journal of Quantum Information (IJQI) World Scientific, 2002–Current.
2. Advisory Board Member, International Journal On Advances in Security, IARIA, 2009–Current. (Editorial Board Member: 2008-2009)

### Organization of International Meetings

1. Organizing Committee Chair: *ERATO Conference on Quantum Information Science 2003 (EQIS 03)*, Doshiha Univ., Kyoto, Japan, September 4–6, 2003.
2. Organizing Committee and Program Committee: *COE Symposium on Quantum Information Theory*, Doshiha Univ., Kyoto, Japan, September 2–3, 2003.
3. Program Committee Vice-Chair: *ERATO Conference on Quantum Information Science 2004 (EQIS 04)*, Hitotsubashi-memorial hall, Tokyo, Japan, September 1–5, 2004.
4. Program Committee Vice-Chair: *ERATO Conference on Quantum Information Science 2005 (EQIS 05)*, JST-Museum Hall, Tokyo, Japan, August 26–30, 2005.
5. Organizing Committee Chair: *COE-Kakenhi Workshop on Quantum Information Theory and Quantum Statistical Inference*, University of Tokyo, Hongo, Tokyo, November 17–18, (2005)
6. Local Committee: *The 8th International Conference on Quantum Communication, Measurement, and Computing (QCMC2006)*, Tsukuba Epocal, Tsukuba, Japan, November 28–December 3, (2006).
7. Program Committee: *Asian Conference on Quantum Information Science 2006 (AQIS06)*, Beijing, China, September 1–4, 2006.
8. Technical Program Committee, *The First International Workshop on Quantum Security (QSEC2007)*, Guadeloupe, French Caribbean, France, January 2–6, 2007. Organized by IEEE France and IARIA.
9. Organizing Committee Chair: *Special seminar series on quantum information*, National Institute of Informatics (NII), Tokyo, February 1–March 21, 2007.
10. Program Committee: *The 3rd Workshop on Theory of Quantum Computation, Communication, and Cryptography (TQC2008)*, The University of Tokyo, Tokyo Japan, January 30–February 1, 2008.
11. Technical Program Committee Co-Chair: *ICQNM 2008, The Second International Conference on Quantum, Nano, and Micro Technologies*, Sainte Luce,

Martinique, France, February 10–15, 2008. Organized by IEEE France and IARIA.

12. Main Organizer: *GSIS & DEX-SMI Workshop on Quantum statistical inference and entanglement Graduate School of Information Sciences*, Tohoku University, February, 11–12, 2008
13. International Advisory Committee: *International School and Conference on Quantum Information (ISCQI - 2008)*, Institute of Physics (IOP), Bhubaneswar, Orissa, India, March 4–12, 2008
14. Program Committee: *Asian Conference on Quantum Information Science 2008 (AQIS08)*, Seoul, Korea, August 25–26, 2008.
15. Organizing Committee: *International Workshop on Statistical-Mechanical Informatics 2008 (IW-SMI2008)*, Sendai, Japan, September 14–17, 2008.
16. Organizing Committee Chair: *GSIS Workshop on Quantum Information Theory*, Sendai, Japan, November 5–7, 2008
17. Organizing Committee: *Multicritical Behaviour of Spin Glasses and Quantum Error Correcting Codes (MBQEC)*, Centennial Hall, Ookayama campus, Tokyo Institute of Technology, Tokyo, Japan, November 17–19 (2008)
18. Organizing committee chair: *DEX-SMI workshop on quantum statistical inference*, National Institute of Informatics (NII), Tokyo, Japan March, 2–4 (2009).
19. Program Committee: *ICQNM 2009, The Third International Conference on Quantum*, Nano and Micro Technologies, Cancun, Mexico, February 1–6 (2009)
20. Program Committee: *Workshop on Theory of Quantum Computation, Communication, and Cryptography (TQC2009)*, Waterloo, Canada, May 11–13 (2009)
21. Program Committee: *ICQNM 2010, The Fourth International Conference on Quantum, Nano and Micro Technologies*, St. Maarten, Netherlands Antilles, February 10–16 (2010).@
22. Program Committee (Special Area Chair): *ICQNM 2011, The Fifth International Conference on Quantum, Nano and Micro Technologies*, Nice, France, August 21–27 (2011).
23. Technical Program Committee: *ICQNM 2012, The Sixth International Conference on Quantum, Nano and Micro Technologies*, Rome, Italy, August 19–24 (2012)
24. Program Committee: *12th Asian Quantum Information Science conference (AQIS12)*, School of Physical Science and Technology, Soochow University, Suzhou, China, August 23–26 (2012).
25. Workshop Co-chair: *Japan-Singapore Workshop on Multi-user Quantum Networks*, Centre for Quantum Technologies, National University of Singapore, Singapore, September 17–20 (2012)
26. Special Area Chair: *ICQNM 2013, The Seventh International Conference on Quantum, Nano and Micro Technologies*, Barcelona, Spain, August 25–31 (2013)
27. Program Committee: *13th Asian Quantum Information Science conference (AQIS 13)*, IMSC Chennai, India, August 25–30 (2013)
28. Program Committee: The *7th International Conference on Information Theoretic Security (ICITS)*, NTU, Singapore, November, 28–30 (2013)

29. Program Committee: *2014 International Symposium on Information Theory (ISIT)*, Honolulu, Hawaii, June 29–July 4 (2014)
30. Program Committee: *14th Asian Quantum Information Science conference (AQIS 14)*, Kyoto, Japan, August 21–25 (2014)
31. Special Area Chair: *ICQNM 2014, The Eighth International Conference on Quantum, Nano and Micro Technologies*, Lisbon, Portugal, November 16–20 (2014)
32. Program Committee: *The 9th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC2014)*, National University of Singapore, Singapore, May, 21-23 (2014)
33. Organizer: *Australia-Japan Workshop on Multi-user Quantum Networks*, University of Technology, Sydney, October, 22–24 (2014)
34. Program Committee: *Quantum Information Processing (QIP 2015)*, Sydney January (2015)
35. Program Committee: *2015 International Symposium on Information Theory (ISIT)*, Hong Kong, June 14–19 (2015)
36. Special Area Chair: *ICQNM 2015, The Seventh International Conference on Quantum, Nano and Micro Technologies*, Venice, Italy, August 23–28, 2015
37. Program Committee: *2015 IEEE Information Theory Workshop (ITW)*, Jeju Island, Koreag, October 11–15, 2015
38. Special Area Chair: *ICQNM 2016, The Tenth International Conference on Quantum, Nano and Micro Technologies*, Nice, France, July 24–28, 2016
39. Program Committee: *16th Asian Quantum Information Science conference (AQIS'16)*, Academia Sinica, Taipei, Taiwan, August 29-September 1, 2016
40. Conference Co-Chair: *Beyond I.I.D. in Information Theory 2017*, Institute for Mathematical Sciences, National University of Singapore, Singapore, July 24–28, 2017

**Review Committee for Ph.D. Thesis**

1. 2005, Review Committee Member for Two Ph.D. Candidates, Graduate School of Information Science and Technology, The University of Tokyo, Japan.
2. 2006, Review Committee Member for a Ph.D. Candidate, Graduate School of Information Science and Technology, The University of Tokyo, Japan.
3. 2008, External examiner (referent), Faculty of Mathematics and Natural Sciences, University of Leiden, Netherlands
4. 2008, External examiner, National Institute of Education, Nanyang Technological University, Singapore.
5. 2008, External examiner, Department of Communication and Integrated Systems, Graduate School of Science and Engineering, Tokyo Institute of Technology.
6. 2012, External examiner, School of Computer Science, McGill University, Canada.
7. 2013, External examiner, Graduate School of Information Sciences, Tohoku University, Japan.
8. 2014, External examiner, Department of Electrical and Computer Engineering, National University of Singapore.

### Other Kinds of Review Committees

1. 2015, External Reviewer of Research Committee, the Czech Science Foundation.
2. 2014, External Reviewer of Research Committee, University of Macau (UM) for the 2014 Multi-Year Research Grant (MYRG).
3. 2012, External Reviewer of TATA INSTITUTE OF FUNDAMENTAL RESEARCH.

### Other Professional Activities

1. I am a founder of the *Asian Conference on Quantum Information Science (AQIS)* conference series, which is a major international conference series on quantum information and computation.
2. As Research Manager, I oversaw the ERATO Quantum Computation and Information Project, which at the time was the largest research group for quantum information and computation in Japan.
3. I have served as a referee for renowned international journals including *Nature Photonics, IEEE Transactions on Information Theory, Communications in Mathematical Physics, Physical Review Letters, Physical Review A, Journal of Physics A, New Journal of Physics, Quantum Information and Compu- tation, Journal of Mathematical Physics*, and many prestigious journals.

### Membership of Academic Societies

IEEE (The Institute of Electrical and Electronics Engineers): Senior Member.
The Mathematical Society of Japan.
JPS (The Physical Society of Japan).
IEICE (The Institute of Electronics, Information and Communication Engineers).

# Index





M. Hayashi, *Group Representation for Quantum Theory*,
DOI 10.1007/978-3-319-44906-7

**S**



**T**