Answers to Selected Exercises

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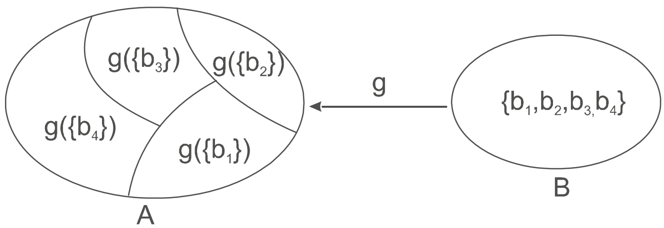
Chapter 1

3. Suppose that $f\colon FC \to FC'$ is an isomorphism. Since $F\colon \hom_{\mathcal{C}}(C, C') \leftrightarrow \hom_{\mathcal{D}}(FC, FC')$ is a bijection, there is a $g\colon C \to C'$ for which $Fg = f$. Moreover, the same argument applied to f^{-1} shows that $f^{-1} = Fh$ for some $h\colon C' \to C$. Thus,

$$F(h \circ g) = Fh \circ Fg = f^{-1} \circ f = 1_{FC} = F(1_C)$$

and since $F\colon \hom_{\mathcal{C}}(C, C) \leftrightarrow \hom_{\mathcal{D}}(FC, FC)$, it follows that $h \circ g = 1_C$. Similarly, $g \circ h = 1_{C'}$ and so g is an isomophism.
5. The morphisms form a monoid under composition.
8. **Poset**$(\mathcal{P}, \leq)$ where $\mathcal{P}$ is nontrivial.
10. In $\mathbf{Mod}_R$ the zero module $\{0\}$ is a zero object. In **Rng**, where we postulate that a ring morphism must send 1 to 1, the trivial ring $\{0\}$ (in which $1 = 0$) is not initial but it is terminal.
13. Consider all subcategories of $\mathcal{C}$ with the same objects as that of $\mathbb{D}$ and that contain all of the morphisms of $\mathbb{D}$. The full subcategory is one such category. The intersection of all such categories is the smallest such category. $\mathcal{D}$ is the subcategory of $\mathcal{C}$ with objects the same as the objects of $\mathbb{D}$ and whose morphisms are the identity morphisms of objects in $\mathcal{D}$, the morphisms in $\mathbb{D}$ and the compositions of finite sequences of morphisms of $\mathbb{D}$.
15. We must show that $g = f^{-1}$. With reference to the figure below,



where $B = \{b_1, b_2, b_3, b_4\}$, note that if $b_i \neq b_j$ in B, then

$$g(\{b_i\}) \cap g\big(\{b_j\}\big) = g\big(\{b_i\} \cap \{b_j\}\big) = g(\emptyset) = \emptyset$$

Also,

$$\bigcup g(\{b_i\}) = g\Big(\bigcup\{b_i\}\Big) = g(B) = A$$

Hence, the sets $g(\{b_i\})$ form a partition of A. Now we can define $f\colon A \to B$ to send the elements of $g(\{b_i\})$ to b_i for all $b_i \in B$. Clearly, $f^{-1} = g$.

17. To show that monic does not necessarily imply injective, intuitively speaking, if the images of all morphisms in a category $\mathcal{C}$ are confined to a "restricted" set, then f need only be well-behaved on this set in order to be left-cancellable. With this guidance, consider the category $\mathcal{C}$ whose objects are the subsets of $\mathbb{Z}$ and for which $\hom_{\mathcal{C}}(A, B)$ is the set of all *nonnegative* set functions from A to B, along with the identity function when $A = B$. In this case, the images of all nonidentity morphisms are contained in the natural numbers $\mathbb{N}$. Now, the absolute value function $\alpha\colon \mathbb{Z} \to \mathbb{N}$ is monic since $\alpha \circ f = f$ for all morphisms f in $\mathcal{C}$. On the other hand, α is clearly not injective.

    To show that injective (and therefore also monic) does not necessarily imply left-invertible, consider the inclusion map $\kappa\colon \mathbb{Z} \to \mathbb{Q}$ between rings, which is injective. However, κ is not left-invertible, since a left-inverse $\sigma\colon \mathbb{Q} \to \mathbb{Z}$ would satisfy $\sigma \circ \kappa = 1$, that is, $\sigma(k) = k$ for all integers k. This is not possible since, for example, it would imply that $\sigma(1/2) = \sigma(1)/\sigma(2) = 1/2$, which is not an integer.

    To see that epics are not necessarily surjective, if we can find a category in which each morphism leaving an object A is completely determined by its values on a proper subset S of A, then the inclusion map $i\colon S \to A$, which is not surjective, will be right-cancellable. To this end, the monoids $\mathbb{N}$ and $\mathbb{Z}$ are additive monoids. Moreover, the inclusion map $\kappa\colon \mathbb{N} \to \mathbb{Z}$ is not surjective. However, it is epic since if

$$g \circ \kappa = h \circ \kappa$$

for $g\colon \mathbb{Z} \to C$ and $h\colon \mathbb{Z} \to C$ then g and h agree on all nonnegative integers. This implies that $g = h$, since for $n > 0$, we have (where e is the identity in C)

$$\begin{aligned}
g(-n) &= g(-n) * e \\
&= g(-n) * h(n-n) \\
&= g(-n) * [h(n) * h(-n)] \\
&= [g(-n) * h(n)] * h(-n) \\
&= [g(-n) * g(n)] * h(-n) \\
&= g(-n+n) * h(-n) \\
&= g(0) * h(-n) \\
&= h(-n)
\end{aligned}$$

Finally, to see that surjective (and therefore epic) maps are not always right-invertible, let $C = \langle a \rangle$ be a cyclic group and let $H = \langle a^2 \rangle$. Consider the canonical projection map $\pi\colon C \to C/H = \{H, aH\}$. This map is surjective, but it is not right-invertible. In fact, any group morphism $\sigma\colon C/H \to C$ must send aH, which has order 2 to an element $\sigma(aH)$ of exponent 2 and so $\sigma(aH) = 1$. Since $\sigma(H) = 1$ as well, the only group morphism from C/H to C is the zero map.

19. Suppose that f is epic. Define α, β: $B \to B \otimes B$ by

$$\alpha(b) = 1 \otimes b \quad \text{and} \quad \beta(b) = b \otimes 1$$

Then

$$\alpha \circ f(a) = 1 \otimes f(a) = f(a)(1 \otimes 1) = f(a) \otimes 1 = \beta \circ f(a)$$

and so $\alpha \circ f = \beta \circ f$, whence $\alpha = \beta$, that is, $1 \otimes b = b \otimes 1$ for all $b \in B$. For the converse, suppose that $1 \otimes b = b \otimes 1$ for all $b \in B$. Let $\alpha \circ f = \beta \circ f$, where α, β: $B \to R$. The ring map $\alpha \circ f = \beta \circ f$: $A \to R$ makes R into an A-module via

$$a * r = (\alpha f(a)) \cdot r = (\beta f(a)) \cdot r$$

Now, define a map $\alpha \times \beta$: $B \times B \to R$ by

$$(\alpha \times \beta)(b, c) = (\alpha b)(\beta c)$$

which is A-bilinear since

$$(\alpha \times \beta)(ab, c) = \alpha(ab) \cdot \beta c = \alpha(f(a)) \cdot \alpha b \cdot \beta c = a*(\alpha \times \beta)(b, c)$$

and so there is a unique θ: $B \otimes B \to R$ for which

$$\theta(x \otimes y) = (\alpha \otimes \beta)(x, y)$$

In particular,

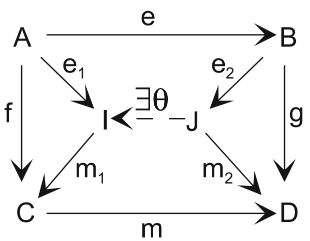
$$\theta(1 \otimes b) = (\alpha \otimes \beta)(1, b) = \beta b$$

and

$$\theta(b \otimes 1) = (\alpha \otimes \beta)(b, 1) = \alpha b$$

and so $\alpha = \beta$.

23. The figure below shows the factorization of f and g.



Since $(m \circ m_1) \circ e_1 = m_2 \circ (e_2 \circ e)$ and since each of these morphisms is in $\mathcal{E}$ or $\mathcal{M}$, there is an isomorphism $\theta: J \to I$ such that

$$e_1 = \theta \circ (e_2 \circ e)$$

Hence

$$f = m_1 \circ e_1 = m_1 \circ \theta \circ (e_2 \circ e) = (m_1 \circ \theta \circ e_2) \circ e$$

and we may take $f = h \circ e$.

Chapter 2

1. Let $F: \mathcal{C} \Rightarrow \mathcal{D}$ be a contravariant functor. Then since $\mathcal{C}$ and its opposite category $\mathcal{C}^{\mathrm{op}}$ have the same objects and the same morphisms (they even have the same hom-sets, but they are associated to different pairs of objects), F can also be thought of as mapping the objects and morphisms of $\mathcal{C}^{\mathrm{op}}$ to the objects and morphisms of $\mathcal{D}$. Moreover, if $f: A \to B$ and $g: B \to C$ are morphisms in the opposite category $\mathcal{C}^{\mathrm{op}}$, then

$$F\left(g \circ_{\mathrm{op}} f\right) = F(f \circ g) = Fg \circ Ff = Ff \circ_{\mathrm{op}} Fg$$

and so $F: \mathcal{C}^{\mathrm{op}} \Rightarrow \mathcal{D}$ is a *covariant* functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. Thus, a contravariant functor $F: \mathcal{C} \Rightarrow \mathcal{D}$ is a covariant functor $F: \mathcal{C}^{\mathrm{op}} \Rightarrow \mathcal{D}$ and conversely.

3. For part a), to see that F is well defined, if $aG' = bG'$, then $b^{-1}a \in G'$ and so $\sigma(b^{-1}a) \in H'$ (since σ takes commutators to commutators). Hence, $(\sigma a)H' = (\sigma b)H'$. Also, $F1 = 1$ and if $\sigma: G \to H$ and $\tau: H \to K$, then

$$F(\tau\sigma)(aG') = (\tau\sigma a)K' = F\tau(\sigma a H') = F\tau(F\sigma a G')$$

and so $F(\tau\sigma) = F\tau F\sigma$. For part b), the canonical projection is natural.

5. First, we must show that a group homomorphism $f: G \to H$ maps $C(G)$ to $C(H)$. But $f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} \in C(H)$ and so the subgroup of G that maps into $C(H)$ contains all commutators, and therefore contains $C(G)$. Clearly C preserves the identity and composition.

7. This map

$$\hom_{\mathcal{C}}(B, \cdot): f \mapsto f^{\leftarrow}$$

for $f: X \to Y$ defines a natural transformation

$$\hom_{\mathcal{C}}(B, \cdot)^*: \hom_{\mathcal{C}}(A, \cdot) \stackrel{\cdot}{\to} \hom_{\mathbf{Set}}(\hom_{\mathcal{C}}(B, A), \hom_{\mathcal{C}}(B, \cdot))$$

by

$$[\hom_{\mathcal{C}}(B, \cdot)^*]_X: \hom_{\mathcal{C}}(A, X) \stackrel{\cdot}{\to} \hom_{\mathbf{Set}}(\hom_{\mathcal{C}}(B, A), \hom_{\mathcal{C}}(B, X))$$

where

$$[\hom_{\mathcal{C}}(B, \,\cdot\,)^*]_X(g_{A,X}) = g^{\leftarrow}$$

In this case, naturalness is

$$f^{\leftarrow\leftarrow} \circ [\hom_{\mathcal{C}}(B, \,\cdot\,)^*]_X = [\hom_{\mathcal{C}}(B, \,\cdot\,)^*]_Y \circ f^{\leftarrow}$$

for $f{:}X \to Y$ in $\mathcal{C}$. For any g: $A \to X$, this is

$$f^{\leftarrow\leftarrow}(g^{\leftarrow}) = (f \circ g)^{\leftarrow}$$

which is true, since

$$f^{\leftarrow\leftarrow}(g^{\leftarrow}) = f^{\leftarrow} \circ g^{\leftarrow} = (f \circ g)^{\leftarrow}$$

9. The identity map 1_C is sent to the map

$$\begin{pmatrix} 1_C \circ \rho_{C\times A,1} \\ 1_A \circ \rho_{C\times A,2} \end{pmatrix} = \begin{pmatrix} \rho_{C\times A,1} \\ \rho_{C\times A,2} \end{pmatrix} = 1_{C\times A}$$

If f: $C \to D$ and g: $D \to E$ then

$$\Pi_A(g \circ f) = \begin{pmatrix} g \circ f \circ \rho_{C\times A,1} \\ 1_A \circ \rho_{C\times A,2} \end{pmatrix}$$

that is,

$$\rho_{E\times A,1} \circ \Pi_A(g \circ f) = g \circ f \circ \rho_{C\times A,1}$$

and

$$\rho_{E\times A,2} \circ \Pi_A(g \circ f) = \rho_{C\times A,2}$$

But

$$\rho_{E\times A,1} \circ \Pi_A g \circ \Pi_A f = g \circ \rho_{D\times A,1} \circ \Pi_A f = g \circ f \circ \rho_{C\times A,1}$$

and

$$\rho_{E\times A,1} \circ \Pi_A g \circ \Pi_A f = g \circ \rho_{D\times A,1} \circ \Pi_A f = g \circ f \circ \rho_{C\times A,1}$$

11. For part 1), let λ: $F \dot{\approx} G$ be a natural isomorphism. Assume that G is faithful. To see that F is faithful, if $f, g \in \hom_{\mathcal{C}}(A, B)$ then

$$\begin{aligned} Ff = Fg &\Rightarrow \lambda_B \circ Ff \circ \lambda_A^{-1} = \lambda_B \circ Fg \circ \lambda_A^{-1} \\ &\Rightarrow Gf = Gg \\ &\Rightarrow f = g \end{aligned}$$

To see that F is full, let $h \in \hom_{\mathcal{C}}(FA, FB)$. Let

$$k = \lambda_B \circ h \circ \lambda_A^{-1} : GA \to GB$$

Since G is full, there is an f: $A \to B$ such that $Gf = k$ and so

$$h = \lambda_B^{-1} \circ k \circ \lambda_A \to \lambda_B^{-1} \circ Gf \circ \lambda_A = Ff$$

Hence F is full. For part 2), if $G \circ F$ is faithful then F is faithful, for if $f, g \in \hom_{\mathcal{C}}(A, B)$, we have

$$Ff = Fg \Rightarrow GFf = GFg \Rightarrow f = g$$

Also, if $G \circ F$ is full then G is full, for if $h \in \hom_{\mathcal{C}}(A, B)$ then there is an f: $A \to B$ such that $h = GFf = G(Ff)$.

13. A functor F: $M \Rightarrow$ **Grp** picks out a single group FM. A morphism $m \in M$ is sent to a group morphism Fm: $FM \to FM$ in End(FM). The definition of functor is equivalent to saying that $m \mapsto Fm$ is a group homomorphism.
15. For part a), we have Card $\circ$ $I(n) = \text{Card}(n) = n$ and for $f{:}n \to m$, Card $\circ$ $I(f) = \text{Card}(f)$ $= \theta_m \circ f \circ \theta_n = f$. For part b), let $F = I \circ$ Card. Note that $FS = I \circ \text{Card}(S) = \text{Card}(S)$, thought of as a set and for f: $S \to T$, $Ff = \theta_T \circ f \circ \theta_S^{-1}$. For any set S, let $\lambda_S = \theta_S$. Then

$$Ff \circ \theta_S = \theta_T \circ f \circ \theta_S^{-1} \circ \theta_S = \theta_T \circ f$$

which is the condition for F to be natural.
17. For part a), let f: $A \to B$ be a morphism in $\mathcal{C}$. Then there must exist morphisms Ff: $FA \to FB$ and Gf: $GA \to GB$ in $\mathcal{P}$, which implies that $FA \prec FB$ and $GA \prec GB$. Moreover, there are morphisms $\lambda(A)$: $FA \to GA$ and $\lambda(B)$: $FB \to GB$ in $\mathcal{P}$ if and only if $FA \prec GA$ and $FB \prec GB$, in which case transitivity implies that

$$Gf \circ \lambda(A) = \lambda(B) \circ Ff$$

Thus, there is a natural transformation from F to G if and only if $FA \prec GA$ for all objects A of $\mathcal{C}$.

For part b), by part a), there is a natural transformation λ from F to G if and only if $FA \prec GA$ for all objects A in $\mathcal{C}$, in which case there is exactly one natural transformation.
19. Since Obj(2) = {0, 1} and Mor(2) = $\{1_0, 1_1, 01{:}\ 0 \to 1\}$, a functor F from 2 to $\mathcal{D}$ sends 0 and 1 to a pair of objects $F(0)$ and $F(1)$ in $\mathcal{D}$ and the morphism 01 to a morphism $F(01)$: $F(0) \to F(1)$. Moreover, every arrow f: $A \to B$ in $\mathcal{D}$ gives rise to such a distinct functor F_f from 2 to $\mathcal{D}$. Hence, there is a bijection between the objects (functors) of $\mathcal{D}^2$ and the arrows of $\mathcal{D}$.

21. Define a functor F: $\mathrm{Set}_* \Rightarrow \mathrm{Set}_0$ as follows: F sends $A_* = A + \{*\}$ to A and if f: $A \to B$ is a partial function, then Ff: $A_* \to B_*$ sends all elements of $A_* \setminus \mathrm{dom}(f)$ to $*$. To see that F is a functor, note that $F1 = 1_*$. Also, let f: $A \to B$ and g: $B \to C$ be partial functions. We have

    a) if $x \in \mathrm{dom}(g \circ f)$ then $(g \circ f)_*(x) = x = g_*(f_*(x))$
    b) if $x \in \mathrm{dom}(f) \setminus \mathrm{dom}(g \circ f)$ then $(g \circ f)_*(x) = * = g_*(f_*(x))$
    c) if $x \notin \mathrm{dom}(f)$ then $(g \circ f)*(x) = * = g_*(f_*(x))$

    and so $(g \circ f)_* = g_* \circ f_*$. Now, any pointed function σ: $A_* \to B_*$ is the image under F of the partial function σ: $\sigma^{-1}(B) \to B$ obtained by restricting σ to $\sigma^{-1}(B)$. Finally, if f, g: $A \to B$ are partial functions for which $f_* = g_*$, then clearly $f = g$. hence, F is an isomorphism.

25. This follows from the contravariant version of Yoneda's lemma. Alternatively, the contravariant functors $\hom_{\mathcal{C}}(\,\cdot\,, A)$ and $\hom_{\mathcal{D}}(F\,\cdot\,, B)$ from $\mathcal{C}$ to Set are covariant functors from $\mathcal{C}^{\mathrm{op}}$ to Set. Hence, for any f: $Y \to X$ in $\mathcal{C}$, the condition of naturalness is

$$\lambda_{A,B}(Y) \circ \hom_{\mathcal{C}}(f, A) = \hom_{\mathcal{D}}(Ff, B) \circ \lambda_{A,B}(X)$$

But $\hom_{\mathcal{C}}(f, A) = f^{\to}$ and $\hom_{\mathcal{D}}(Ff, B) = (Ff)^{\to}$, and so

$$\lambda_{A,B}(Y) \circ f^{\to} = (Ff)^{\to} \circ \lambda_{A,B}(X)$$

Taking $X = A$ and applying to 1_A gives

$$\lambda_{A,B}(Y) \circ f^{\to} 1_A = (Ff)^{\to} \circ \lambda_{A,B}(A) 1_A$$

for any f: $Y \to A$. But $f^{\to} 1_A = 1_A f = f$ and so we get

$$\lambda_{A,B}(Y) f = \lambda_{A,B}(A) 1_A \circ Ff$$

Finally, suppose that

$$\lambda_{A,B}(X) f = (Ff)^{\to} g = g \circ Ff$$

for all f: $X \to A$, where $g \in \hom_{\mathcal{D}}(FA, B)$. Then if f: $Y \to X$ in $\mathcal{C}$, we have for any h: $X \to A$,

$$\begin{aligned}(\lambda_{A,B}(Y) \circ f^{\to})h &= \lambda_{A,B}(Y)(h \circ f)\\ &= g \circ F(h \circ f)\\ &= g \circ Fh \circ Ff\\ &= (Ff)^{\to} \circ (g \circ Fh)\\ &= (Ff)^{\to} \circ \lambda_{A,B}(X)h\end{aligned}$$

and so $\lambda_{A,B}(Y) \circ f^{\to} = (Ff)^{\to} \circ \lambda_{A,B}(X)$, showing that $\lambda_{A,B}$ is natural.

Chapter 3

1. The pair

$$(F(X), j\colon X \to F(X))$$

where $F(X)$ is the field generated by the elements of X, that is, the field of all rational functions in the variables X over F, and j is inclusion, is universal for X and U.

5. Let $U\colon \mathrm{Alg}(F) \Rightarrow \mathrm{Set}$ be the forgetful functor, where $\mathrm{Alg}(F)$ is the category of F-algebras. The pair $(F[x], j\colon \{x\} \to UF[x])$ is universal from $\{x\}$ to U. For given any pair $(A, g\colon \{x\} \to UA)$, we define $\tau\colon F[x] \to A$ by $\tau(p(x)) = p(g(x))$.

7. A couniversal pair $(V, h\colon S \to V)$ for a set S has the property that for every set function $f\colon W \to S$, there is a unique linear map $\tau\colon W \to V$ for which $h \circ \tau = f$. But $\tau 0 = 0$ and so $h0 = f0$ for all f. Hence, if $|S| \geq 2$ then there is no couniversal pair. If $S = \{s\}$, then take $V = \langle v \rangle$ and h to be the constant function. If $S = \emptyset$, take $V = \{0\}$ and h to be the empty function.

9. To see that a) and b) are equivalent, note that

$$\tau_{C,E}^{-1}(g \circ h) = Gg \circ \tau_{C,D}^{-1}(h)$$

for all $h\colon U \to D$ and $g\colon D \to D'$ is equivalent to

$$\tau_{C,D'}\left(Gg \circ \tau_{C,D}^{-1}(h)\right) = g \circ h$$

for all $h\colon U \to D$ and $g\colon D \to D'$. But $\alpha = \tau_{C,D}^{-1}(h)$ runs through all morphisms from C to GD as h runs through all morphisms from U to D and so this is equivalent to

$$\tau_{C,D'}(Gg \circ \alpha) = g \circ \tau_{C,D}(\alpha)$$

for all $\alpha\colon C \to GD$, which is in turn equivalent to

$$\tau_{C,D'} \circ (Gg)^{\leftarrow} = g^{\leftarrow} \circ \tau_{C,D}$$

Part c) comes directly from b) by setting $h = 1_U$. Conversely, if 3) holds then

$$\tau_{C,D'}^{-1}(g \circ h) = G(g \circ h) \circ \tau_{C,D}^{-1}(1_U) = Gg \circ Gh \circ \tau_{C,D}^{-1}(1_U) = Gg \circ \tau_{C,D'}^{-1}(h)$$

which is 2).

11. By a theorem, the family of bijections

$$\{\tau_{C,D}\colon \hom_{\mathcal{C}}(C, FD) \leftrightarrow \hom_{\mathcal{D}}(S, D)\}_{D \in \mathcal{D}}$$

is a natural isomorphism and so $\hom_{\mathcal{C}}(C, F\,\cdot\,)$ is representable if and only if the pair

$$\mathcal{U} = (S, u\colon C \to FS) \quad \text{where} \quad u = \tau_{C,S}^{-1}(1_S)$$

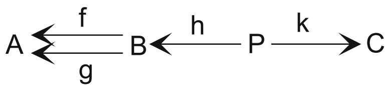
is universal.

Chapter 4

3. Let $f: A \to B$. A cone is any morphism $g: X \to A$. If $t: T \to A$ is terminal, then the mediating morphism θ for the identity map $1: A \to A$ satisfies $t \circ \theta = 1$ and so t is right-invertible. Also, t is monic and so an isomorphism. Conversely, an isomorphism $t: T \approx A$ is terminal.
5. The cones over $\{f: A \to B\}$ are the same as the cones over the diagram $\{f: A \to B, f: A \to B\}$ and so a terminal cone is an equalizer of $\{f, f\}$, that is, an isomorphism.
7. A morphism in **Field** is either the zero map or an embedding. Let $\mathcal{P} = (P, \rho_1: P \to A, \rho_2: P \to B)$ be a product. Since $\hom(A, B) \neq \emptyset$, projections are surjective and so isomorphisms. Thus, $P = A \times B$ is isomorphic to A and to B. The product construction implies that for any $(X, f: X \to A, g: X \to B)$ there is a unique $\theta: X \to P$ such that

$$\rho_1 \circ \theta = f \quad \text{and} \quad \rho_2 \circ \theta = g$$

Now, if $f = 0$ then $\theta = 0$ since ρ_1 is an isomorphism. Hence, $g = 0$. Thus, the cone $(B, 0, 1_B)$ has no mediating morphism.
9. Lexicographic order does not work because the second projection is not monotone. However, product order works just fine.
13. Consider the category with 4 objects and 9 morphisms, as shown in the commutative diagram below What is the dual of this result?



Since the only cone over $\{B, C\}$ is (P, h, k), this must be the product of B and C. However, h is not epic since $f \circ h = g \circ h$ but $f \neq g$.
15. For part a), let $f, g: M \to N$ be R-maps. Let $A = \text{im}(f - g)$ and let $\pi: N \to N/A$ be the canonical projection map. Then for $m \in M$,

$$\pi \circ (f - g)(m) = A$$

and so $\pi \circ f(m) = \pi \circ g(m)$, whence $\pi \circ f = \pi \circ g$. Now, if $h: N \to X$ satisfies $h \circ f = h \circ g$, it follows that $h \circ (f - g) = 0$ and so $A = \text{im}(f - g) \leq \ker(h)$. Hence, there is a unique map $\theta: N/A \to X$ for which $\theta \circ \pi = h$, as desired.

For part b), let $f, g: R \to S$ be ring homomorphisms. Let $A = \text{im}(f - g)$ and let $\pi: S \to S/I$ where $I = \langle A \rangle$ is the ideal generated by A and π is the canonical projection map. Then for $r \in R$,

$$\pi \circ (f - g)(r) = I$$

and so $\pi \circ f(r) = \pi \circ g(r)$, whence $\pi \circ f = \pi \circ g$. Now, if $h: S \to X$ satisfies $h \circ f = h \circ g$, it follows that $h \circ (f - g) = 0$ and so $A = \text{im}(f - g) \leq \ker(h)$. Hence, there is a unique map $\theta: S/I \to X$ for which $\theta \circ \pi = h$, as desired.

17. Since u_x: $\{x\} \to S_x$, we have

$$G\kappa_x \circ u_x\colon \{x\} \to GC$$

and so $u|_{\{x\}} = Gk_x \circ u_x$. Now, let f: $X \to GE$ for $E \in \mathcal{D}$. Then $f|_{\{x\}}$: $\{x\} \to GE$ and so there is a unique τ_x: $S_x \to E$ for which

$$G\tau_x \circ u_x = f\big|_{\{x\}}$$

The definition of coproduct implies that there is a unique θ: $C \to E$ for which

$$\theta \circ \kappa_x = \tau_x$$

for all $x \in X$ and so there is a unique θ for which

$$G(\theta \circ \kappa_x) \circ u_x = f\big|_{\{x\}}$$

(this follows from $\theta \circ \kappa_x = \theta' \circ \kappa_x$ for all x implies $\theta = \theta'$). This is equivalent to

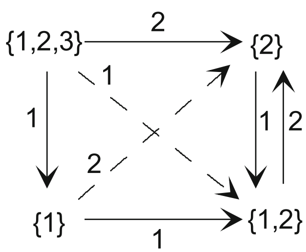
$$(G\theta \circ G\kappa_x \circ u_x)(x) = f(x)$$

that is,

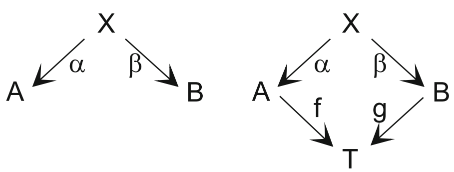
$$(G\theta \circ u)(x) = f(x)$$

and so $G\theta \circ u = f$, as desired.

19. Consider the category shown below.

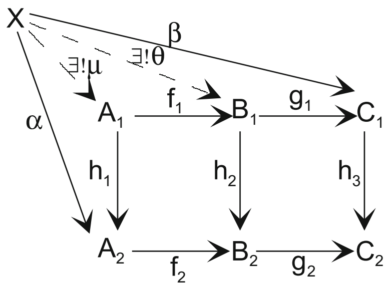


21. The essence of the proof is that there is no real difference between the two diagrams in the figure below



where f and g are the unique morphisms associated to the terminal object T. This follows from the fact that $f \circ \alpha$ and $g \circ \beta$ must be the unique morphism from X to T, which makes the right-hand diagram commute. Thus, a limit in the cones over the left-hand diagram, that is, a product, is a limit in the cones over the right-hand diagram, which is a pullback for this diagram.

23. Consider the figure below



where $g_2 f_2\alpha = h_3\beta$. Since the right-hand square is a pullback and since $(f_2\alpha, \beta)$ is a cone, there is a unique θ: $X \to B_1$ for which

$$g_1\theta = \beta \quad \text{and} \quad h_2\theta = f_2\alpha \tag{*}$$

Since (α, θ) is a cone for the square on the left, there is a unique μ: $X \to A_1$ for which

$$h_1\mu = \alpha \quad \text{and} \quad f_1\mu = \theta \tag{**}$$

Now, μ is a mediating arrow for the entire rectangle, for we have

$$h_1\mu = \alpha \quad \text{and} \quad g_1 f_1\mu = g_1\theta = \beta$$

As to uniqueness, if

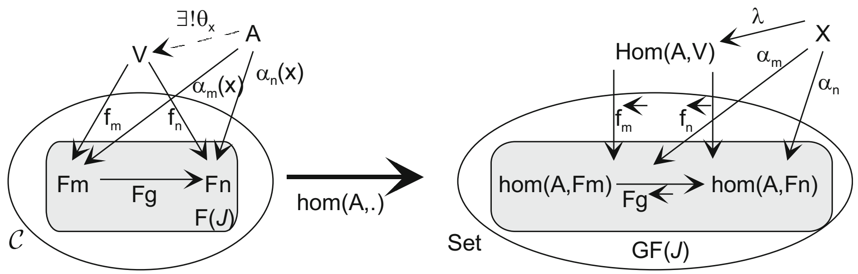
$$h_1\mu_0 = \alpha \quad \text{and} \quad g_1 f_1\mu_0 = \beta$$

then we claim that μ_0 satisfies $(**)$, in which case $\mu_0 = \mu$. To see this, we have already that $h_1\mu_0 = \alpha$. Also, we claim that $f_1\mu_0 = \theta$, in which case, $g_1 f_1\mu_0 = g_1\theta = \beta$ and we are done. To show that $f_1\mu_0 = \theta$, we show that $f_1\mu_0$ satisfies $(*)$. First, $g_1 f_1\mu_0 = \beta$ by assumption. Second,

$$h_2 f_1 \mu_0 = f_2 h_1 \mu_0 = f_2 \alpha$$

which is the second condition in $(*)$.

25. As shown in the figure below,



let $\mathcal{X} = \big(X, \{\alpha_n \colon X \to F_n\}\big)$ be a cone over $G \circ F$. Then

$$(Fg)^{\leftarrow} \circ \alpha_m = \alpha_n$$

that is, for any $x \in X$,

$$(Fg)[\alpha_m(x)] = \alpha_n(x)$$

This shows that

$$\mathcal{A}_x = (A, \{\alpha_n(x) \colon A \to Fn\})$$

is a cone over F. Hence, there is a unique $\theta_x \colon A \to V$ for which

$$f_n \circ \theta_x = \alpha_n(x)$$

Let $\lambda \colon X \to \hom(A, V)$ be defined by $\lambda(x) = \theta_x$. Then

$$\big(f_n^{\leftarrow} \circ \lambda\big)(x) = f_n^{\leftarrow}(\theta_x) = f_n \circ \theta_x = \alpha_n(x)$$

and so λ is a mediating morphism. As to uniqueness, if

$$\big(f_n^{\leftarrow} \circ \mu\big)(x) = \alpha_n(x)$$

then

$$f_n \circ \mu(x) = \alpha_n(x)$$

which implies that $\mu(x) = \theta_x = \lambda(x)$.

As to the dual, first we note that for $A \in \mathcal{C}^{\text{op}}$, the hom functor

$$\hom_{\mathcal{C}^{\text{op}}}(A, \,\cdot\,)\colon \mathcal{C}^{\text{op}} \Rightarrow \textbf{Set}$$

preserves limits, that is,

$$\hom_{\mathcal{C}}(\,\cdot\,, A)\colon \mathcal{C}^{\text{op}} \Rightarrow \textbf{Set}$$

preserves limits. But a limit in $\mathcal{C}^{\text{op}}$ is a colimit in $\mathcal{C}$ and so the dual is that $\hom_{\mathcal{C}}(\,\cdot\,, A)$ sends colimits in $\mathcal{C}$ to limits in **Set**.

27. Suppose that $\mathcal{K} = \left(V, \{f_n \mid n \in \mathcal{J}\}\right)$ is a cone of F, whose image $G\mathcal{K} = \left(GV, \{Gf_n \mid n \in \mathcal{J}\}\right)$ is a limit of $F \circ G$. If $\mathcal{X} = \left(X, \{g_n \mid n \in \mathcal{J}\}\right)$ is a cone over F, then its image $G\mathcal{X} = (GX, \{Gg_n \mid n \in \mathcal{J}\})$ is a cone over $F \circ G$ and so there is a unique map $\theta\colon GX \to GV$ for which $Gf_n \circ \theta = Gg_n$. Since G is fully faithful, there is a unique $\lambda\colon X \to V$ for which $\theta = G\lambda$ and so $G(f_n \circ \lambda) = Gg_n$, whence $f_n \circ \lambda = g_n$. Thus, λ is a unique cone morphism from $\mathcal{K}$ to $\mathcal{X}$.
30. The elements of the quotient S/N have the form $s + N$, where $s \in S$ has finite support. If $\rho_i(s) \neq 0$ and $\rho_j(s) \neq 0$ for $i < j$, then since $[\rho_i(s)]_{i,j} \in N$, we have

$$sN = \left(s - [\rho_i(s)]_{i,j}\right) + N$$

where the latter has ith coordinate equal to 0 and support that is properly contained in the support of $s + N$. It follows that any $x \in S/N$ has the form $s + N$, where $|\operatorname{supp}(s)| = 1$, that is, $x = [a_i]_i + N$ for $a_i \in M_i$.
Now let us examine the elements of N. The generators of N are

$$[a_i]_{i,j} = [a_i]_i - \left[f_{i,j}(a_i)\right]_j$$

where $a_i \in M_i$ and $i < j$. If $k > j$, then we can write

$$\begin{aligned} [a_i]_{i,j} &= [a_i]_i - \left[f_{i,j}(a_i)\right]_j \\ &= [a_i]_i - \left[f_{i,k}(a_i)\right]_k + \left(\left[-f_{i,j}(a_i)\right]_j - \left[-f_{i,k}(a_i)\right]_k\right) \end{aligned}$$

If $b_j = -\,f_{i,j}(a_i)$, then

$$f_{j,k}(b_j) = -f_{j,k}\, f_{i,j}(a_i) = -f_{i,k}(a_i)$$

and so

$$[a_i]_{i,j} = [a_i]_i - \left[f_{i,k}(a_i)\right]_k + \left([b_j]_j - \left[f_{j,k}(b_j)\right]_k\right)$$

which is the sum of two generators, each of which has last term of index k. Thus, since any $x \in N$ is a finite sum of generators, and since the index set I is directed, there is an index k for which x is the finite sum of generators whose last terms all have index k. Moreover, since we can add generators that have the same pair of indices, we may assume that the first indices in this sum are distinct. Thus,

$$\begin{aligned} x &= \left([a_{i_1}]_{i_1} - \left[f_{i_1,k}(a_{i_1})\right]_k + \cdots + \left([a_{i_n}]_{i_n} - \left[f_{i_n,k}(a_{i_n})\right]_k\right)\right. \\ &= \left([a_{i_1}]_{i_1} + \cdots + [a_{i_n}]_{i_n} - \left(\left[f_{i_1,k}(a_{i_1})\right]_k + \cdots + \left[f_{i_n,k}(a_{i_n})\right]_k\right)\right) \end{aligned}$$

where the i_j are distinct and less than k.

Now, if $|\text{supp}(x)| = 1$, that is, if x has exactly one nonzero coordinate, then x must have the form $x = \left[a_{i_j}\right]_{i_j}$ for some j, and $f_{i_j,k}\left(a_{i_j}\right) = 0$. In simpler notation, the elements of N that have support of size 1 are of the form $x = [a_i]_i$ for $a_i \in M_i$ and for which $f(a_{i,k}) = 0$ for some $k > i$.

The pair $\left(\text{dirlim}(\mathcal{M}), \{\pi \circ \kappa_i \mid i \in I\}\right)$ is an initial object in the category of cones under the diagram $\mathcal{M}$, that is, it is the colimit in the categorical sense. For if $(X, \{g_i\})$ is a cone under $\mathcal{M}$, then from the definition of direct sum, there is a unique mediating arrow $\theta: S \to X$ for which

$$\theta \circ \kappa_i = g_i$$

Now, if $x = [a_i]_{i,j}$ is a generator of N, then

$$\begin{aligned} \theta(x) &= \theta\left([a_i]_{i,j}\right) \\ &= \theta([a_i]_i) - \theta\left(\left[f_{i,j}(a_i)\right]_j\right) \\ &= \theta\kappa_i(a_i) - \theta\kappa_j\left(f_{i,j}(a_i)\right) \\ &= g_i(a_i) - g_j\left(f_{i,j}(a_i)\right) \end{aligned}$$

and since $(X, \{g_i\})$ is a cone under $\mathcal{M}$, we have

$$\theta(x) = g_i(a_i) - g_j\left(f_{i,j}(a_i)\right) = g_i(a_i) - g_i(a_i) = 0$$

Thus, $x \in \ker(\theta)$, from which we get $N \le \ker(\theta)$. it follows that θ induces a map $\overline{\theta}: S/N \to X$ defined by

$$\overline{\theta}([a_i]_i + N) = \theta([a_i]_i) = \theta\kappa_i(a_i) = g_i(a_i)$$

Thus,

$$\left(\overline{\theta} \circ \pi_N \circ \kappa_i\right)(a_i) = g_i(a_i)$$

and so $\overline{\theta} \circ (\pi_N \circ \kappa_i) = g_i$. Moreover, if $\mu \circ (\pi_N \circ \kappa_i) = g_i$, then

$$\mu([a_i]_i + N) = \overline{\theta}([a_i]_i + N)$$

and so $\mu = \overline{\theta}$, which shows that $\overline{\theta}$ is unique.

Chapter 5

1. In terms of adjunctions, we have

$$\tau_{C,D}\colon \hom_{\mathcal{C}}(C, GD) \overset{\cdot}{\leftrightarrow} \hom_{\mathcal{D}}(FC, D)$$

and

$$\sigma_{E,C}\colon \hom_{\mathcal{E}}(E, G'C) \overset{\cdot}{\leftrightarrow} \hom_{\mathcal{C}}(F'E, C)$$

The composition

$$\lambda_{E,D} = \tau_{F'E,D} \circ \sigma_{GD,E}$$

maps

$$\hom_{\mathcal{E}}(E, G'GD) \overset{\cdot}{\leftrightarrow} \hom_{\mathcal{C}}(F'E, GD) \overset{\cdot}{\leftrightarrow} \hom_{\mathcal{D}}(FF'E, D)$$

and the map is natural since

$$\lambda_{E,\cdot} = \tau_{F'E,\cdot} \circ \sigma_{G\cdot,E} = \tau_{F'E,\cdot} \circ (\sigma_{\cdot,E} * G)$$

and

$$\lambda_{\cdot,D} = \tau_{F'\cdot,D} \circ \sigma_{GD,\cdot} = (\tau_{\cdot,D} * F') \circ \sigma_{GD,\cdot}$$

3. $F(X) = F[X]$, the ring of polynomials in X. $f\colon X \to Y$ can be extended to polynomials.
5. Let $u_C = \pi_{[C,C]}$ be projection modulo the derived subgroup $[C, C]$. Then if A is an abelian group and $f\colon C \to A$ is a group morphism, it is easy to see that $[C, C] \leq \ker(f)$ and so there is a unique $\tau\colon C/[C, C] \to A$ for which

$$\tau \circ \pi_{[C,C]} = f$$

This shows that $(C/[C, C], \pi_{[C,C]}\colon C \to C/[C, C])$ is a universal pair from C to U.
7. Let $u_A\colon a \mapsto 1 \otimes a$. Then if M is an R-module and $f\colon A \to UM$ is a morphism of abelian groups, let $\tau\colon R \times A \to M$ be defined by

$$\tau(r, a) = rf(a)$$

Since this map is $\mathbb{Z}$-bilinear, there is a unique R-map $\tau'\colon R \otimes A \to M$ for which $\tau'(r \otimes a) = rf(a)$. Moreover,

$$\tau' \circ u_A\colon a \to \tau'(1 \otimes a) = f(a)$$

9. Note first that

$$\text{fix}(X, v) := \{x \in X | vx = x\} = vX$$

To see that F is a left adjoint of I, we show that there is a universal pair

$$(F(X, v), u_{(X,v)}: (X, v) \to IF(X, v))$$

that is,

$$(vX, u_{(X,v)}: (X, v) \to (vX, 1))$$

for every $(X, v) \in$ **Idem**. This amounts to showing that for every $f: (X, v) \to (Y, 1)$, that is, for every set function $f: X \to Y$ for which $f \circ v = f$, there is a unique $\tau: vX \to Y$ for which

$$\tau \circ u_{(X,v)} = f$$

If $u_{(X,v)}(x) = vX$, then this becomes

$$\tau \circ v = f$$

and so this uniquely defines τ as equal to $f|_{vX}$.
To see that F is a right adjoint of I, we must show that there is a couniversal pair

$$((X, 1), v_X: FIX \to X)$$

that is,

$$((X, 1), v_X: X \to X)$$

for every set X. Let $v_X = 1_X$. Given $f: vY \to X$, let $\tau: (Y, v) \to (X, 1)$ be defined by setting $\tau(y) = f(vy)$. Then $\tau(y) = \tau(vy)$ and so τ is a morphism. Also, $\tau(vy) = f(vy)$ and so $1 \circ F\tau = f$.

11. a) If $F: P \to Q$ is isotone then $p \le p'$ implies that $Fp \le Fp'$. Put in categorical terms, if $f: p \to p'$ then $Ff: Fp \to Fp'$. b) If

$$p \le GFp \quad \text{and} \quad q \le FGq$$

and if $p \le Gq$ then applying F gives

$$q \le FGq \le Fp$$

Also, if $q \le Fp$ then applying G gives

$$p \le GFp \le Gq$$

Conversely, if $p \leq Gq$ iff $q \leq Fp$ then since $Fp \leq Fp$, we have for $q = Fp$, $p \leq GF\,p$. The other is similar. c) Since $p \leq GFp$ there is a unique morphism $u_p: p \to GFp$. If $q \in Q$ and $f: p \to Gq$ in $\mathcal{P}$ then $p \leq Gq$ which implies that $q \leq Fp$ in $\mathcal{Q}$, whence there is a unique $\tau: q \to Fp$ in $\mathcal{Q}$ and so $\tau: Fp \to q$ in $\mathcal{Q}^{\mathrm{op}}$. Moreover, $G\tau: GFp \to Gq$ in $\mathcal{P}$ and so

$$p \leq GFp \leq Gq$$

which is the statement that $G\tau \circ u_p = f$. Thus, $(Fp, u_p: p \to GFp)$ is universal from p to G. d) If

$$u = \{u_p\}: I_{\mathcal{P}} \dot{\to} GF$$

is the unit of the adjoint pair, then $u_p: p \to GFp$ and so $p \leq GFp$. Similarly, if

$$v = \{v_q\}: FG \dot{\to} I_{\mathcal{Q}}$$

is the counit then $v_q: q \to FGq$ and so $q \leq FGq$. Thus, (F, G) is a Galois connection. (The fact that F and G are functors implies that they are order-reversing between $\mathcal{P}$ and $\mathcal{Q}$.)

13. For any $f: X \otimes V \to W$, we seek a unique $\tau_f: X \to \mathcal{L}(V, W)$ with the property that

$$e \circ (\tau_f \otimes 1_V) = f$$

that is,

$$\tau_f(x)(v) = f(x \otimes v)$$

But this uniquely defines τ_f as $\tau_f\,(x) = f\,(x \otimes \cdot\,)$. Note that

$$\begin{aligned}\tau_f(ax + by)(v) &= f((ax + by) \otimes v)\\ &= f(ax \otimes v + by \otimes v)\\ &= [a\tau_f(x) + b\tau_f(y)](v)\end{aligned}$$

and so τ_f is linear.

Index of Symbols

S. Roman, *An Introduction to the Language of Category Theory*, Compact Textbooks in Mathematics, DOI 10.1007/978-3-319-41917-6, © The Author(s) 2017

$\Rightarrow$ Functor
$\leftrightarrow$ Bijection
$\dot{\rightarrow}$ Natural transformation
$\dot{\leftrightarrow}$ Natural bijection
$\approx$ Isomorphism
$\dot{\approx}$ Natural isomorphism
$\dashv$ Left adjoint
$\vdash$ Right adjoint
1_A Identity morphism
$(A \to \mathcal{C})$ Comma category of arrows leaving A
$(\mathcal{C} \to A)$ Comma category of arrows entering A
$(A \to G)$ Comma category of arrows leaving A entering G
$(G \to A)$ Comma category of arrows entering A leaving G
$\mathcal{A}(v, w)$ Set of arcs between v and w in a digraph
$\mathcal{B} \times \mathcal{C}$ Product category
$\mathcal{C}^{\mathrm{op}}$ Opposite category
$\mathcal{C}, \mathcal{D}, \mathcal{E}$ Categories
$\mathcal{C}^{\rightarrow}$ Category of arrows
$\mathbf{Cone}_{\mathcal{C}}(F)$ *or* $\mathbf{Cone}_{\mathcal{C}}(\mathbb{D})$ Category of cones
$\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, etc. Diagrams
$\mathbb{D}(F\colon \mathcal{J} \Rightarrow \mathcal{C})$ Diagram in $\mathcal{C}$ with functor F and index category $\mathcal{J}$
$\mathbf{dia}_{\mathcal{J}}(\mathcal{C})$ Category of diagrams
$\mathcal{D}^{\mathcal{C}}$ Functor category
$f^{\leftarrow}$ Follow by f
$f^{\rightarrow}$ Preceed by f
$\hom_{\mathcal{C}}(A, B)$ Hom-set
$\hom_{\mathcal{C}}(A, -)$ Hom-set category
$\hom_{\mathcal{C}}(A, \cdot\,)$ Hom-set functor
$\mathcal{K}, \mathcal{L}$ Cones and cocones
$\mathbf{Mor}(\mathcal{C})$ Morphisms of $\mathcal{C}$
$\mathbf{Obj}(\mathcal{C})$ Objects of $\mathcal{C}$
$\mathcal{V}(D)$ Vertex class of a digraph

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