

List of Symbols

Symbol	Description
$\text{conv}(A)$	Convex hull of the set A
$\text{diag}(A)$	Vector with diagonal entries of the matrix A
$\text{Diag}(A_1, \dots, A_n)$	Block diagonal matrix with matrices A_i on the main diagonal
\mathbf{e}_i	i th standard unit vector
I_n	The $n \times n$ identity matrix
$\lambda_i(X)$	i th smallest eigenvalue of a symmetric matrix X
$\lambda_{\min}(X)$ ($\lambda_{\max}(X)$)	The smallest (largest) eigenvalue of a symmetric matrix X
\mathcal{M}_n	The vector space of $n \times n$ real matrices
$\mathcal{M}_{m,n}$	The vector space of $m \times n$ real matrices
\mathbb{S}_n	The vector space of $n \times n$ symmetric matrices
\mathbb{S}_n^+	The cone of positive semidefinite $n \times n$ matrices
\mathbb{S}_n^{++}	The cone of positive definite $n \times n$ matrices
$\text{tr}(A)$	Normalized trace of a square matrix A
$X \succeq Y$	$X - Y \in \mathbb{S}_n^+$ ($X - Y \succeq 0$)
$\langle x y \rangle$	$x^T y$
$\langle X Y \rangle$	$\text{tr}(X^T Y)$
$\langle \underline{X} \rangle$	The monoid, freely generated by the letters X_1, \dots, X_n
$\mathbb{R}\langle \underline{X} \rangle$	The free algebra with generating set $\{X_1, \dots, X_n\}$
$\text{Sym } \mathbb{R}\langle \underline{X} \rangle$	The set of all symmetric elements form $\mathbb{R}\langle \underline{X} \rangle$
Σ^2	Sums of hermitian squares
Σ_{2d}^2	Sums of hermitian squares of degree $\leq 2d$
$(\Sigma_{2d}^2)^\vee$	The dual cone to Σ_{2d}^2

\mathbf{W}_d	Vector with all words of degree $\leq d$
M_S	Quadratic module generated by S
$M_{S,2d}$	Truncated quadratic module generated by S with elements of degree $\leq 2d$
\mathcal{D}_S	Semialgebraic set associated to $S \subseteq \text{Sym} \mathbb{R}\langle \underline{X} \rangle$
$\mathcal{D}_S(k)$	$\mathcal{D}_S \cap \mathbb{S}_k^n$
\mathcal{D}_S^∞	Operator semialgebraic set
$\mathcal{D}_S^{\mathcal{F}}$	\mathcal{F} -semialgebraic set
\mathcal{N}_ε	An nc ε -neighborhood of 0
$\mathcal{B}(\underline{A}, \varepsilon)$	An nc ε -neighborhood of \underline{A}
$\text{cc}(f)$	Commutative collapse of f
$\text{deg}_\alpha w$	α -degree of f
$\underset{\text{cyc}}{\sim}$	Cyclic equivalence on $\mathbb{R}\langle \underline{X} \rangle$
$[f]$	Canonical representative of $f \in \mathbb{R}\langle \underline{X} \rangle$
$\lambda_{\min}(f)$	Smallest eigenvalue $f \in \text{Sym} \mathbb{R}\langle \underline{X} \rangle$ attains
$\lambda_{\min}(f, S)$	Smallest eigenvalue of $f \in \text{Sym} \mathbb{R}\langle \underline{X} \rangle$ on \mathcal{D}_S^∞
$f_{\text{sohs}}^{(s)}, L_{\text{sohs}}^{(s)}$	sth approximation for $\lambda_{\min}(f, S)$
\mathbb{B}, \mathbb{D}	Nc ball, nc polydisc (respectively)
$\text{tr}_{\min}(f)$	Smallest trace of f on \mathbb{S}^n
$\text{tr}_{\Theta^2}(f)$	SDP approximation for $\text{tr}_{\min}(f)$
$\text{tr}_{\min}(f, S)$	Smallest trace of f on \mathcal{D}_S
$\text{tr}_{\min}^{\text{II}_1}(f, S)$	Smallest trace of f on $\mathcal{D}_S^{\text{II}_1}$
$\text{tr}_{\Theta^2}^{(s)}(f, S), L_{\Theta^2}^{(s)}(f, S)$	SDP approximations for $\text{tr}_{\min}^{\text{II}_1}(f, S)$ of order s

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