Appendix A
Construction of the Euclidean Dirac Algebra

For a concise presentation of the construction of the Euclidean Dirac algebra as a specific case of Clifford algebras, see, for instance, [93, Chapter 11].

Definition A.1 Given $A = (a_{ij})_{i,j\in\{1,\dots,n\}} \in \mathbb{C}^{n\times n}$ and $B = (b_{ij})_{i,j\in\{1,\dots,m\}} \in \mathbb{C}^{m\times m}$, one defines their *Kronecker product* $A \circ B$ by

$$A \circ B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & \ddots & & \vdots \\ \vdots & & & \\ a_{n1}B & \cdots & & a_{nn}B \end{pmatrix}$$

$$= \left(a_{\lceil\frac{p}{m}\rceil\lceil\frac{q}{m}\rceil} b_{(((p-1) \bmod m)+1)(((q-1) \bmod m)+1)}\right)_{p,q\in\{1,\dots,mn\}} \in \mathbb{C}^{nm\times nm},$$

where $\lceil x\rceil := \min\{z \in \mathbb{Z} \,|\, z \geqslant x\}$ for all $x \in \mathbb{R}$ and $k \bmod \ell$ denotes the nonnegative integer $j \in \{0, \dots, \ell-1\}$ such that $k - j$ is divisible by ℓ, with $\ell, k \in \mathbb{Z}$.

Proposition A.2 *Let $n, m, \ell, k \in \mathbb{N}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times m}$, $C \in \mathbb{C}^{\ell\times\ell}$, $D \in \mathbb{C}^{k\times k}$. Then one concludes that*

$$A \circ (B \circ C) = (A \circ B) \circ C,$$

$$(A \circ B)^* = A^* \circ B^*,$$

$$\operatorname{tr}(A \circ B) = \operatorname{tr}(A)\operatorname{tr}(B),$$

$$\text{if } n = m \text{ and } \ell = k \text{ then, } AB \circ CD = (A \circ C)(B \circ D).$$

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Proof We only sketch a proof for the first assertion. It boils down to the following equations,

$$\left\lceil \frac{\left\lceil \frac{j}{k} \right\rceil}{m} \right\rceil = \left\lceil \frac{j}{m\,k} \right\rceil,$$

$$\left(\left(\left\lceil \frac{j}{k} \right\rceil - 1\right) \bmod m\right) + 1 = \left\lceil \frac{(j-1 \bmod mk) + 1}{k} \right\rceil,$$

$$(j-1 \bmod mk) \bmod k = j-1 \bmod k, \quad j \in \{1, \ldots, mnk\}.$$

The expressions on the left-hand side correspond to the indices of the entries of A, B and C, respectively, in $(A \circ B) \circ C$ and, similarly, the expressions on the right-hand sides correspond to the respective indices of the entries of A, B and C in $A \circ (B \circ C)$. □

Definition A.3 Introduce the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

in addition, define

$$\gamma_{1,2} := \sigma_1, \quad \gamma_{2,2} := \sigma_2.$$

Let $\hat{n} \in \mathbb{N}$. Recursively, one sets

$$\gamma_{k,2\hat{n}+1} := \gamma_{k,2\hat{n}}, \quad k \in \{1, \ldots, 2\hat{n}\},$$

$$\gamma_{2\hat{n}+1,2\hat{n}+1} := (-\mathrm{i})^{\hat{n}}\, \gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}},$$

and

$$\gamma_{k,2\hat{n}+2} := \sigma_1 \circ \gamma_{k,2\hat{n}}, \quad k \in \{1, \ldots, 2\hat{n}\},$$

$$\gamma_{2\hat{n}+1,2\hat{n}+2} := \mathrm{i}^{\hat{n}} \sigma_1 \circ \left(\gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}}\right),$$

$$\gamma_{2\hat{n}+2,2\hat{n}+2} := \sigma_2 \circ I_{2^{\hat{n}}},$$

with I_r the identity matrix in $\mathbb{C}^r$, $r \in \mathbb{N}$.

Remark A.4 By induction, one obtains

$$\gamma_{k,2\hat{n}}, \gamma_{k,2\hat{n}+1}, \gamma_{2\hat{n}+1,2\hat{n}+1} \in \mathbb{C}^{2^{\hat{n}} \times 2^{\hat{n}}}, \quad k \in \{1, \ldots, 2\hat{n}\}. \tag{A.1}$$

◇

Lemma A.5 *Let $\gamma_1, \ldots, \gamma_k \in \mathscr{B}(\mathscr{K})$ for some Hilbert space $\mathscr{K}$ and such that for all $j, k \in \{1, \ldots, k\}, j \neq k$, one has $\gamma_j \gamma_k + \gamma_k \gamma_j = 0$. Then*

$$\gamma_k \gamma_{k-1} \cdots \gamma_1 = (-1)^{k(k-1)/2} \gamma_1 \gamma_2 \cdots \gamma_k.$$

Proof The assertion being obvious for $k = 1$, we assume that the assertion of the lemma holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned} \gamma_{k+1} \gamma_k \gamma_{k-1} \cdots \gamma_1 &= (-1)^k \gamma_k \gamma_{k-1} \cdots \gamma_1 \gamma_{k+1} \\ &= (-1)^{[k(k-1)/2]+k} \gamma_1 \gamma_2 \cdots \gamma_k \gamma_{k+1} \\ &= (-1)^{k(k+1)/2} \gamma_1 \gamma_2 \cdots \gamma_k \gamma_{k+1}. \end{aligned}$$

□

Corollary A.6 *For all $k, l \in \{1, \ldots, n\}$, $n \in \mathbb{N}_{\geqslant 2}$, one has*

$$\gamma_{k,n} \gamma_{l,n} + \gamma_{l,n} \gamma_{k,n} = 2\delta_{kl} I_{2^{\hat{n}}},$$

where $\gamma_{j,n}$ is given in Definition A.3, $j \in \{1, \ldots, n\}$, and $\hat{n} \in \mathbb{N}$ is such that $n = 2\hat{n}$ or $n = 2\hat{n} + 1$.

Proof The assertion holds for $n = 2$. Assume that the assertion is valid for $n = 2\hat{n}$ for some $\hat{n} \in \mathbb{N}$. Then Lemma A.5 implies

$$\begin{aligned} \gamma_{2\hat{n}+1,2\hat{n}+1}^2 &= (-i)^{2\hat{n}} \left(\gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}}\right) \left(\gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}}\right) \\ &= (-1)^{\hat{n}} \left(\gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}}\right) (-1)^{2\hat{n}(2\hat{n}-1)/2} \left(\gamma_{2\hat{n},2\hat{n}} \cdots \gamma_{1,2\hat{n}}\right) \\ &= (-1)^{\hat{n}+2\hat{n}^2-\hat{n}} I_{2^{\hat{n}}} = I_{2^{\hat{n}}}. \end{aligned}$$

For $k \in \{1, \ldots, 2\hat{n} - 1\}$ one computes

$$\begin{aligned} \gamma_{k,2\hat{n}+1} \gamma_{2\hat{n}+1,2\hat{n}+1} &= \gamma_{k,2\hat{n}} (-i)^{\hat{n}} \left(\gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}}\right) \\ &= (-1)^{2\hat{n}-1} (-i)^{\hat{n}} \left(\gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}}\right) \gamma_{k,2\hat{n}} \\ &= -\gamma_{2\hat{n}+1,2\hat{n}+1} \gamma_{k,2\hat{n}+1}. \end{aligned}$$

Hence, the assertion is established for $\gamma_{k,2\hat{n}+1}$, $k \in \{1, \ldots, 2\hat{n} + 1\}$.

For $k, l \in \{1, \dots, 2\hat{n}\}$ one computes with the help of Proposition A.2,

$$\begin{aligned}\gamma_{k,2\hat{n}+2}\gamma_{l,2\hat{n}+2} + \gamma_{l,2\hat{n}+2}\gamma_{k,2\hat{n}+2} &= \left(\sigma_1 \circ \gamma_{k,2\hat{n}}\right)\left(\sigma_1 \circ \gamma_{l,2\hat{n}}\right) \\ &\quad + \left(\sigma_1 \circ \gamma_{l,2\hat{n}}\right)\left(\sigma_1 \circ \gamma_{k,2\hat{n}}\right) \\ &= \sigma_1^2 \circ \gamma_{k,2\hat{n}}\gamma_{l,2\hat{n}} + \sigma_1^2 \circ \gamma_{l,2\hat{n}}\gamma_{k,2\hat{n}} \\ &= I_2 \circ \left(\gamma_{k,2\hat{n}}\gamma_{l,2\hat{n}} + \gamma_{l,2\hat{n}}\gamma_{k,2\hat{n}}\right) \\ &= I_2 \circ 2\delta_{kl} I_{2^{\hat{n}}} = 2\delta_{kl} I_{2^{\hat{n}+1}}.\end{aligned}$$

One observes that

$$\begin{aligned}\gamma_{2\hat{n}+1,2\hat{n}+2}^2 &= \left(\mathrm{i}^{\hat{n}}\right)^2 \sigma_1^2 \circ \left(\gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}}\right)^2 \\ &= \left(\mathrm{i}^{\hat{n}}\right)^2 \sigma_1^2 \circ (-\mathrm{i})^{-2\hat{n}} I_{2^{\hat{n}}} = \left(\mathrm{i}^{\hat{n}}\right)^2 \sigma_1^2 \circ (-1)^{-2\hat{n}} \left(\mathrm{i}^{\hat{n}}\right)^{-2} I_{2^{\hat{n}}} = I_{2^{\hat{n}+1}},\end{aligned}$$

using $\gamma_{2\hat{n}+1,2\hat{n}+1}^2 = I_{2^{\hat{n}}}$. Moreover, $\gamma_{2\hat{n}+2,2\hat{n}+2}^2 = \sigma_2^2 \circ I_{2^{\hat{n}}} = I_{2^{\hat{n}+1}}$. In addition, one notes that

$$\begin{aligned}\gamma_{2\hat{n}+2,2\hat{n}+2}\gamma_{2\hat{n}+1,2\hat{n}+2} &= \sigma_2 \mathrm{i}^{\hat{n}} \sigma_1 \circ \gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}} \\ &= -\mathrm{i}^{\hat{n}} \sigma_1 \sigma_2 \circ \gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}} = -\gamma_{2\hat{n}+1,2\hat{n}+2}\gamma_{2\hat{n}+2,2\hat{n}+2}, \\ \gamma_{2\hat{n}+2,2\hat{n}+2}\gamma_{k,2\hat{n}+2} &= \sigma_2\sigma_1 \circ \gamma_{k,\hat{n}} = -\gamma_{k,2\hat{n}+2}\gamma_{2\hat{n}+2,2\hat{n}+2},\end{aligned}$$

and

$$\begin{aligned}\gamma_{2\hat{n}+1,2\hat{n}+2}\gamma_{k,2\hat{n}+2} &= \mathrm{i}^{\hat{n}} \sigma_1\sigma_1 \circ \gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}}\gamma_{k,2\hat{n}} \\ &= \sigma_1 \mathrm{i}^{\hat{n}} \sigma_1 \circ (-1)^{2\hat{n}-1} \gamma_{k,2\hat{n}}\gamma_{1,2\hat{n}} \cdots \gamma_{2\hat{n},2\hat{n}} \\ &= -\gamma_{k,2\hat{n}+2}\gamma_{2\hat{n}+1,2\hat{n}+2}\end{aligned}$$

for all $k \in \{1, \dots, 2\hat{n}\}$, implying the assertion. □

Corollary A.7 *For all $k \in \mathbb{N}$, $n \in \mathbb{N}_{\geqslant 2}$, and $k \leqslant n$, one has*

$$\gamma_{k,n}^* = \gamma_{k,n},$$

where $\gamma_{j,n}$ is given in Definition A.3, $j \in \{1, \dots, n\}$.

Proof We will proceed by induction. Before doing so, we note that due to Corollary A.6 and Lemma A.5, for all $k \in \{1, \ldots, n\}$,

$$\gamma_{k,n}\gamma_{k-1,n}\cdots\gamma_{1,n} = (-1)^{k(k-1)/2}\gamma_{1,n}\gamma_{2,n}\cdots\gamma_{k,n}.$$

One observes that $\gamma_{1,2}$ and $\gamma_{2,2}$ are self-adjoint. We assume that $\gamma_{k,2\hat{n}}$ is self-adjoint for all $k \in \{1, \ldots, 2\hat{n}\}$ for some $\hat{n} \in \mathbb{N}$. The only matrices not obviously self-adjoint using the induction hypothesis and Proposition A.2 are $\gamma_{2\hat{n}+1,2\hat{n}+2}$ and $\gamma_{2\hat{n}+1,2\hat{n}+1}$. Since the proof for either case follows along similar lines, it suffices to prove the self-adjointness of $\gamma_{2\hat{n}+1,2\hat{n}+2}$. For this purpose one computes,

$$\begin{aligned}
\gamma^*_{2\hat{n}+1,2\hat{n}+2} &= \left(\mathrm{i}^{\hat{n}}\sigma_1 \circ \left(\gamma_{1,2\hat{n}}\cdots\gamma_{2\hat{n},2\hat{n}}\right)\right)^* \\
&= \mathrm{i}^{\hat{n}}(-1)^{\hat{n}}\sigma_1^* \circ \left(\gamma_{1,2\hat{n}}\cdots\gamma_{2\hat{n},2\hat{n}}\right)^* \\
&= \mathrm{i}^{\hat{n}}(-1)^{\hat{n}}\sigma_1 \circ \left(\gamma_{2\hat{n},2\hat{n}}\cdots\gamma_{1,2\hat{n}}\right) \\
&= \mathrm{i}^{\hat{n}}(-1)^{\hat{n}+[2\hat{n}(2\hat{n}-1)/2]}\sigma_1 \circ \left(\gamma_{1,2\hat{n}}\cdots\gamma_{2\hat{n},2\hat{n}}\right) \\
&= \mathrm{i}^{\hat{n}}(-1)^{\hat{n}+2\hat{n}^2-\hat{n}}\sigma_1 \circ \left(\gamma_{1,2\hat{n}}\cdots\gamma_{2\hat{n},2\hat{n}}\right) \\
&= \gamma_{2\hat{n}+1,2\hat{n}+2}.
\end{aligned}$$

□

Next, we proceed to establish the following result on traces:

Proposition A.8 *Let $\hat{n} \in \mathbb{N}$ and suppose that $\gamma_{j,2\hat{n}}$, $\gamma_{j',2\hat{n}+1}$, $j \in \{1, \ldots, 2\hat{n}\}$, $j' \in \{1, \ldots, 2\hat{n}+1\}$, are given as in Definition A.3. Then,*

$$\begin{aligned}
&\mathrm{tr}\left(\gamma_{i_1,2\hat{n}+1}\cdots\gamma_{i_{2k+1},2\hat{n}+1}\right) = 0, \text{ if } i_1, \ldots, i_{2k+1} \in \{1, \ldots, 2\hat{n}+1\} \text{ and } k < \hat{n}, \\
&\mathrm{tr}\left(\gamma_{i_1,2\hat{n}}\cdots\gamma_{i_{2k+1},2\hat{n}}\right) = 0, \text{ if } i_1, \ldots, i_{2k+1} \in \{1, \ldots, 2\hat{n}\} \text{ and } k \in \mathbb{N}, \\
&\mathrm{tr}\left(\gamma_{i_1,2\hat{n}+1}\cdots\gamma_{i_{2\hat{n}+1},2\hat{n}+1}\right) = (2i)^{\hat{n}}\varepsilon_{i_1\cdots i_{2\hat{n}+1}}, \text{ if } i_1, \ldots, i_{2\hat{n}+1} \in \{1, \ldots, 2\hat{n}+1\},
\end{aligned}$$

where $\varepsilon_{i_1\cdots i_{2\hat{n}+1}}$ is the fully anti-symmetric symbol in $2\hat{n}+1$ dimensions, that is, $\varepsilon_{i_1\ldots i_{2\hat{n}+1}} = 0$ whenever $|\{i_1, \ldots, i_{2\hat{n}+1}\}| < 2\hat{n}+1$ and if the map $\pi\colon \{1, \ldots, 2\hat{n}+1\} \to \{1, \ldots, 2\hat{n}+1\}$ is bijective, then $\varepsilon_{\pi(1)\cdots\pi(2\hat{n}+1)} = \mathrm{sgn}(\pi)$.

Proof The first formula can be seen as follows. Since $k < \hat{n}$, there exists $i \in \{1, \ldots, 2\hat{n}+1\}\backslash\{i_1, \ldots, i_{2k+1}\}$, and one computes

$$\begin{aligned}
\mathrm{tr}\left(\gamma_{i_1,2\hat{n}+1}\cdots\gamma_{i_{2k+1},2\hat{n}+1}\right) &= \mathrm{tr}\left(\gamma_{i_1,2\hat{n}+1}\cdots\gamma_{i_{2k+1},2\hat{n}+1}\gamma^2_{i,2\hat{n}+1}\right) \\
&= \mathrm{tr}\left(\gamma_{i,2\hat{n}+1}\gamma_{i_1,2\hat{n}+1}\cdots\gamma_{i_{2k+1},2\hat{n}+1}\gamma_{i,2\hat{n}+1}\right) \\
&= -\,\mathrm{tr}\left(\gamma_{i_1,2\hat{n}+1}\gamma_{i,2\hat{n}+1}\cdots\gamma_{i_{2k+1},2\hat{n}+1}\gamma_{i,2\hat{n}+1}\right)
\end{aligned}$$

$$= \ldots = (-1)^{2k+1} \operatorname{tr}\left(\gamma_{i_1,2\hat{n}+1} \cdots \gamma_{i_{2k+1},2\hat{n}+1} \gamma_{i,2\hat{n}+1} \gamma_{i,2\hat{n}+1}\right)$$
$$= -\operatorname{tr}\left(\gamma_{i_1,2\hat{n}+1} \cdots \gamma_{i_{2k+1},2\hat{n}+1}\right).$$

Hence, $\operatorname{tr}\left(\gamma_{i_1,2\hat{n}+1} \cdots \gamma_{i_{2k+1},2\hat{n}+1} \gamma_{i,2\hat{n}+1} \gamma_{i,2\hat{n}+1}\right) = 0$.

The second assertion can be proved along the same lines.

The third assertion follows upon taking into account the cancellation and anti-commuting properties of the algebra in conjunction with the first statement, once the following equality has been established:

$$\operatorname{tr}\left(\gamma_{1,2\hat{n}+1} \cdots \gamma_{2\hat{n}+1,2\hat{n}+1}\right) = (2\mathrm{i})^{\hat{n}}.$$

To verify the latter identity one computes

$$\begin{aligned}
&\operatorname{tr}\left(\gamma_{1,2\hat{n}+3} \cdots \gamma_{2\hat{n}+3,2\hat{n}+3}\right) \\
&\quad = \operatorname{tr}\left(\gamma_{1,2\hat{n}+2} \cdots \gamma_{2\hat{n}+2,2\hat{n}+2} (-\mathrm{i})^{\hat{n}+1} \gamma_{1,2\hat{n}+2} \cdots \gamma_{2\hat{n}+2,2\hat{n}+2}\right) \\
&\quad = (-\mathrm{i})^{\hat{n}+1} \operatorname{tr}\Big(\left(\sigma_1 \circ \gamma_{1,2\hat{n}}\right) \cdots \left(\sigma_1 \circ \gamma_{2\hat{n},2\hat{n}}\right) \mathrm{i}^{\hat{n}} \left(\sigma_1 \circ \gamma_{1,2\hat{n}} \ldots \gamma_{2\hat{n},2\hat{n}}\right) \left(\sigma_2 \circ I_{2^{\hat{n}}}\right) \\
&\qquad \times \left(\sigma_1 \circ \gamma_{1,2\hat{n}}\right) \cdots \left(\sigma_1 \circ \gamma_{2\hat{n},2\hat{n}}\right) \mathrm{i}^{\hat{n}} \left(\sigma_1 \circ \gamma_{1,2\hat{n}} \ldots \gamma_{2\hat{n},2\hat{n}}\right) \left(\sigma_2 \circ I_{2^{\hat{n}}}\right)\Big) \\
&\quad = \left(\mathrm{i}^2\right)^{\hat{n}} (-\mathrm{i})^{\hat{n}+1} \operatorname{tr}\left(\sigma_1^{2\hat{n}+1} \sigma_2 \sigma_1^{2\hat{n}+1} \sigma_2 \circ \left(\gamma_{1,2\hat{n}} \ldots \gamma_{2\hat{n},2\hat{n}}\right)^4\right) \\
&\quad = (-1)^{\hat{n}+1} (-\mathrm{i})^{\hat{n}+1} \operatorname{tr}\left(\sigma_1 \sigma_1 \sigma_2 \sigma_2 \circ I_{2^{\hat{n}}}\right) = \mathrm{i}^{\hat{n}+1} 2^{\hat{n}+1}.
\end{aligned}$$

□

We conclude with the following result.

Corollary A.9 *Let $n \in \mathbb{N}_{\geqslant 2}$ be odd, V be a complex vector space, $k \in \mathbb{N}_0$, with $k+1 < n$, $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Let $\Phi \colon \{1, \ldots, n\}^n \to V$ be satisfying the property*

$$\begin{aligned}
&\sum_{(i_{k+3},\ldots,i_n)\in\{1,\ldots,n\}^{n-k-2}} \Phi(i_1, \ldots, i_k, i, j, i_{k+3}, \ldots, i_n) \\
&\quad = \sum_{(i_{k+3},\ldots,i_n)\in\{1,\ldots,n\}^{n-k-2}} \Phi(j_1, \ldots, j_k, j, i, i_{k+3}, \ldots, i_n), \quad i, j \in \{1, \ldots, n\}.
\end{aligned}$$

Then

$$\sum_{(i_{k+1},i_{k+2},i_{k+3},\ldots,i_n)\in\{1,\ldots,n\}^{n-k}} \operatorname{tr}\left(\gamma_{i_1,n} \cdots \gamma_{i_n,n}\right) \Phi(i_1, \ldots, i_n) = 0,$$

where $\gamma_{j,n}$, $j \in \{1, \ldots, n\}$, are given by Definition A.3.

Proof In the course of this proof we shall suppress the index n in $\gamma_{i,n}$.

$$\begin{aligned}
&\sum_{(i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\in\{1,\dots,n\}^{n-k}} \gamma_{i_1}\cdots\gamma_{i_n}\Phi(i_1,\dots,i_k,i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\\
&= \frac{1}{2}\sum_{(i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\in\{1,\dots,n\}^{n-k}} \gamma_{i_1}\cdots\gamma_{i_n}\Phi(i_1,\dots,i_k,i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\\
&\quad + \frac{1}{2}\sum_{(i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\in\{1,\dots,n\}^{n-k}} \gamma_{i_1}\cdots\gamma_{i_n}\\
&\qquad\qquad \times \Phi(i_1,\dots,i_k,i_{k+2},i_{k+1},i_{k+3},\dots,i_n)\\
&= \frac{1}{2}\sum_{(i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\in\{1,\dots,n\}^{n-k}}\\
&\quad \left(\gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+1}}\gamma_{i_{k+2}}\gamma_{i_{k+3}}\cdots\gamma_{i_n} + \gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+2}}\gamma_{i_{k+1}}\gamma_{i_{k+3}}\cdots\gamma_{i_n,n}\right)\\
&\quad \times \Phi(i_1,\dots,i_n)\\
&= \frac{1}{2}\sum_{(i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\in\{1,\dots,n\}^{n-k},\, i_{k+1}\neq i_{k+2}}\\
&\quad \left(\gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+1}}\gamma_{i_{k+2}}\gamma_{i_{k+3}}\cdots\gamma_{i_n} + \gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+2}}\gamma_{i_{k+1}}\gamma_{i_{k+3}}\cdots\gamma_{i_n}\right)\\
&\quad \times \Phi(i_1,\dots,i_n)\\
&\quad + \frac{1}{2}\sum_{(i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\in\{1,\dots,n\}^{n-k},\, i_{k+1}= i_{k+2}}\\
&\quad \left(\gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+1}}\gamma_{i_{k+2}}\gamma_{i_{k+3}}\cdots\gamma_{i_n} + \gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+2}}\gamma_{i_{k+1}}\gamma_{i_{k+3}}\cdots\gamma_{i_n}\right)\\
&\quad \times \Phi(i_1,\dots,i_n)\\
&= \frac{1}{2}\sum_{(i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\in\{1,\dots,n\}^{n-k},\, i_{k+1}= i_{k+2}}\\
&\quad \left(\gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+3}}\cdots\gamma_{i_n} + \gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+3}}\cdots\gamma_{i_n}\right)\Phi(i_1,\dots,i_n)\\
&= \sum_{(i_{k+1},i_{k+2},i_{k+3},\dots,i_n)\in\{1,\dots,n\}^{n-k},\, i_{k+1}= i_{k+2}} \gamma_{i_1}\cdots\gamma_{i_k}\gamma_{i_{k+3}}\cdots\gamma_{i_n}\Phi(i_1,\dots,i_n).
\end{aligned}$$

Applying the internal trace to the latter sum, one infers that each term vanishes by Proposition A.8. □

Appendix B
A Counterexample to [22, Lemma 5]

In this appendix we shall provide a counterexample for the trace class property asserted in [22, Lemma 5]. The counterexample is constructed in dimension $n = 3$ and recorded in Theorem B.5.

Analogously to Example 4.8, we let Φ assume values in the 2×2 matrices and denote the Pauli matrices (see also Example 4.8) again by $\sigma_j, j \in \{1, 2, 3\}$. Before we give an explicit formula for Φ, we need the following definitions. Let $\phi_1 \in C^\infty(\mathbb{R})$ be a function interpolating between 0 and 1 with

$$\phi_1(x) = \begin{cases} 0, & x \leqslant 0, \\ 1, & x \geqslant 1, \end{cases} \quad x \in \mathbb{R}, \quad \phi_2 := \phi_1(-(\cdot + 1))$$

and let

$$\phi_{1,r,t} := \phi_1\big(t^{-1}(\cdot) - t^{-1}r\big), \quad \phi_{2,r,t} := \phi_2\big(t^{-1}(\cdot) - t^{-1}r\big), \quad r, t > 0.$$

For $r_1, r_2, t_1, t_2 \in (0, \infty)$ with $r_1 + t_1 < r_2 - t_2$, this yields the following variant of a smooth "cut-off" function

$$\psi_{r_1,r_2,t_1,t_2} := \phi_{1,r_1,t_1}\phi_{2,r_2-t_2,t_2}. \tag{B.1}$$

One notes that $\psi_{r_1,r_2,t_1,t_2} \in C^\infty(\mathbb{R})$. We will use the following properties of ψ_{r_1,r_2,t_1,t_2} (all of them are easily checked):

$$0 \leqslant \psi_{r_1,r_2,t_1,t_2} \leqslant 1, \tag{B.2}$$

$$\psi_{r_1,r_2,t_1,t_2}|_{[r_1+t_1,r_2-t_2]} = 1, \tag{B.3}$$

$$\psi_{r_1,r_2,t_1,t_2}|_{\mathbb{R}\setminus[r_1,r_2]} = 0, \tag{B.4}$$

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$$|\psi'_{r_1,r_2,t_1,t_2}| \leqslant d_1 \left(\frac{1}{t_1} \vee \frac{1}{t_2}\right) \text{ on } [r_1, r_1 + t_1] \cup [r_2 - t_2, r_2], \tag{B.5}$$

$$|\psi^{(\ell)}_{r_1,r_2,t_1,t_2}| \leqslant d_\ell \left(\frac{1}{t_1^\ell} \vee \frac{1}{t_2^\ell}\right), \quad \ell \in \mathbb{N}_{\geqslant 2}, \tag{B.6}$$

with $d_1 := \|\phi'_1\|_\infty := \sup_{x\in\mathbb{R}} |\phi'_1(x)|$ and $d_\ell := \|\phi_1^{(\ell)}\|_\infty$, $\ell \in \mathbb{N}_{\geqslant 2}$. For $k \in \mathbb{N}_{>1}$ define

$$r_k := \sum_{j=1}^{k-1} 2^j = 2^k - 2,$$

$$\psi_{1,k} := \psi_{r_k, r_{k+1}, \frac{1}{2}2^k, \frac{1}{20}2^k}, \quad \psi_{2,k} := \psi_{r_k, r_{k+1}, \frac{1}{36}2^k, \frac{17}{18}2^k}.$$

One observes that

$$r_k + \frac{1}{2}2^k = r_{k+1} - \frac{1}{2}2^k < r_{k+1} - \frac{1}{20}2^k, \quad r_k + \frac{1}{36}2^k = r_{k+1} - \frac{35}{36}2^k < r_{k+1} - \frac{17}{18}2^k,$$

so that $\psi_{1,k}$ and $\psi_{2,k}$ are well-defined. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we let $\Phi\colon \mathbb{R}^3 \to \mathbb{C}^{2\times 2}$ be defined as follows,

$$\Phi(x) := \sum_{j=1}^{3} \sigma_j + \sum_{k=2}^{\infty} \frac{1}{k^{1/3}} \sum_{j=1}^{3} \sigma_j\, \xi_{k,j}(x), \quad x \in \mathbb{R}^3, \tag{B.7}$$

where

$$\xi_{k,j}(x) := \frac{1}{r_{k+1}} \psi_{1,k}(|x|)(x_j - r_k)\psi_{2,k}(x_j), \quad x \in \mathbb{R}^3. \tag{B.8}$$

One observes that $\Phi \in C^\infty(\mathbb{R})$. Next, we introduce the sets

$$B_k := \{x \in \mathbb{R}^3 \,|\, r_k \leqslant |x| \leqslant r_{k+1}\} \cap \bigcup_{j\in\{1,2,3\}} \{x \in \mathbb{R}^3 \,|\, r_k \leqslant x_j \leqslant r_{k+1}\}, \quad k \in \mathbb{N}, \tag{B.9}$$

and

$$\widetilde{B}_k := \left\{x \in \mathbb{R}^3 \,\middle|\, r_k + \frac{1}{2}2^k \leqslant |x| \leqslant r_{k+1} - \frac{1}{20}2^k\right\} \bigcap \left\{x \in \mathbb{R}^3 \,\middle|\, r_k + \frac{1}{36}2^k \leqslant x_1, x_2, x_3 \leqslant r_{k+1} - \frac{17}{18}2^k\right\}, \quad k \in \mathbb{N}. \tag{B.10}$$

Before turning to the properties of Φ, we study $\xi_{k,j}$ first.

Lemma B.1 *Let $j \in \{1, 2, 3\}$, $\ell \in \{1, 2, 3\}$, $\xi_{k,j}$ as in* (B.8)*, B_k, $\widetilde{B}_k$ as in* (B.9) *and* (B.10)*, respectively. Then the following assertions* (α)–(γ) *hold:*

(α) *For all $k \in \mathbb{N}$, $x \in \mathbb{R}^3$, $\xi_{k,j}(x) \neq 0$ implies $x \in B_k$.* (B.11)

(β) *For all $\alpha \in \mathbb{N}_0^3$, there exists $\kappa > 0$ such that for all $k \in \mathbb{N}$,*

$$|\partial^\alpha \xi_{k,j}(x)| \leqslant \kappa(1 + |x|)^{-|\alpha|}, \quad x \in B_k. \tag{B.12}$$

(γ) *For all $\ell \in \{1, 2, 3\}$, and all $k \in \mathbb{N}$, $\partial_\ell \xi_{k,j}(x) = \delta_{\ell j}$, $x \in \widetilde{B}_k$.* (B.13)

Proof (B.11): The assertion follows from (B.4) and the definition of B_k.
(B.12): One observes that $\psi_{2,k} \neq 0$ on (r_k, r_{k+1}) and that $0 \leqslant \psi_{2,k} \leqslant 1$ by (B.4) and (B.2); hence,

$$|(x_j - r_k)\psi_{2,k}(x_j)| \leqslant 2^k, \quad j \in \{1, 2, 3\},\ k \in \mathbb{N}_{\geqslant 2}.$$

One recalls,

$$r_k = \sum_{j=1}^{k-1} 2^j = 2^k - 2 < r_{k+1} = 2^{k+1} - 2 = 2(2^k - 1),$$

in particular, $(1/r_{k+1}) \leqslant \kappa_0 2^{-k}$ for some $\kappa_0 > 0$. Hence,

$$\left\| \frac{1}{r_{k+1}} \psi_{1,k}(|x|) \sum_{j=1}^{3} \sigma_j (x_j - r_k)\psi_{2,k}(x_j) \right\| \leqslant \chi_{B_k}(x)\kappa_0, \quad x \in \mathbb{R}^3,$$

with B_k introduced in (B.9). Thus, (B.12) holds for $\ell = 0$. Next, for the first partial derivatives in item (B.12) one computes for $\ell \neq j$,

$$(\partial_\ell \xi_{k,j})(x) = \frac{1}{r_{k+1}} \psi'_{1,k}(|x|) \frac{x_\ell}{|x|} (x_j - r_k)\psi_{2,k}(x_j)$$

and for $\ell = j$,

$$\begin{aligned}(\partial_j \xi_{k,j})(x) = \frac{1}{r_{k+1}} \psi'_{1,k}(|x|) \frac{x_j}{|x|} (x_j - r_k)\psi_{2,k}(x_j) + \frac{1}{r_{k+1}} \psi_{1,k}(|x|)\sigma_j \psi_{2,k}(x_j) \\ + \frac{1}{r_{k+1}} \psi_{1,k}(|x|)\sigma_j (x_j - r_k)\psi'_{2,k}(x_j), \quad j \in \{1, 2, 3\}.\end{aligned}$$

For $x \in B_k$, one has $|(x_j - r_k)\psi_k'(x_j)| \leqslant c$ by (B.5), $|\psi_k'(|x|)(x_\ell - r_k)\psi_k(x_\ell)| \leqslant c$ by (B.2) and (B.5) and for some $\kappa, c > 0$ and all $k \in \mathbb{N}_{\geqslant 2}$,

$$\left\| \frac{1}{r_{k+1}} \psi_k(|x|)\sigma_j \psi_k(x_j) \right\| \leqslant \left\| \frac{1}{r_{k+1}} \psi_k(|x|) \right\| \leqslant \kappa(1 + |x|)^{-1}$$

since for all $x \in B_k$ one has $r_{k+1} \geqslant |x|$. Higher-order derivatives can be treated similarly, using (B.6), proving assertion (B.12).

(B.13): This is obvious. □

The next lemma gives an account of the asymptotic properties of Φ and its derivatives.

Lemma B.2 *Let Φ be given by* (B.7)*. Then the following assertions* (α)–(γ) *hold:*

(α) *Φ is bounded, pointwise self-adjoint, $\Phi \in C^\infty(\mathbb{R}^3; \mathbb{C}^{2\times 2})$, $\Phi(x)^{-1}$ exists for all $x \in \mathbb{R}^3$, and $\Phi(x)^2 \underset{|x|\to\infty}{\longrightarrow} I_2$.*

(β) *There exists $\kappa > 0$ such that*

$$|(\partial_j \Phi)(x)| \leqslant \kappa(1 + |x|)^{-1}, \quad x \in \mathbb{R}^3,\ j \in \{1, 2, 3\},$$

and the formula

$$(\partial_j \Phi)(x) = k^{-1/3}\sigma_j \quad x \in \widetilde{B}_k,\ j \in \{1, 2, 3\},\ k \in \mathbb{N},$$

holds, where $\widetilde{B}_k$ is given by (B.10)*.*

(γ) *For all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geqslant 2$, there exists $\kappa' > 0$, such that*

$$|(\partial^\alpha \Phi)(x)| \leqslant \kappa' (1 + |x|)^{-|\alpha|}, \quad x \in \mathbb{R}^3.$$

Proof For item (α), we use Lemma B.1 (B.12) with $\ell = 0$ together with the fact that $B_k \cap B_{k'} = \emptyset$ for $k' > k+1$, so Φ is bounded. Φ is easily verified to be pointwise self-adjoint. For showing invertibility of Φ, one computes for $x \in B_k$,

$$\begin{aligned}
\Phi(x)\Phi(x) &= \left(\sum_{j=1}^{3} \left(\sigma_j + \frac{1}{k^{1/3}} \xi_{k,j}(x)\sigma_j \right) \right)^2 \\
&= \sum_{j=1}^{3} \left(1 + \frac{1}{k^{1/3}} \xi_{k,j}(x) \right)^2 I_2 \\
&= \sum_{j=1}^{3} \left(1 + 2\frac{1}{k^{1/3}} \xi_{k,j}(x) + \left(\frac{1}{k^{1/3}} \xi_{k,j}(x) \right)^2 \right) I_2 \geqslant I_2,
\end{aligned}$$

implying (α). Item (β) follows from Lemma B.1, (B.12), and (B.13), whereas item (γ) follows from (B.1), (B.12). □

In order to prove that $\mathrm{tr}_4\left((L^*L+z)^{-1}-(LL^*+z)^{-1}\right)$ for $L=\mathcal{Q}+\Phi$ (with $\mathcal{Q}$ as in (6.3)) is *not* trace class for z in a neighborhood of 0, we need to invoke the following general statement:

Theorem B.3 ([16, Theorem 3.1]) *Let $K\in\mathscr{B}(L^2(\mathbb{R}^n))$ be an operator induced by a continuous integral kernel $k\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$. Assume that $K\in\mathscr{B}_1\big(L^2(\mathbb{R}^n)\big)$. Then the function $x\mapsto k(x,x)$ defines an element of $L^1(\mathbb{R}^n)$.*

Before we state and prove the main result of this chapter, we need to study the volume of $\widetilde{B}_k$:

Lemma B.4 *Let $\widetilde{B}_k$, $k\in\mathbb{N}$, be as in (B.10). Then there exists $k_0\in\mathbb{N}$, such that for all $k\in\mathbb{N}_{\geqslant k_0}$,*

$$\mathrm{vol}\left(\widetilde{B}_k\right)=2^{3k}/(36)^3.$$

Proof Let $k\in\mathbb{N}$. One observes that if

$$x\in\left\{x\in\mathbb{R}^3\,\middle|\, r_k+\frac{1}{36}2^k\leqslant x_1,x_2,x_3\leqslant r_{k+1}-\frac{17}{18}2^k\right\},$$

then

$$\sqrt{3}\left(r_k+\frac{1}{36}2^k\right)\leqslant|x|\leqslant\sqrt{3}\left(r_{k+1}-\frac{17}{18}2^k\right).$$

Since $16/10\leqslant\sqrt{3}\leqslant 18/10$, for sufficiently large $k\in\mathbb{N}$, the estimates

$$\sqrt{3}\left(r_k+\frac{1}{36}2^k\right)\geqslant\left(\frac{16}{10}+\frac{16}{10}\frac{1}{36}\right)2^k-\frac{16}{10}2\geqslant\frac{3}{2}2^k-2=r_k+\frac{1}{2}2^k,$$

and

$$\sqrt{3}\left(r_{k+1}-\frac{17}{18}2^k\right)\leqslant\frac{18}{10}\left(2-\frac{17}{18}\right)2^k-2\frac{18}{10}\leqslant\frac{19}{10}2^k-2=r_{k+1}-\frac{1}{20}2^k,$$

hold. Consequently, for sufficiently large $k\in\mathbb{N}$,

$$\left\{x\in\mathbb{R}^3\,\middle|\, r_k+\frac{1}{36}2^k\leqslant x_1,x_2,x_3\leqslant r_{k+1}-\frac{17}{18}2^k\right\}\subseteq\widetilde{B}_k.$$

Hence, there exists $k_0\in\mathbb{N}$, such that for all $k\in\mathbb{N}_{\geqslant k_0}$,

$$\mathrm{vol}\left(\widetilde{B}_k\right)=\left(r_{k+1}-\frac{17}{18}2^k-(r_k+\frac{1}{36}2^k)\right)^3.$$

□

Theorem B.5 *Let $n = 3$ and $\mathscr{Q}$ and Φ be given by* (6.3) *and* (B.7)*, respectively. Then there exists $\delta > 0$ such that for $L = \mathscr{Q} + \Phi$, and for any real $z \in B(0, \delta)\backslash\{0\}$,*

$$\operatorname{tr}_4\left((L^*L + z)^{-1} - (LL^* + z)^{-1}\right) \notin \mathscr{B}_1\big(L^2(\mathbb{R}^3)\big).$$

Proof In view of Remark 11.3 and Lemma 7.7 it suffices to check whether or not

$$\widetilde{T} := \operatorname{tr}_4\left((R_{1+z}C)^3 R_{1+z}\right)$$

is a trace class operator, where $C = [\mathscr{Q}, \Phi]$, and R_{1+z} are given by (2.2) and (4.6), respectively.

Arguing by contradiction, we shall assume that $\widetilde{T} \in \mathscr{B}_1\big(L^2(\mathbb{R}^3)\big)$. One observes,

$$\begin{aligned}(R_{1+z}C)^3 R_{1+z} &= R_{1+z}CR_{1+z}CR_{1+z}CR_{1+z}\\ &= [R_{1+z}, C]R_{1+z}CR_{1+z}CR_{1+z} + CR_{1+z}R_{1+z}CR_{1+z}CR_{1+z}\\ &= [R_{1+z}, C]R_{1+z}CR_{1+z}CR_{1+z} + CR_{1+z}[R_{1+z}, C]R_{1+z}CR_{1+z}\\ &\quad + CR_{1+z}CR_{1+z}R_{1+z}CR_{1+z}\\ &= [R_{1+z}, C]R_{1+z}CR_{1+z}CR_{1+z} + CR_{1+z}[R_{1+z}, C]R_{1+z}CR_{1+z}\\ &\quad + CR_{1+z}CR_{1+z}[R_{1+z}, C]R_{1+z} + CR_{1+z}CR_{1+z}CR_{1+z}R_{1+z}.\end{aligned} \tag{B.14}$$

By Lemmas 4.5 and B.2, one gets $CR_{1+z}, R_{1+z}C \in \mathscr{B}_4\big(L^2(\mathbb{R}^3)\big)$ and $[R_{1+z}, C] \in \mathscr{B}_2\big(L^2(\mathbb{R}^3)\big)$. Hence, by Theorem 4.2, one infers that despite the last term in (B.14), all operators are trace class. In addition, one computes

$$\begin{aligned}CR_{1+z}CR_{1+z}CR_{1+z}R_{1+z} &= C[R_{1+z}, C]R_{1+z}CR_{1+z}R_{1+z}\\ &\quad + C^2R_{1+z}R_{1+z}CR_{1+z}R_{1+z}\end{aligned} \tag{B.15}$$

$$\begin{aligned}&= C[R_{1+z}, C]R_{1+z}CR_{1+z}R_{1+z} + C^2R_{1+z}[R_{1+z}, C]R_{1+z}R_{1+z}\\ &\quad + C^2R_{1+z}CR_{1+z}^3\\ &= C[R_{1+z}, C]R_{1+z}CR_{1+z}R_{1+z} + C^2R_{1+z}[R_{1+z}, C]R_{1+z}R_{1+z}\\ &\quad + C^2[R_{1+z}, C]R_{1+z}^3 + C^3R_{1+z}^4.\end{aligned} \tag{B.16}$$

Next, one notes that Lemma 4.4 implies the relation

$$[R_{1+z}, C] = R_{1+z}\,(\Delta C)\,R_{1+z} + 2R_{1+z}\,(\mathscr{Q}C)\,\mathscr{Q}R_{1+z}.$$

With the help of Lemma B.2, there exists $\kappa > 0$ such that

$$\max\{\|C(x)^2\|, \|(\Delta C)(x)\|, \|(\mathscr{Q}C)(x)\|\} \leqslant \kappa(1 + |x|)^{-2}, \quad x \in \mathbb{R}^3.$$

Therefore, Lemma 4.5 and Theorem 4.2 imply

$$\begin{aligned}C[R_{1+z}, C]R_{1+z}CR_{1+z} &= C(R_{1+z}(\Delta C)R_{1+z} \\ &\qquad + 2R_{1+z}(\mathscr{Q}C)\mathscr{Q}R_{1+z})R_{1+z}CR_{1+z} \\ &= CR_{1+z}(\Delta C)R_{1+z}CR_{1+z} + 2CR_{1+z}\,(\mathscr{Q}C)\,R_{1+z}\mathscr{Q}R_{1+z}CR_{1+z} \\ &\in \mathscr{B}_4 \cdot \mathscr{B}_2 \cdot \mathscr{B}_4 + \mathscr{B}_4 \cdot \mathscr{B}_2 \cdot \mathscr{B} \cdot \mathscr{B}_4 \subseteq \mathscr{B}_1,\end{aligned}$$

and,

$$C^2R_{1+z}[R_{1+z}, C]R_{1+z} \in \mathscr{B}_2 \cdot \mathscr{B}_2 \cdot \mathscr{B} \subseteq \mathscr{B}_1,$$

as well as,

$$\begin{aligned}C^2[R_{1+z}, C]R^3_{1+z} &= C^2R_{1+z}\,((\Delta C)\,R_{1+z} + 2R_{1+z}\,(\mathscr{Q}C)\,\mathscr{Q}R_{1+z})\,R^3_{1+z} \\ &= C^2R_{1+z}\,(\Delta C)\,R_{1+z}R^3_{1+z} + 2C^2R_{1+z}R_{1+z}\,(\mathscr{Q}C)\,QR_{1+z}R^3_{1+z} \\ &\in \mathscr{B}_2 \cdot \mathscr{B}_2 \cdot \mathscr{B} + \mathscr{B}_2 \cdot \mathscr{B}_2 \cdot \mathscr{B} \subseteq \mathscr{B}_1.\end{aligned}$$

Noting that the inner trace maps trace class operators to trace class operators (cf. Remark 3.2), and combining (B.14) and (B.16) together with our assumption that $\widetilde{T}$ is trace class, one concludes that

$$T := \mathrm{tr}_4\left(C^3R^4_{1+z}\right) = \mathrm{tr}_4\left(C^3\right)R^4_{1+z} \in \mathscr{B}_1\left(L^2(\mathbb{R}^3)\right).$$

Next, one observes that T is an integral operator induced by the following integral kernel

$$\begin{aligned}t\colon (x, y) \mapsto \int_{(\mathbb{R}^3)^3} &\mathrm{tr}_4\left(C^3\right)(x)r_{1+z}(x - x_1)r_{1+z}(x_1 - x_2)r_{1+z}(x_2 - x_3)r_{1+z}(x_3 - y) \\ &\times\ \mathrm{d}^3x_1\,\mathrm{d}^3x_2\,\mathrm{d}^3x_3,\end{aligned}$$

where r_{1+z} is the Helmholtz Green's function, see (5.11) associated with $(-\Delta + (1 + z))^{-1}$. By Theorem 5.1 (and Proposition 5.4), t is continuous. As T is trace class, Theorem B.3 implies that the map $x \mapsto t(x, x)$ generates an $L^1(\mathbb{R}^3)$-function. Hence,

$$\begin{aligned}&\int_{\mathbb{R}^3} |t(x, x)|\,\mathrm{d}^3x \\ &\quad = \int_{\mathbb{R}^3} \bigg| \int_{(\mathbb{R}^3)^3} \mathrm{tr}_4\left(C^3\right)(x)r_{1+z}(x - x_1)r_{1+z}(x_1 - x_2)r_{1+z}(x_2 - x_3)r_{1+z}(x_3 - x) \\ &\qquad\qquad \times\ \mathrm{d}^3x_1\,\mathrm{d}^3x_2\,\mathrm{d}^3x_3 \bigg|\,\mathrm{d}^3x\end{aligned}$$

$$= \int_{\mathbb{R}^3} \left| \int_{(\mathbb{R}^3)^3} \operatorname{tr}_4 \left(C^3\right)(x) r_{1+z}(x_1) r_{1+z}(x_1 - x_2) r_{1+z}(x_2 - x_3) r_{1+z}(x_3) \right.$$

$$\left. \times \; \mathrm{d}^3 x_1 \, \mathrm{d}^3 x_2 \, \mathrm{d}^3 x_3 \right| \mathrm{d}^3 x$$

$$= \int_{\mathbb{R}^3} \left| \operatorname{tr}_4 \left(C^3\right)(x) \right| \mathrm{d}^3 x \left\langle \delta_{\{0\}}, R^4_{1+z} \delta_{\{0\}} \right\rangle < \infty.$$

In other words,

$$\operatorname{tr}_4 \left(C^3\right) \in L^1(\mathbb{R}^3). \tag{B.17}$$

The rest of the proof aims at showing that the statement (B.17) is false. For this purpose we need to compute $\operatorname{tr}_4 \left([\mathscr{Q}, \Phi]^3\right)$ on $\bigcup_{k \in \mathbb{N}_{\geqslant 2}} \widetilde{B}_k$, with $\widetilde{B}_k$ given in (B.10). We recall from Lemma B.2 (*ii*),

$$(\partial_j \Phi)(x) = \frac{1}{k^{1/3}} \frac{1}{r_{k+1}} \sigma_j, \quad x \in \widetilde{B}_k, \; j \in \{1, 2, 3\}.$$

Hence,

$$\begin{aligned} \operatorname{tr}_4 \left([\mathscr{Q}, \Phi]^3\right)(x) &= \sum_{j,m,\ell=1}^{3} 2\mathrm{i}\varepsilon_{jm\ell} \operatorname{tr}_2 \left((\partial_j \Phi)(x)(\partial_m \Phi)(x)(\partial_\ell \Phi)(x)\right) \\ &= \sum_{j,m,\ell=1}^{3} 2\mathrm{i}\varepsilon_{jm\ell} \frac{1}{k} \frac{1}{r^3_{k+1}} \operatorname{tr}_2 \left(\sigma_j \sigma_m \sigma_\ell\right) \\ &= -\sum_{j,m,\ell=1}^{3} 4\varepsilon^2_{jm\ell} \frac{1}{k} \frac{1}{r^3_{k+1}} \\ &= -24 \frac{1}{k} \frac{1}{r^3_{k+1}}, \end{aligned}$$

implying,

$$\left| \operatorname{tr}_4 \left([\mathscr{Q}, \Phi]^3\right)(x) \right| \geqslant 24 \frac{1}{k} \frac{1}{r^3_{k+1}}, \quad x \in \widetilde{B}_k, \; k \in \mathbb{N}_{\geqslant 2}. \tag{B.18}$$

However, employing Lemma B.4 one infers with the help (B.18) that for some $k_0 \in \mathbb{N}$,

$$\begin{aligned}\operatorname{tr}_4\left(C^3\right) &= \left\| \operatorname{tr}_4\left([\mathcal{Q}, \Phi]^3\right)\right\|_{L^1(\mathbb{R}^3)} \geqslant \sum_{k=k_0}^{\infty} \frac{1}{k}\frac{1}{r_{k+1}^3}\operatorname{vol}\left(\widetilde{B}_k\right) \\ &= \frac{1}{(36)^3}\sum_{k=k_0}^{\infty}\frac{1}{k}\frac{1}{r_{k+1}^3}2^{3k} = \frac{1}{(36)^3}\sum_{k=k_0}^{\infty}\frac{1}{k}\frac{1}{(2^k-2)^3}2^{3k} = \infty,\end{aligned}$$

contradicting (B.17). □

Remark B.6 It might be of interest to compute the index of $\mathcal{Q}+\Phi$, with the potential Φ constructed in this chapter: One notes that Φ is a $\mathcal{Q}$-compact perturbation of the operator

$$\mathcal{Q} + U \text{ in } L^2(\mathbb{R}^n), \quad \text{where } U := \sum_{j=1}^{3} \sigma_j.$$

Since $U^2 = I_2$ and $\partial_j U = 0, j \in \{1, 2, 3\}$, one infers that U is admissible. The index formula in Theorem 10.1 leads to $\operatorname{ind}(\mathcal{Q} + U) = 0$, and hence to $\operatorname{ind}(\mathcal{Q} + \Phi) = 0$.

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