Appendix A
Describing Function

Describing Function (DF) is a classical tool for analyzing the existence of limit cycles in nonlinear systems based in the frequency-domain approach. Although this method is not as general as the analysis for linear system is, it gives good approximated results for relay feedback systems.

The idea of the method using the scheme presented in Fig. A.1 is to obtain the DF of the single-input–single-output control block, which is assumed nonlinear, and according to [39] the definition of DF is the complex fundamental-harmonic gain of a nonlinearity in the presence of a driving sinusoid. Consider the input signal of the control block as $\sigma(t) = A\sin(\omega t)$ and its output signal, presented by its Fourier representation, as

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega t) + b_n \sin(n\omega t)].$$

A restriction for this approach is that in the above scheme only one block can be nonlinear, so a common case might be considering the plant to be linear and allowing the control block to be the nonlinear part, covering in such a way a wide variety of real engineering problem like saturation, dead zones, Coulomb friction or backlash. Also linear plant has to be time invariance and regarding the approximation of the output $u(t)$ via its Fourier representation, the analysis is done with null offset ($a_0 = 0$) and taking only the first harmonic ($n = 1$) of the Fourier series of this signal:

$$\frac{u(t)}{\sigma(t)} \simeq \frac{a_1 \cos(\omega t) + b_1 \sin(\omega t)}{A\sin(\omega t)} \tag{A.1}$$

due to this last consideration, most of the information of $u(t)$ has to be contained in the first harmonic term, otherwise approximation will be extremely poor, for achieving this, the linear element must have low-pass properties (filtering hypothesis), it

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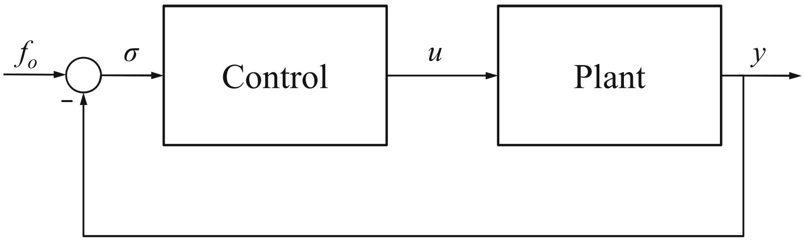

Fig. A.1 Basic scheme for DF analysis

means that

$$|G(j\omega)| \gg |G(jn\omega)| \qquad n = 2, 3, \ldots$$

Using the trigonometric identity $a_1 \cos(\omega t) + b_1 \sin(\omega t) = \sqrt{a_1^2 + b_1^2}\sin(\omega t + \phi)$, where $\phi = \arctan(a_1/b_1)$, in Eq. (A.1), we get

$$\frac{u(t)}{\sigma(t)} \simeq \frac{\sqrt{a_1^2 + b_1^2}\sin(\omega t + \phi)}{A\sin(\omega t)}.$$

According to [39] the describing function $N(A, \omega)$ is defined as the ratio of the phasor representation of output component at frequency ω and the phasor representation of input component at frequency ω, that is

$$N(A, \omega) = \frac{\left(\sqrt{a_1^2 + b_1^2}\right) e^{j(\omega t + \phi)}}{Ae^{j\omega t}} = \frac{1}{A}(b_1 + ja_1) \tag{A.2}$$

where the coefficients a_1 and b_1 of the first harmonic of Fourier representation are given by

$$a_1 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} u(t)\cos(\omega t)dt, \qquad b_1 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} u(t)\sin(\omega t)dt.$$

Finally, substituting a_1 and b_1 into (A.2) yields to

$$N(A, \omega) = \frac{\omega}{\pi A}\int_0^{2\pi/\omega} u(t)\sin(\omega t)dt + j\frac{\omega}{\pi A}\int_0^{2\pi/\omega} u(t)\cos(\omega t)dt. \tag{A.3}$$

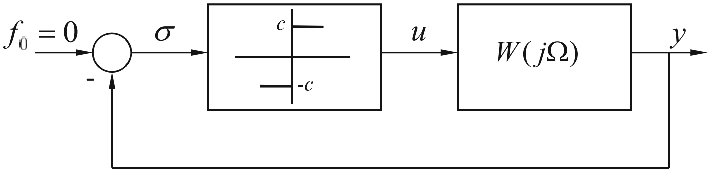

Fig. A.2 Relay feedback using single-relay control

A.1 Describing Function of a Single-Relay

Let us first investigate the existence of periodic solution of a single-relay control given by (see Fig. A.2)

$$u(t) = \begin{cases} -c & \text{if } \frac{\pi}{\omega} < t < 0 \\ c & \text{if } 0 < t < \frac{\pi}{\omega}. \end{cases} \tag{A.4}$$

Using (A.3) yields

$$N(A, \omega) = \frac{4c}{\pi A} \tag{A.5}$$

where A and ω are the amplitude and frequency of the output $y(t)$, respectively.

Appendix B
The locus of a perturbed relay system (LPRS)

B.1 Asymmetric oscillations in relay feedback systems

The *locus of a perturbed relay system* (LPRS) method of analysis is similar from the methodological point of view to the DF method. It is designed to imitate the methodology of analysis used in the DF-based approach. Some concepts (like the notion of the *equivalent gain*) are also similar. However, the LPRS method is exact and the notions that are traditionally used within the DF method are redefined, so that in the LPRS analysis similar notions are used in the exact sense.

Let us consider the SISO relay feedback system, that has a constant input, described by the following equations:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{B.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times 1}$ and $\mathbf{C} \in \mathbb{R}^{1\times n}$ are matrices, $\mathbf{A}$ is nonsingular, $\mathbf{x} \in \mathbb{R}^{n\times 1}$ is the state vector, $y \in \mathbb{R}^1$ is the system output and $u \in \mathbb{R}^1$ is the control defined as follows:

$$u(t) = \begin{cases} +h & \text{if } e(t) = r_0 - y(t) \geq b \\ & \text{or} \quad e(t) > -b\,,\ u(t-) = h \\ -h & \text{if } e(t) = r_0 - y(t) \leq -b \\ & \text{or} \quad e(t) < b\,,\ u(t-) = -h \end{cases} \tag{B.2}$$

where r_0 is a constant input to the system, e is the error signal, h is the relay amplitude, $2b$ is the hysteresis value of the relay and $u(t-) = \lim\limits_{\epsilon\to 0, \epsilon>0} u(t-\epsilon)$ is the control at the time instant immediately preceding time t. We shall consider that

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time $t = 0$ corresponds to the time of the error signal becoming equal to the positive half-hysteresis value (subject to $\dot{e} > 0$): $e(0) = b$ and call this time the *time of relay switch from* $-h$ *to* h.

We can also represent the linear part of the relay feedback system (B.1) by the transfer function $W_l(s)$:

$$W_l(s) = \mathbf{C}(\mathbf{I}s - \mathrm{A})^{-1}\mathbf{B}. \tag{B.3}$$

We shall assume that the linear part is strictly proper, i.e., the relative degree of $W_l(s)$ is 1 or higher, which is a valid assumption for all physically realisable systems.

If the input to the system is a constant value $r_0 : r(t) \equiv r_0$, then an asymmetric periodic motion occurs in the relay feedback system, so that each signal has a periodic term with zero mean value and a nonzero constant term: $u(t) = u_0 + u_p(t)$, $y(t) = y_0 + y_p(t)$, $e(t) = e_0 + e_p(t)$, where subscript 0 refers to the constant term, and subscript p refers to the periodic term of the function. The periodic term represents the sum of all periodic terms (harmonics) in the Fourier series expansion for respective signal. The constant term is the averaged value of the signal on the period.

The constant input signal r_0 can be quasi-statically (slowly) slewed from a certain negative value to a positive value, so that at each value of the input signal the system establishes a stable oscillation, and the values of the constant terms of the error signal and of the control signal are measured. Then the constant term of the control signal can be considered as a function of the constant term of the error signal. This dependance would give the *bias function* $u_0 = u_0(e_0)$, which would be not a discontinuous but a smooth function. The described smoothing effect is known as the *chatter smoothing* phenomenon. The derivative of the mean control with respect to the mean error taken in the point of zero mean error $e_0 = 0$ (corresponding to zero constant input) provides the *equivalent gain* of the relay k_n. The *equivalent gain* of the relay can be used as a local approximation of the *bias function*: $k_n = \mathrm{d}u_0/\mathrm{d}e_0|_{e_0=0} = \lim_{r_0 \to 0}(u_0/e_0)$.

B.2 Computation of the LPRS

B.2.1 Computeation of LPRS from matrix state space description

A complex function of frequency ω named the locus of a perturbed relay system (LPRS) was introduced in [13] for analysis of self-excited oscillations in and external signal propagation through the relay system as follows:

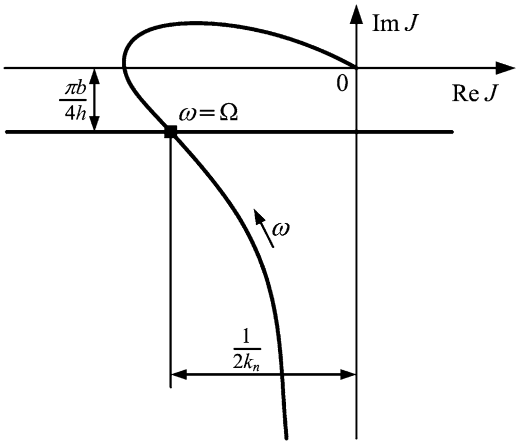

Fig. B.1 LPRS and analysis of relay feedback system

$$J(\omega) = -0.5\mathbf{C}\left[\mathbf{A}^{-1} + \tfrac{2\pi}{\omega}\left(\mathbf{I} - e^{\frac{2\pi}{\omega}\mathbf{A}}\right)^{-1} e^{\frac{\pi}{\omega}\mathbf{A}}\right]\mathbf{B} \\ +j\tfrac{\pi}{4}\mathbf{C}\left(\mathbf{I} + e^{\frac{\pi}{\omega}\mathbf{A}}\right)^{-1}\left(\mathbf{I} - e^{\frac{\pi}{\omega}\mathbf{A}}\right)\mathbf{A}^{-1}\mathbf{B}, \tag{B.4}$$

where $\omega \in [0, \infty)$. An LPRS plot is presented in Fig. B.1.

It was proved in [13] and [14] that the frequency Ω of the self-excited oscillations can be computed through solving the equation:

$$\operatorname{Im} J(\Omega) = -\frac{\pi b}{4h}, \tag{B.5}$$

and the equivalent gain k_n, which describes propagation of constant signals (or signals slowly varied with respect to the self excited oscillation) can be computed as:

$$k_n = -\frac{1}{2\operatorname{Re} J(\Omega)}. \tag{B.6}$$

Both values provided by formulas (B.5) and (B.6) are exact. LPRS also offers a convenient graphic interpretation of finding the frequency Ω and the equivalent gain k_n (Fig. B.1). The point of intersection of the LPRS and the horizontal line, which lies at the distance of $\pi b/(4h)$ below (if $b > 0$) or above (if $b < 0$) the horizontal axis (line "$-\pi b/4h$"), allows for computation of the frequency of the oscillations and of the equivalent gain k_n of the relay.

B.2.2 Computation of the LPRS from transfer function

A different formula for $J(\omega)$ was derived in [14] for the case of the linear part given by a transfer function. Through application of the Fourier series, it was proved that LPRS can also be computed as the following infinite series:

$$J(\omega) = \sum_{k=1}^{\infty}(-1)^{k+1}\mathrm{Re}W_l(k\omega) + j\sum_{k=1}^{\infty}\frac{1}{2k-1}\mathrm{Im}W_l[(2k-1)\omega]. \tag{B.7}$$

Another technique of LPRS computation is based on the possibility of derivation of analytical formulas for the LPRS of low-order dynamics. It is a result of the additivity property that LPRS possesses, which can be formulated as follows.

Additivity property. If the transfer function $W_l(s)$ of the linear part is a sum of n transfer functions: $W_l(s) = W_1(s) + W_2(s) + \cdots + W_n(s)$ then the LPRS $J(\omega)$ can be calculated as a sum of the n component LPRS: $J(\omega) = J_1(\omega) + J_2(\omega) + \cdots + J_n(\omega)$, where $J_i(\omega)$ $(i = 1, \ldots, n)$ is the LPRS of the relay system with the transfer function of the linear part being $W_i(s)$.

The considered property offers a technique of the LPRS computation based on the expansion of the process transfer function into partial fractions. Therefore, if $W_l(s)$ is expanded into the sum of first and second order dynamics then LPRS $J(\omega)$ can be calculated through the summation of the component LPRS $J_i(\omega)$ corresponding to each of the component transfer functions, subject to analytical formulas for the LPRS of first and second order dynamics. Formulas for $J(\omega)$ of first and second order dynamics were derived in [14]. They are presented in Table B.1.

B.2.3 Some properties of the LPRS

Some important properties of the LPRS as a frequency-domain characteristic are related with the boundary points corresponding to zero frequency and infinite frequency. The initial point of the LPRS (which corresponds to zero frequency) can be found through formula (B.4). It can be noted that the limit of function $J(\omega)$ for ω tending to zero can be found as follows. First the following two limits must be evaluated: $\lim_{\omega\to 0}\left[\frac{2\pi}{\omega}((\mathbf{I} - e^{\frac{2\pi}{\omega}\mathbf{A}})^{-1}e^{\frac{\pi}{\omega}\mathbf{A}}\right] = \mathbf{0}$, $\lim_{\omega\to 0}\left[(\mathbf{I} + e^{\frac{\pi}{\omega}\mathbf{A}})^{-1}(\mathbf{I} - e^{\frac{\pi}{\omega}\mathbf{A}})\right] = \mathbf{I}$. Then the limit for the LPRS can be written as follows:

$$\lim_{\omega\to 0} J(\omega) = \left[-0.5 + j\frac{\pi}{4}\right]\mathbf{CA}^{-1}\mathbf{B} \tag{B.8}$$

The product of matrices $\mathbf{CA}^{-1}\mathbf{B}$ in (B.8) is the negative value of the gain of the transfer function. Therefore, for a *nonintegrating linear part of the relay feedback system, the initial point of the corresponding LPRS is* $(0.5K, -j\pi/4K)$*, where* K *is the static gain of the linear part.*

Table B.1 Formulas of the LPRS $J(\omega)$

Transfer fun. $W(s)$	LPRS $J(\omega)$
$\frac{K}{s}$	$0 - j\frac{\pi^2 K}{8\omega}$
$\frac{K}{Ts+1}$	$\frac{K}{2}(1-\alpha\ \mathrm{csch}\alpha) - j\frac{\pi K}{4}\tanh(\alpha/2)$ $\alpha = \pi/(T\omega)$
$\frac{Ke^{-\tau s}}{Ts+1}$	$\frac{K}{2}(1-\alpha e^{\gamma}\ \mathrm{csch}\alpha) + j\frac{\pi K}{4}\left(\frac{2e^{-\alpha}e^{\gamma}}{1+e^{-\alpha}} - 1\right)$ $\alpha = \frac{\pi}{T\omega},\ \ \gamma = \frac{\tau}{T}$
$\frac{K}{(T_1 s+1)(T_2 s+1)}$	$\frac{K}{2}[1 - T_1/(T_1-T_2)\alpha_1\ \mathrm{csch}\alpha_1 - T_2/(T_2-T_1)\alpha_2\ \mathrm{csch}\alpha_2)]$ $-j\frac{\pi K}{4}/(T_1-T_2)[T_1\tanh(\alpha_1/2) - T_2\tanh(\alpha_2/2)]$ $\alpha_1 = \pi/(T_1\omega),\ \ \alpha_2 = \pi/(T_2\omega)$
$\frac{K}{s^2+2\xi s+1}$	$\frac{K}{2}[(1-(B+\gamma C)/(\sin^2\beta + \sinh^2\alpha)]$ $-j\frac{\pi K}{4}(\sinh\alpha - \gamma\sin\beta)/(\cosh\alpha + \cos\beta)$ $\alpha = \pi\xi/\omega,\ \ \beta = \pi(1-\xi^2)^{1/2}/\omega,\ \ \gamma = \alpha/\beta$ $B = \alpha\cos\beta\sinh\alpha + \beta\sin\beta\cosh\alpha,$ $C = \alpha\sin\beta\cosh\alpha - \beta\cos\beta\sinh\alpha$
$\frac{Ks}{s^2+2\xi s+1}$	$\frac{K}{2}[\xi(B+\gamma C) - \pi/\omega\cos\beta\sinh\alpha]/(\sin^2\beta + \sinh^2\alpha)]$ $-j\frac{\pi K}{4}(1-\xi^2)^{-1/2}\sin\beta/(\cosh\alpha + \cos\beta)$ $\alpha = \pi\xi/\omega,\ \ \beta = \pi(1-\xi^2)^{1/2}/\omega,\ \ \gamma = \alpha/\beta$ $B = \alpha\cos\beta\sinh\alpha + \beta\sin\beta\cosh\alpha,$ $C = \alpha\sin\beta\cosh\alpha - \beta\cos\beta\sinh\alpha$
$\frac{Ks}{(s+1)^2}$	$\frac{K}{2}[\alpha(-\sinh\alpha + \alpha\cosh\alpha)/\sinh^2\alpha - j0.25\pi\alpha/(1+\cosh\alpha)]$ $\alpha = \pi/\omega$
$\frac{Ks}{(T_1 s+1)(T_2 s+1)}$	$\frac{K}{2}/(T_2-T_1)[\alpha_2\ \mathrm{csch}\alpha_2 - \alpha_1\ \mathrm{csch}\alpha_1]$ $-j\frac{\pi K}{4}/(T_2-T_1)[\tanh(\alpha_1/2) - \tanh(\alpha_2/2)]$ $\alpha_1 = \pi/(T_1\omega),\ \ \alpha_2 = \pi/(T_2\omega)$

To find the limit of $J(\omega)$ for ω tending to infinity, the following two limits of the expansion into power series of the exponential function must be considered:

$$\lim_{\omega\to\infty}\exp\left(\frac{\pi}{\omega}\mathbf{A}\right) = \lim_{\omega\to\infty}\sum_{n=0}^{\infty}\frac{(\pi/\omega)^n}{n!}\mathbf{A}^n = \mathbf{I}$$

and

$$\lim_{\omega\to\infty}\left\{\frac{2\pi}{\omega}\left[\mathbf{I} - \exp\left(\frac{2\pi}{\omega}\mathbf{A}\right)\right]^{-1}\right\} = \lim_{\lambda=\frac{2\pi}{\omega}=\to 0}\{\lambda[\mathbf{I} - \exp(\lambda\mathbf{A})]^{-1}\} = -\mathbf{A}^{-1}.$$

These limits show that the end point of the LPRS for $\omega \to \infty$ for nonintegrating linear parts is the origin:

$$\lim_{\omega\to\infty} J(\omega) = 0 + j0. \tag{B.9}$$

Appendix C
Poincaré map

C.1 Basic concepts in Poincaré maps

Consider a time-invariant system

$$\dot{x} = f(x) \tag{C.1}$$

where $x(t) \in \mathbb{R}^n$ is the state. Let $\bar{x}(t)$ be its T-periodic solution starting from x_0. Introduce a smooth surface S by the equation $s(x) = 0$ where $s : \mathbb{R}^n \mapsto \mathbb{R}$ is a smooth scalar function and assume it intersects the trajectory in x_0 transversely, that is, $s(x_0) = 0$, $\nabla s(x_0)^T F(x) \neq 0$. We will call such a surface the transverse surface or cross-section across x_0. It can be shown that the solution starting from $x \in S = \{x : s(x) = 0\}$ close to x_0 will cross the surface $s(x) = 0$ again at least once. Let $\bar{t}(x)$ be the time of the first return and $P(x) \in S$ be the point of the first return.

Let us recall, that a *periodic orbit* of a nonlinear system $\dot{x} = f(x)$, with a vector state $x(t) \in \mathbb{R}^n$, is an invariant set which is determined by an initial condition x_p and a period T. Here T is defined as the smallest time $T > 0$ for which $\Phi(x_p, T) = x_p$ where $\Phi(x, t)$ stand for the solution operator ([26, p. 49]). A periodic orbit that is isolated (there no exist any other periodic orbit in its neighborhood) is named *limit cycle*.

Definition C.1 (Poincaré map [35]). The mapping $x \mapsto P(x)$ is called Poincaré map or return map.

For later use, the following Theorem for asymptotic orbital stability based on Poincaré maps read as follows.

Theorem C.1. *Let $\bar{x}(t)$ be a T-periodic solution, $x(t_0) = x_0$; S be the smooth cross-section across x_0 and P be the corresponding Poincaré map.*

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If for any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $x(t) \in S$, $|x - x_0| < \delta$ *implies* $|P(x) - x_0| < \varepsilon$*, then* $\bar{x}(t)$ *is orbitally stable. If, in addition,* $P^n(x) \to x_0$ *as* $n \to \infty$*, then* $\bar{x}(t)$ *is AOS.*

If all $n - 1$ *eigenvalues of the linearized Poincaré map* $\xi \mapsto \partial P(\xi)/\partial x$, $\xi \in S$ *have the absolute values less than 1, then* $\bar{x}(t)$ *is asymptotically orbitally stable.*

If the linearized Poincaré map has at least one eigenvalue with an absolute value greater than one then $\bar{x}(t)$ *is orbitally unstable.*

Proof. The proof is provided by Fradkov and Pogromsky [35] and it is therefore omitted.

Readers can review supplementary material and examples about Poincaré maps in [79][Chap. 7].

Appendix D
Output Feedback

D.1 State observer design

To maintain the estimation error bounded, the following linear observer for (6.4), (6.9)

$$\dot{\tilde{x}} = A\tilde{x} + B_2\tau + L(y - \tilde{y}) \tag{D.1}$$

is considered. Here, $\tilde{y} = C\tilde{x}$ and L must be designed such that the matrix $\tilde{A} := (A - LC)$ is Hurwitz. Let $e(t) := x(t) - \tilde{x}(t)$. Thus, $e(t)$ enters to a ball $\mathscr{B}_\delta(E_0)$ centered at the equilibrium point E_0 with radius $\delta > 0$ in finite-time T_e, such that

$$\|e(t)\| \leq e^+ \qquad \text{for all } t > T_e.$$

Now, let us decouple the unknown inputs from the successive derivatives of the output of the linear estimation error system defined as $y_e = y - \tilde{y}$.

Definition D.1 (Strong observability [86]). The system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2u(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{D.2}$$

it is called strongly observable if for all $x_0 \in \chi \subseteq \mathbb{R}^n$ and for every input function u, the following holds:

$$y_u(t, x_0) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}B_2u(\tau)d\tau + Du(t) = 0 \tag{D.3}$$

for all $t \geq 0$ implies $x_0 = 0$.

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Consider the output distribution matrix C and deriving a linear combination of the output y_e, ensuring that the derivative of this combination is unaffected by the uncertainties, that is,

$$\frac{d}{dt}(CB_1)^{\perp} y_e(t) = (CB_1)^{\perp} C\tilde{A} e(t), \tag{D.4}$$

and construct the extended vector

$$\begin{bmatrix} \frac{d}{dt}(CB_1)^{\perp} y_e \\ y_e \end{bmatrix} = \underbrace{\begin{bmatrix} (CB_1)^{\perp} C\tilde{A} \\ C \end{bmatrix}}_{M} e.$$

Rearranging terms, the following equation is obtained

$$Me = \frac{d}{dt} \begin{bmatrix} (CB_1)^{\perp} & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} y_e(t) \\ \int y_e(t) dt \end{bmatrix}.$$

Since (A, C, B_2) is strongly observable, the matrix M has full row rank (see Molinari [62]). This implies that the above algebraic equation has an unique solution for $e(t)$, that is

$$e(t) = \frac{d}{dt} \underbrace{M^{+} \begin{bmatrix} (CB_1)^{\perp} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y_e(t) \\ \int y_e(t) dt \end{bmatrix}}_{\theta(t)} \tag{D.5}$$

where $M^{+} := (M^T M)^{-1} M^T$. From the above expression, the reconstruction of $x(t)$ is equivalent to the reconstruction of $e(t)$. Hence, a real time high-order sliding mode differentiator will be used. The HOSM differentiator is given by

$$\begin{aligned}
\dot{z}_0 &= -\lambda_k \Gamma^{\frac{1}{k+1}} |z_0 - \theta|^{\frac{k}{k+1}} \operatorname{sign}(z_0 - \theta) + z_1 \\
\dot{z}_1 &= -\lambda_{k-1} \Gamma^{\frac{1}{k}} |z_1 - \dot{z}_0|^{\frac{k-1}{k}} \operatorname{sign}(z_1 - \dot{z}_0) + z_2 \\
&\vdots \\
\dot{z}_{k-1} &= -\lambda_1 \Gamma^{\frac{1}{2}} |z_{k-1} - \dot{z}_{k-2}|^{\frac{1}{2}} \operatorname{sign}(z_{k-1} - \dot{z}_{k-2}) + z_k \\
\dot{z}_k &= -\lambda_0 \Gamma \operatorname{sign}(z_k - \dot{z}_{k-1}).
\end{aligned} \tag{D.6}$$

The values of the parameters $\lambda_0, \lambda_1, \ldots, \lambda_k$ are chosen separately by recursive methods provide for the convergence of the $(k-1)$th-order differentiator commonly obtained by computer simulation [53], obtaining a finite-time T such that the identity

$$z_i(t) = \frac{d^i\theta(t)}{dt^i} \tag{D.7}$$

holds for every $i = 0, 1, \ldots, k$. The parameter Γ is a Lipschitz constant of $\ddot{\theta}(t)$, which is defined as

$$\Gamma \leq \|\tilde{A}\|e^+ + \|B\|w^+. \tag{D.8}$$

The vector $e(t)$ can be reconstructed from the first order sliding dynamics. Thus, we achieve the identity $z_1(t) = e(t)$, and consequently

$$\hat{x}(t) := z_1(t) + \tilde{x}(t) \text{ for all } t \geq T$$

where $\hat{x}(t)$ represents the estimated value of $x(t)$. Therefore, the identity

$$\hat{x}(t) \equiv x(t), \tag{D.9}$$

is achieved for all $t \geq T$.

Definition D.2 ([63]). Given the system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \qquad x(0) = x_0, \tag{D.10}$$

the invariant zeros of the above system are the set of all the eigenvalues λ such that

$$\begin{bmatrix} \lambda - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{D.11}$$

as a solution for some scalar u and non-zero x.

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