Appendix A
The Spectral Function

We briefly mention some properties of the spectral function.

A.1 Sum Rules

Expressed with operators, we have

$$\varrho_{AB}(x) = \frac{1}{Z_0}\,\mathrm{Tr}\left[e^{-\beta H}[A(x),B(0)]_\alpha\right], \tag{A.1}$$

i.e., this is the expectation of the (anti)commutator of the two operators. For fundamental fields, the (anti)commutator is just a Dirac delta (cf. (2.8)), which implies

$$\varrho_{A\Pi}(t=0,\mathbf{x}) = i\delta(\mathbf{x}), \qquad \int \frac{dk_0}{2\pi}\varrho_{A\Pi}(k_0,\mathbf{k}) = i; \tag{A.2}$$

this is the basic form of sum rules. In general, the (anti)commutator of a local operator at equal times must be proportional to $\delta(\mathbf{x})$, but the proportionality constant can be complicated. Thus in the generic sum rule, the right-hand side is not just i, but in addition, a (temperature-dependent) constant can appear.

A.2 Positivity

Using the time translation-invariance of the equilibrium and the properties of the (anti)commutators, it is easy to show that $\varrho^*_{AB}(k) = \varrho_{B^\dagger A^\dagger}(k)$, which means that $\varrho^*_{AA^\dagger}(k) = \varrho_{AA^\dagger}(k)$, i.e., the spectral function is real for $B = A^\dagger$. Moreover,

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$\varrho_{AB}(-k) = -\alpha\varrho_{BA}(k)$, which means that the spectral function is antisymmetric for $B = A$.

Continuing with the real spectral functions $\varrho_{AA^\dagger}$, we can insert into the definition a complete system with states $\langle Pn|$, where P is the four-vector of the energy and momentum, and n is the eigenvalue of N, the operator of a conserved charge. We obtain

$$\begin{aligned}
\mathrm{Tr}\, e^{-\beta(H-\mu N)}[A(x), A^\dagger(0)]_\alpha &= \sum_{Pn} \left\langle Pn \left| e^{-\beta(H-\mu N)}[A(x), A^\dagger(0)]_\alpha \right| Pn \right\rangle \\
&= \sum_{PP'nn'} \left(e^{-\beta(E-\mu n)} - \alpha e^{-\beta(E'-\mu n')} \right) \left\langle Pn \left| A(x) \right| P'n' \right\rangle \left\langle P'n' \left| A^\dagger(0) \right| Pn \right\rangle \\
&= \sum_{PP'nn'} \left(e^{-\beta(E-\mu n)} - \alpha e^{-\beta(E'-\mu n')} \right) e^{i(P-P')x} \left| \left\langle Pn \left| A(0) \right| P'n' \right\rangle \right|^2,
\end{aligned} \tag{A.3}$$

where we have used $A(x) = e^{iPx}A(0)e^{-iPx}$. If A has a definite charge q, then $n' + q = n$ must be true. In Fourier space, we have

$$\varrho_{AA^\dagger}(k) = \sum_{PP'nn'} e^{-\beta(E-\mu n)} \left(1 - \alpha e^{-\beta(k_0+\mu q)}\right) (2\pi)^4 \delta(k + P - P') \left| \left\langle Pn \left| A(0) \right| P'n' \right\rangle \right|^2. \tag{A.4}$$

This form shows that for $k_0 + \mu q > 0$, the spectral function is strictly positive.

The same formula also demonstrates that the spectral function is a sum of Dirac deltas, at values of k, where two energy eigenstates satisfy the equality $P' = P + k$. At zero temperature and chemical potential, only the $E = 0$ ground state remains, where

$$\varrho_{AA^\dagger}(k_0 > 0)|_{T=\mu=0} = \sum_P C_k (2\pi)^4 \delta(k - P), \tag{A.5}$$

where $C_k = |\langle kq | A(0) | 0 \rangle|^2$. This is proportional to the density of states.

Appendix B
Computation of the Basic Diagrams

Here we summarize the results of the computation of the tadpole and the bubble diagrams in a one-component bosonic or fermionic theory.

B.1 Tadpole Integral

The contribution of the tadpole diagram (see the first diagram of Fig. 2.2) is momentum-independent, and so it yields the same result in every formalism. In real space, the diagram is proportional to $G^{(33)}(x=0)$. This will be considered the tadpole integral with the definition

$$\mathscr{T} = \alpha G^{(33)}(x=0). \tag{B.1}$$

We can write it in the imaginary time formalism as

$$\mathscr{T} = \alpha T \sum\!\!\!\!\!\!\int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + \omega_p^2}, \tag{B.2}$$

where $\omega_n = 2\pi n$ or $(2n+1)\pi$ for the bosonic or fermionic case, respectively. We do not perform the sum, instead we use the line of argument of Eq. (2.82): using the definition in (2.42), we obtain

$$\begin{aligned} \mathscr{T} &= \int \frac{d^4p}{(2\pi)^4} \left(\frac{\alpha}{2} + n_\alpha(p_0 - \mu)\right) \varrho_0(p) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_p} \left(\alpha + n_\alpha(\omega_p - \mu) + n_\alpha(\omega_p + \mu)\right). \end{aligned} \tag{B.3}$$

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It naturally splits into a zero- and finite-temperature/chemical potential part:

$$\mathscr{T} = \alpha \mathscr{T}_0 + \frac{1}{2}\left[\mathscr{T}_{T,\mu} + \mathscr{T}_{T,-\mu}\right], \qquad \mathscr{T}_0 = \mathscr{T}\big|_{\mu=T=0}. \tag{B.4}$$

Here $\mathscr{T}_0$ is UV divergent. We need to regularize the integral in order to be able to handle it. With momentum cutoff, we have

$$\mathscr{T}_0 = \frac{1}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{p^2+m^2}} = \frac{1}{4\pi^2}\int_0^{\Lambda}\frac{dpp^2}{\sqrt{p^2+m^2}} = \frac{\Lambda^2}{8\pi^2} - \frac{m^2}{16\pi^2}\ln\frac{4\Lambda^2}{em^2}. \tag{B.5}$$

We can also apply dimensional regularization (cf. Appendix B.3) to compute its value:[1]

$$\frac{\mu^{2\varepsilon}}{2}\int\frac{d^{3-2\varepsilon}p}{(2\pi)^{3-2\varepsilon}}\left(p^2+m^2\right)^{-1/2} = \frac{m^2}{16\pi^2}\left(-\frac{1}{\varepsilon}+\gamma_E - 1 + \ln\frac{m^2}{4\pi\mu^2} + \mathscr{O}(\varepsilon)\right), \tag{B.6}$$

where γ_E is the Euler constant, $\gamma_E = 0.577216$.

The quantity $\mathscr{T}_{T,\mu}$ is UV finite, due to the exponential suppression of the distribution functions. For the evaluation, we change the integration variable $p \to x = \beta p$, and obtain

$$\mathscr{T}_{T,\mu} = \frac{T^2}{2\pi^2}\int_0^{\infty} dx\,\frac{x^2}{\gamma}\frac{1}{e^{\gamma-\beta\mu}-\alpha}, \qquad \gamma = \sqrt{x^2+(\beta m)^2}. \tag{B.7}$$

The only analytic problem with this integral is a pole for the bosonic ($\alpha = 1$) case for $\mu > m$. Therefore, the bosonic expression $\mathscr{T}$ is defined only in the range $-m < \mu < m$, just as for the thermodynamics itself.

In the convergent regimes, one can study different limits. The small-temperature regime is common for the bosonic and fermionic cases:

$$\begin{aligned}\mathscr{T}_{T\ll\mu,m} &= \frac{T^2}{2\pi^2}\int_{\beta m}^{\infty} dx\, e^{-x+\beta\mu}\sqrt{x^2-(\beta m)^2} = \frac{mT}{2\pi^2}K_1(\beta m)e^{\beta\mu} \\ &\approx \frac{1}{m}\left(\frac{mT}{2\pi}\right)^{3/2} e^{-\beta(m-\mu)}.\end{aligned} \tag{B.8}$$

[1] Usually, it is computed in $4-2\varepsilon$ dimensions, using Wick rotation; but the result is the same.

For the leading order of the small-mass expansion, we also can get analytic expressions by formally substituting $m = 0$:

$$\mathcal{T}_{T,\mu}(m=0) = \frac{T^2}{2\pi^2}\int\limits_0^\infty dx\,\frac{x}{e^{-\beta\mu}e^x - \alpha} = \frac{\alpha T^2}{2\pi^2}\mathrm{Li}_2(\alpha e^{\beta\mu}), \tag{B.9}$$

where Li_2 is the dilogarithm (Spence's) function. In the bosonic case, the pole is regularized by principal value integration. Using the identity [1]

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1/z) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z), \tag{B.10}$$

one obtains

$$\frac{1}{2}\left[\mathcal{T}_{T,\mu}(m=0) + \mathcal{T}_{T,-\mu}(m=0)\right] = \begin{cases} \dfrac{T^2}{12} - \dfrac{\mu^2}{8\pi^2} & \text{for } \alpha = 1 \\ \dfrac{T^2}{24} + \dfrac{\mu^2}{8\pi^2} & \text{for } \alpha = -1. \end{cases} \tag{B.11}$$

Note that for the $\alpha = 1$ bosonic case, only $|\mu| < m$ is the physical domain.

The next terms of the small-mass expansion come from the expansion of the Bose–Einstein or Fermi–Dirac distribution. We will compute the function [2]

$$I_\alpha^\varepsilon(z) = \int\limits_0^\infty \frac{dx\, e^{-\varepsilon x}}{\gamma}\,\frac{1}{e^{-\beta\mu}e^\gamma - \alpha}, \qquad \gamma^2 = x^2 + z^2. \tag{B.12}$$

It is connected to the tadpole as

$$\frac{d\mathcal{T}_{T,\mu}}{dm^2} = -\frac{1}{4\pi^2}I_\alpha(\beta m). \tag{B.13}$$

We work out only the $\mu = 0$ case.

In the bosonic case, the expansion of the distribution function reads

$$\frac{1}{e^\gamma - 1} = \frac{1}{\gamma} - \frac{1}{2} + 2\sum_{n=1}^\infty \frac{\gamma}{\gamma^2 + (2\pi n)^2}. \tag{B.14}$$

We note that the first term is the contribution of the zeroth (static) Matsubara mode. We have

$$I_+^\varepsilon(z) = \int_0^\infty dx \left(\frac{x^\varepsilon}{y^2} - \frac{x^\varepsilon}{2y} + 2\sum_{n=1}^\infty \frac{x^\varepsilon}{x^2 + z^2 + (2\pi n)^2} \right). \tag{B.15}$$

In the first term, we can take $\varepsilon = 0$, which yields $\pi/(2z)$. The second term is similar to the zero-temperature result

$$-\frac{1}{2}\int_0^\infty dx\, x^\varepsilon (x^2 + z^2)^{-1/2} = -\frac{z^\varepsilon}{4\sqrt{\pi}}\Gamma(-\frac{\varepsilon}{2})\Gamma(\frac{1+\varepsilon}{2}) = \frac{1}{2\varepsilon} + \frac{1}{2}\ln\frac{z}{2}. \tag{B.16}$$

In the last term, we can take $z = 0$ and obtain

$$\begin{aligned} 2\sum_{n=1}^\infty \int_0^\infty dx \frac{x^\varepsilon}{x^2 + (2\pi n)^2} &= \sum_{n=1}^\infty \frac{\pi^\varepsilon (2n)^{-1+\varepsilon}}{\cos \varepsilon\pi/2} \\ &= \frac{(2\pi)^\varepsilon \zeta(1-\varepsilon)}{2\cos\varepsilon\pi/2} = -\frac{1}{2\varepsilon} + \frac{\gamma_E}{2} - \frac{1}{2}\ln(2\pi), \end{aligned} \tag{B.17}$$

where ζ is the Riemann zeta function. We then have

$$I_+(z) = \frac{\pi}{2z} + \frac{1}{2}\ln\frac{z}{4\pi} + \frac{\gamma_E}{2}. \tag{B.18}$$

In the fermionic case, we write

$$\frac{1}{e^y + 1} = \frac{1}{2} - 2\sum_{n=1}^\infty \frac{y}{y^2 + ((2n+1)\pi)^2}. \tag{B.19}$$

Here there is no $1/y$ term, since there is no zeroth Matsubara mode. The first term is the same as before; the second term yields

$$\begin{aligned} 2\sum_{n=1}^\infty \int_0^\infty dx \frac{x^\varepsilon}{x^2 + ((2n+1)\pi)^2} &= \sum_{n=1}^\infty \frac{\pi^\varepsilon (2n+1)^{-1+\varepsilon}}{\cos\varepsilon\pi/2} = \frac{\pi^\varepsilon (2 - 2^\varepsilon)\zeta(1-\varepsilon)}{2\cos\varepsilon\pi/2} \\ &= -\frac{1}{2\varepsilon} + \frac{\gamma_E}{2} - \frac{1}{2}\ln\frac{\pi}{2}. \end{aligned} \tag{B.20}$$

Then we have

$$I_-(z) = -\frac{\gamma_E}{2} - \frac{1}{2}\ln\frac{z}{\pi}. \tag{B.21}$$

To have the result for the tadpole integral, we integrate (B.13). We obtain for bosons

$$\mathscr{T}_T^+ = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{m^2}{16\pi^2}\left[\ln\frac{m^2}{c_+^2 T^2} - 1\right], \tag{B.22}$$

for fermions

$$\mathscr{T}_T^- = \frac{T^2}{24} + \frac{m^2}{16\pi^2}\left[\ln\frac{m^2}{c_-^2 T^2} - 1\right]. \tag{B.23}$$

The constant terms are

$$\begin{aligned} \ln c_+ &:= -\gamma_E + \ln 4\pi = 1.95381, \qquad c_+ = 7.05551 \\ \ln c_- &:= -\gamma_E + \ln \pi = 0.567514, \qquad c_- = 1.7638. \end{aligned} \tag{B.24}$$

Therefore in the high-temperature expansion, we have

$$\begin{aligned} &\text{bosons with cutoff :} && \mathscr{T} = \frac{\Lambda^2}{8\pi^2} + \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{m^2}{16\pi^2}\ln\frac{\Lambda^2}{d_+^2 T^2} \\ &\text{bosons with dim.reg. :} && \mathscr{T} = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{m^2}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{c_+^2 T^2}\right], \\ &\text{fermions with cutoff :} && \mathscr{T} = -\frac{\Lambda^2}{8\pi^2} + \frac{T^2}{24} + \frac{m^2}{16\pi^2}\ln\frac{\Lambda^2}{d_-^2 T^2} \\ &\text{fermions with dim.reg. :} && \mathscr{T} = \frac{T^2}{24} + \frac{m^2}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{c_-^2 T^2}\right], \end{aligned} \tag{B.25}$$

where

$$\begin{aligned} \ln d_+ &= -\gamma_E + 1 + \ln 2\pi = 2.26066, \quad d_+ = 9.58943, \\ \ln d_- &= -\gamma_E + 1 + \ln\frac{\pi}{2} = 0.874367, \quad d_- = 2.39736. \end{aligned} \tag{B.26}$$

B.2 The Bubble Diagram

Similarly to the tadpole case, we just define the basic integral. In imaginary time, this corresponds to the expression

$$\mathscr{I}_E(x) = \alpha G_{33}(m_1, x) G_{33}(m_2, x), \tag{B.27}$$

where $\alpha = -1$ if both propagating particles are fermions. This formula is true if two different particles propagate on the two lines. In this note we discuss only the case in which the particles are identical and there is no chemical potential in the system.

In Fourier space, it reads

$$\mathscr{I}_E(q) = \alpha T \sum\!\!\!\!\!\!\int \frac{d^3p}{(2\pi)^3} \frac{1}{(\omega_n^2 + \omega_{1p}^2)((q_0 - \omega_n)^2 + \omega_{2(\mathbf{q}-\mathbf{p})}^2)}. \tag{B.28}$$

One can tell immediately the result of the integral for the $q = 0$ case. There we have

$$\mathscr{I}_E(q) = \alpha T \sum\!\!\!\!\!\!\int \frac{d^3p}{(2\pi)^3} \frac{1}{(\omega_n^2 + p^2 + m^2)^2} = -\frac{\partial \mathscr{T}}{\partial m^2}. \tag{B.29}$$

At zero temperature, we can use (B.5) or (B.6), and we have with cutoff

$$\mathscr{I}_E^{T=0}(0) = -\frac{\partial}{\partial m^2}\left[\frac{m^2}{16\pi^2}\ln\frac{em^2}{4\Lambda^2}\right] = \frac{1}{16\pi^2}\left[\ln\frac{4\Lambda^2}{em^2} - 1\right] \tag{B.30}$$

and with dimensional regularization

$$\begin{aligned}\mathscr{I}_E^{T=0}(0) &= -\frac{\partial}{\partial m^2}\left[\frac{m^2}{16\pi^2}\left(-\frac{1}{\varepsilon} + \gamma_E - 1 + \ln\frac{m^2}{4\pi\mu^2}\right)\right] \\ &= \frac{1}{16\pi^2}\left(\frac{1}{\varepsilon} - \gamma_E - \ln\frac{m^2}{4\pi\mu^2}\right).\end{aligned} \tag{B.31}$$

The finite-temperature correction reads, using (B.13),

$$\mathscr{I}_E^T(0) = \frac{1}{4\pi^2} I_\alpha(\beta m, \mu = 0). \tag{B.32}$$

At high temperatures, we can use (B.25) and write

$$\begin{aligned}
&\text{bosons with cutoff :} && \mathscr{I}_E(0) = \frac{T}{8\pi m} + \frac{1}{16\pi^2}\ln\frac{\Lambda^2}{d_+^2 T^2} \\
&\text{bosons with dim.reg. :} && \mathscr{I}_E(0) = \frac{T}{8\pi m} + \frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{c_+^2 T^2}\right], \\
&\text{fermions with cutoff :} && \mathscr{I}_E(0) = -\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{d_-^2 T^2} \\
&\text{fermions with dim.reg. :} && \mathscr{I}_E(0) = -\frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{c_-^2 T^2}\right].
\end{aligned} \tag{B.33}$$

To obtain the momentum-dependence, we change to real time, and write the retarded expression

$$i\mathscr{I}^{(ar)}(q) = 2\int \frac{d^4p}{(2\pi)^4} \left[iG^{(rr)}(p)iG^{(ra)}(q-p)\right]. \tag{B.34}$$

This is a causal function, so it can be reproduced from its discontinuity. We obtain

$$\operatorname*{Disc}_{q_0} i\mathscr{I}^{(ar)}(q) = \int \frac{d^4p}{(2\pi)^4} \varrho(p)\varrho(q-p)(1+n(p_0)+n(q_0-p_0)). \tag{B.35}$$

Using the free spectral function and the notation $k = q - p$, we have

$$\begin{aligned} &\operatorname{Disc} i\mathscr{I}(q) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{(2\pi)}{4\omega_p\omega_k} \Big[\left[\delta(q_0-\omega_p-\omega_k)-\delta(q_0+\omega_p+\omega_k)\right]\left[1+n_p+n_k\right] \\ &\qquad + \left[\delta(q_0-\omega_p+\omega_k)-\delta(q_0+\omega_p-\omega_k)\right]\left[n_k-n_p\right]\Big]. \end{aligned} \tag{B.36}$$

We see that $\operatorname{Disc} i\mathscr{I}(-q_0,\mathbf{q}) = \operatorname{Disc} i\mathscr{I}(q_0,\mathbf{q})$, so it is enough to work out the $q_0 > 0$ case. We obtain

$$\begin{aligned} \operatorname{Disc} i\mathscr{I}(q_0>0,\mathbf{q}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{(2\pi)}{4\omega_p\omega_k} \Big[&\delta(q_0-\omega_p-\omega_k)\left[1+n_p+n_k\right] \\ &+2\delta(q_0-\omega_p+\omega_k)\left[n_k-n_p\right]\Big]. \end{aligned} \tag{B.37}$$

This integral depends on the angle between $\mathbf{p}$ and $\mathbf{q}$. We change from $\cos\theta \to k$ with the help of the relation

$$k^2 = q^2+p^2+2qpx \quad \Rightarrow \quad pdx = \frac{k}{q}dk, \qquad |q-p|<k<q+p, \tag{B.38}$$

which means that k, p, q forms a triangle. This means that

$$\begin{aligned} \operatorname{Disc} i\mathscr{I}(k) \; = \frac{1}{8\pi k}\int_0^\infty dqdp\,\Theta_\Delta(k,p,q)\frac{pq}{\omega_p\omega_q}\Big[&\delta(k_0-\omega_p-\omega_q)(1+n_p+n_q) \\ &+2\delta(k_0-\omega_p+\omega_q)(n_q-n_p)\Big] \end{aligned}$$

$$= \frac{1}{8\pi k}\int_0^\infty dp \frac{p}{\omega_p}\int_{\omega_-}^{\omega_+} d\omega_q \Big[\delta(k_0-\omega_p-\omega_q)(1+n_p+n_q)$$
$$+ 2\delta(k_0-\omega_p+\omega_q)(n_q-n_p)\Big]$$
$$= \frac{1}{8\pi k}\int_m^\infty d\omega \Big[\Theta(\omega_+ > k_0-\omega > \omega_-)(1+n(\omega)+n(k_0-\omega))$$
$$+2\Theta(\omega_+ > \omega-k_0 > \omega_-)(n(\omega-k_0)-n(\omega))\Big], \quad \text{(B.39)}$$

where $\omega_\pm^2 = (k \pm p)^2 + m^2$. The constraints taking into account both theta functions can be summarized as

$$\begin{aligned} &\omega_+ > |k_0-\omega| > \omega_- \\ &|K^2 - 2k_0\omega| < 2kp \\ &4K^2\omega^2 - 4K^2k_0\omega + K^4 - 4k^2m^2 < 0. \end{aligned} \quad \text{(B.40)}$$

The positions of the zeros are

$$\Omega^\pm = \frac{1}{2}\left(k_0 \pm k\sqrt{1-\frac{4m^2}{K^2}}\right). \quad \text{(B.41)}$$

This is real if $K^2 > 4m^2$ or $K^2 < 0$. The solution of the inequality then reads

$$\begin{aligned} K^2 > 4m^2 \quad &\Rightarrow \quad \Omega_+ > \omega > \Omega_- \\ K^2 < 0 \quad &\Rightarrow \quad \Omega_+ < \omega \quad \text{or} \quad \omega < \Omega_-. \end{aligned} \quad \text{(B.42)}$$

Since $\Omega_- < 0$ if $K^2 < 0$, this term does not contribute in the integration range $[m,\infty]$. In the other cases, one can prove that for $K^2 > 4m^2$, we have $\Omega_- > m$, and for $K^2 < 0$, we have $\Omega_+ > m$. Taking into account that we still have $k_0 - \Omega_+ = \Omega_-$, we obtain

$$\begin{aligned} \text{Disc}\, i\mathscr{I}(k) \;=\; &\frac{1}{8\pi}\Theta(K^2 > 4m^2)\sqrt{1-\frac{4m^2}{K^2}} \\ &+\frac{1}{4\pi k}\left(\Theta(K^2 > 4m^2) + \Theta(K^2 < 0)\right)\int_{|\Omega_-|}^{\Omega_+} d\omega n(\omega). \end{aligned} \quad \text{(B.43)}$$

The integral can be performed:

$$\begin{aligned}\mathrm{Disc}\, i\mathscr{I}(k) &= \frac{1}{8\pi}\Theta(K^2 > 4m^2)\sqrt{1-\frac{4m^2}{K^2}} + \\ &+\frac{1}{4\pi k}\left(\Theta(K^2 > 4m^2) + \Theta(K^2 < 0)\right)\ln\frac{1-e^{-\beta\Omega_+}}{1-e^{-\beta|\Omega_-|}}.\end{aligned} \tag{B.44}$$

At zero temperature, we have only the first term: this yields a branch cut starting at $2m$ with a square-root threshold behavior. This corresponds to the creation of two real particles. At finite temperature, the amplitude of the branch cut is modified, and we also obtain a contribution below the light cone ($K^2 < 0$). This latter term is allowed because the finite temperature leads to Lorentz invariance-breaking (it singles out a rest frame). The name of this contribution is Landau damping.

We can study some limiting cases:

- For small q_0, the Landau damping reads

$$\mathrm{Disc}\, i\mathscr{I}(q_0 \ll q) = \frac{q_0}{4\pi q} n\left(\frac{q}{2}\sqrt{1+\frac{4m^2}{q^2}}\right). \tag{B.45}$$

- For small q, the finite-temperature correction to the zero-temperature result reads

$$\frac{1}{4\pi}\Theta(Q^2 > 4m^2)\sqrt{1-\frac{4m^2}{Q^2}}\, n\left(\frac{q_0}{2}\right). \tag{B.46}$$

- At high temperature ($T \gg m, q_0, q$), the second term reads

$$\frac{T}{4\pi q}\left(\Theta(Q^2 > 4m^2) + \Theta(Q^2 < 0)\right)\ln\left|\frac{q_0 + q\gamma}{q_0 - q\gamma}\right|, \qquad \gamma = \sqrt{1-\frac{4m^2}{Q^2}}. \tag{B.47}$$

Once we know the discontinuity, we can recover the original function using the Kramers–Kronig relation. The only problem is that the zero-temperature part goes to a constant at large q_0 value, and therefore, the Kramers–Kronig integral does not converge. To overcome this difficulty, we calculate the contribution directly at zero temperature at finite momenta. We write for the 11 part

$$i\mathscr{I}_{T=0}(q) = -\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-m^2+i\varepsilon)((q-p)^2-m^2+i\varepsilon)}. \tag{B.48}$$

Wick rotation is performed with the rule

$$\int\frac{dp_0}{2\pi}f(p_0) = i\int\frac{dp_{0E}}{2\pi}f(ip_{0E}), \tag{B.49}$$

and we apply Feynman parameterization:

$$\mathscr{I}_{T=0}(q) = -\int_0^1 dx \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{(p_E^2 + m^2 + q_E^2 x(1-x))^2}. \tag{B.50}$$

We now use 4D momentum cutoff to evaluate this expression. With $z = p_E^2$ we obtain

$$\begin{aligned}\mathscr{I}_{T=0}(q) &= -\frac{1}{16\pi^2}\int_0^1 dx \int_0^{\Lambda^2} dz \frac{z}{(z + m^2 + q_E^2 x(1-x))^2} \\ &= -\frac{1}{16\pi^2}\left(\ln\frac{\Lambda^2}{m^2} - 1\right) + \frac{1}{16\pi^2}\int_0^1 dx \ln\frac{m^2 + q_E^2 x(1-x))^2}{m^2}.\end{aligned} \tag{B.51}$$

This means that

$$\mathscr{I}_{T=0}(q) = \mathscr{I}_{T=0}(q=0) + \frac{1}{16\pi^2}\int_0^1 dx \ln\frac{m^2 - q^2 x(1-x)}{m^2}. \tag{B.52}$$

This formula should be true independently of the chosen regularization. Therefore, the full expression can be written as

$$\begin{aligned}\mathscr{I}^{(ra)}(q) = {} & \mathscr{I}_{T=0}(q=0) + \frac{1}{16\pi^2}\int_0^1 dx \ln\frac{m^2 - q^2 x(1-x)}{m^2}\Bigg|_{q_0 + i\varepsilon} \\ & + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mathrm{Disc}\, i\mathscr{I}_T(\omega, \mathbf{q})}{q_0 - \omega + i\varepsilon},\end{aligned} \tag{B.53}$$

where in the second line, only the finite-T part of the discontinuity has to be taken into account.

B.3 Dimensional Regularization

The basic idea of dimensional regularization is that we evaluate the integrals in arbitrary dimension and identify the divergences by the poles of the result in going to integer dimensions.

The basic input to dimensional regularization is that the surface of a d-dimensional sphere $K_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is an analytic function of the dimension. Therefore, a rotationally invariant integrand can be rewritten as

$$\begin{aligned} \mu^{2\varepsilon} \int \frac{d^{d-2\varepsilon} p}{(2\pi)^{d-2\varepsilon}} &= \frac{2(4\pi\mu^2)^{\varepsilon}}{(4\pi)^{d/2}\Gamma(d/2-\varepsilon)} \int dp\, p^{d-1-2\varepsilon} \\ &= \frac{(4\pi\mu^2)^{\varepsilon}}{(4\pi)^{d/2}\Gamma(d/2-\varepsilon)} \int dz\, z^{d/2-\varepsilon-1}, \end{aligned} \tag{B.54}$$

where in the last term, we performed the change of integration variable $z = p^2$. Note that $d = 1$ is also a possible choice.

We need the quantity μ of mass dimension to maintain the original engineering dimension of the integral. Its numerical value is arbitrary, and its appearance is an unavoidable consequence of the presence of divergences (in the case of cutoff regularization, the cutoff itself provides this scale).

One of the most common integrals reads

$$\int_0^\infty dz\, z^{a-1} (M^2 + z)^{-b} = (M^2)^{a-b} \frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)}. \tag{B.55}$$

The properties of the gamma functions are

$$\begin{aligned} &\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \\ &\Gamma(z+1) = z\Gamma(z) \\ &\Gamma(1) = \Gamma(2) = 1, \;\; \Gamma(1/2) = \sqrt{\pi} \\ &\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \varepsilon \left(\frac{\gamma_E^2}{2} + \frac{\pi^2}{12} \right) \\ &\Gamma(-1+\varepsilon) = -\frac{1}{\varepsilon} + \gamma_E - 1 - \varepsilon \left(1 - \gamma_E + \frac{\gamma_E^2}{2} + \frac{\pi^2}{12} \right). \end{aligned} \tag{B.56}$$

Finally, we often need the expansion

$$X^{\varepsilon} = e^{\varepsilon \ln X} = 1 + \varepsilon \ln X + \frac{\varepsilon^2}{2} (\ln X)^2. \tag{B.57}$$

Chapter C
Integrals Relevant for Dimensional Reduction

C.1 Nonstatic Sum-Integrals

The high-temperature expansion of the nonstatic contribution to the free-energy density of the massive ideal gas in powers of M^2/T^2 can be related to the mass expansion of the nonstatic tadpole integral up to an unimportant constant term:

$$I(M) = T\sum_{n\neq 0}\int p^{(\Lambda)} \log(\beta^2(\omega_n^2 + p^2 + M^2)) = I_0 + I_1 M^2 + I_2 M^4 + \ldots,$$

$$I'(M) = T\sum_{n\neq 0}\int p^{(\Lambda)} \frac{1}{\omega_n^2 + p^2 + M^2} = I_1 + 2I_2 M^2 + 3I_3 M^4 + \ldots. \qquad \text{(C.1)}$$

The three-dimensional integrals are computed with sharp momentum cutoff Λ with the result [3]

$$I_1 = I_1^{(2)}\Lambda^2 + I_1^{(1)}\Lambda T + I_1^{(0)} T^2, \qquad I_1^{(1)} = \frac{1}{8\pi^2}, \quad I_1^{(1)} = -\frac{1}{2\pi^2}, \quad I_1^{(0)} = \frac{1}{12},$$

$$I_2 = I_2^{(ln)} \ln\frac{\Lambda}{T} + I_2^{(0)}, \qquad I_2^{(ln)} = -\frac{1}{16\pi^2}, \quad I_2^{(0)} = \frac{1}{16\pi^2}[1 + \ln(2\pi) - \gamma_E],$$

$$I_3 = \frac{\zeta(3)}{192\pi^4 T^2}. \qquad \text{(C.2)}$$

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The nonstatic bubble integral can be reduced to a one-variable integral plus some analytic terms:

$$\begin{aligned}
\mathscr{B}^{nonstatic}(\mathbf{k}) &= T\sum_{n\neq 0}\int p^{(\Lambda)}\frac{1}{(\omega_n^2+\mathbf{p}^2)(\omega_n^2+(\mathbf{p}+\mathbf{k})^2)} \\
&= 2I_2^{(0)} + B(k/T) + F(k/\Lambda), \\
B(\kappa) &= \frac{1}{4\pi^2}\int_0^\infty \frac{dy}{y}\left(1-\frac{y}{\kappa}\ln\left|\frac{\kappa+2y}{\kappa-2y}\right|\right)\left(\frac{1}{e^y+1}+\frac{1}{2}-\frac{1}{y}\right) \\
F(x) &= \left(\frac{2}{x}-1\right)\ln\left(1-\frac{x}{2}\right)+1.
\end{aligned} \tag{C.3}$$

The expansion coefficients of the "setting-sun" integral are introduced as

$$\begin{aligned}
&T^2\sum_{l,n,m\neq 0}\int_{p_1}^{(\Lambda)}\int_{p_2}^{(\Lambda)}\int_{p_3}^{(\Lambda)}\frac{(2\pi)^3\delta(\mathbf{p_1}+\mathbf{p_2}+\mathbf{p_3})\delta_{n_1+n_2+n_3,0}}{(\omega_l^2+\omega_{p_1}^2)(\omega_m^2+\omega_{p_2}^2)(\omega)_m^2+\omega_{p_3}^2)} \\
&= K_0 + K_1M^2 + K_2M^4 + \ldots
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
K_0 &= K_0^{(2)}\Lambda^2 + K_0^{(1)}\Lambda T + K_0^{(ln)}T^2\ln\frac{\Lambda}{T} + K_0^{(0)}T^2, \\
K_1 &= K_1^{(2ln)}\left(\ln\frac{\Lambda}{T}\right)^2 + K_1^{(ln)}\ln\frac{\Lambda}{T} + K_1^{(0)},
\end{aligned} \tag{C.5}$$

and they are computed mostly numerically:

$$\begin{aligned}
&K_0^{(2)} = 0.0001041333, \quad K_0^{(1)} = -0.0029850437, \quad K_0^{(ln)} = \frac{5}{32\pi^2} \\
&K_0^{(0)} = -0.0152887686, \quad K_1^{(2ln)} = -\frac{3}{128\pi^4}, \quad K_1^{(ln)} = 0.001087971.
\end{aligned} \tag{C.6}$$

C.2 Three-Dimensional Integrals

The integrals that are relevant for the one-loop computation of the effective potential in three dimensions are the following:

$$\int p^{(\Lambda)} \log(p^2 + M^2) = 2J_1 M^2 \Lambda + 2J_0 M^3, \quad J_1 = \frac{1}{4\pi^2}, \quad J_0 = -\frac{1}{12\pi}$$

$$\int_{p_1}^{(\Lambda)} \int_{p_2}^{(\Lambda)} \int_{p_3}^{(\Lambda)} \frac{(2\pi)^3 \delta(\mathbf{p_1} + \mathbf{p_2} + \mathbf{p_3})}{(p_1^2 + M^2)(p_2^2 + M^2)(p_3^2 + M^2)} = L^{(ln)} \log \frac{\Lambda^2}{9M^2} + L^{(0)},$$

$$L^{(ln)} = \frac{1}{32\pi^2}, \quad L_0 = 0.00670322. \tag{C.7}$$

References

1. A. Besser, Finite and *p*-adic polylogarithms. Compos. Math. **130**, 215–223 (2002)
2. L. Dolan, R. Jackiw, Phys. Rev. D **9**, 3320 (1974)
3. A. Jakovác, Phys. Rev. D **53**, 4538 (1996)