Appendix

I have been impressed with the urgency of doing. Knowing is not enough; we must apply. Being willing is not enough; we must do.

—Leonardo da Vinci

A.1 Painlevé-Gullstrand Coordinates for General Spherically Symmetric Metrics

Beginning from the metric given by Eq. (2.94) and following the notations of Ref. [1], we search for a new time coordinate τ (Painlevé-Gullstrand time). The transformation $t \rightarrow \tau\,(t, R)$ yields

$$d\tau = \frac{\partial \tau}{\partial t}\, dt + \frac{\partial \tau}{\partial R}\, dR$$

and

$$dt = \frac{1}{\partial \tau / \partial t} \left(d\tau - \frac{\partial \tau}{\partial R}\, dR \right) ,$$

which transforms the line element (2.94) into

$$ds^2 = -\frac{\mathrm{e}^{-2\phi}\,(1 - 2M/R)}{(\partial \tau/\partial t)^2}\, d\tau^2 + 2\mathrm{e}^{-2\phi} \left(1 - \frac{2M}{R}\right) \frac{\partial \tau/\partial R}{(\partial \tau/\partial t)^2}\, d\tau dR$$
$$+ \left[-\mathrm{e}^{-2\phi} \left(1 - \frac{2M}{R}\right) \left(\frac{\partial \tau/\partial R}{\partial \tau/\partial t} \right)^2 + \frac{1}{1 - 2M/R} \right] dR^2 + R^2 d\Omega^2_{(2)} \,. \tag{A.1}$$

We now impose that the new time coordinate τ is such that $g_{11} = 1$, which implies that

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$$\frac{\partial \tau}{\partial R} = \pm \frac{e^{\phi}}{1 - 2M/R} \sqrt{\frac{2M}{R}} \frac{\partial \tau}{\partial t} . \tag{A.2}$$

Then, the metric component g_{01} in the new coordinates is

$$g_{01} = \pm \frac{e^{-\phi}}{\partial \tau / \partial t} \sqrt{\frac{2M}{R}} \tag{A.3}$$

and the line element assumes the form (2.95).

A.2 Kodama Vector in FLRW Space

Here we compute the components of the Kodama vector in FLRW space in pseudo-Painlevé-Gullstrand and in comoving coordinates.

A.2.1 Pseudo-Painlevé-Gullstrand Coordinates

In these coordinates the 2-metric h_{ab} of Eq. (2.71) and its inverse are given by

$$(h_{ab}) = \begin{pmatrix} \frac{-\left(1 - H^2R^2 - kR^2/a^2\right)}{1 - kR^2/a^2} & \frac{-HR}{1 - kR^2/a^2} \\ \\ \frac{-HR}{1 - kR^2/a^2} & \frac{1}{1 - kR^2/a^2} \end{pmatrix} , \tag{A.4}$$

$$\left(h^{ab}\right) = \begin{pmatrix} -1 & -HR \\ \\ -HR & \left(1 - H^2R^2 - kR^2/a^2\right) \end{pmatrix} \tag{A.5}$$

by decomposing the metric (3.25). The volume form on the normal 2-space is

$$\epsilon_{ab} = \sqrt{|h|}\, (dt)_a \wedge (dR)_b = \frac{1}{\sqrt{1 - kR^2/a^2}} \left(\delta_{a0}\delta_{b1} - \delta_{a1}\delta_{b0}\right) , \tag{A.6}$$

while

$$\begin{aligned} \epsilon^{ab} &= g^{ac} g^{bd} \frac{\left(\delta_{c0}\delta_{d1} - \delta_{c1}\delta_{d0}\right)}{\sqrt{1 - kR^2/a^2}} \\ &= \frac{\left(h^{a0}h^{b1} - h^{a1}h^{b0}\right)}{\sqrt{1 - kR^2/a^2}} . \end{aligned}$$

The Kodama vector is

$$K^a \equiv \epsilon^{ab}\nabla_b R = \frac{\left(h^{a0}h^{b1} - h^{a1}h^{b0}\right)}{\sqrt{1-kR^2/a^2}}\,\delta_{b1}$$
$$= \frac{\left(h^{a0}h^{11} - h^{a1}h^{10}\right)}{\sqrt{1-kR^2/a^2}}$$

and, therefore,

$$K^0 = \frac{-1}{\sqrt{1-kR^2/a^2}}\left(1 - H^2R^2 - kR^2/a^2 + H^2R^2\right)$$
$$= \frac{-\left(1-kR^2/a^2\right)}{\sqrt{1-kR^2/a^2}} = -\sqrt{1-kR^2/a^2}\,,$$
$$K^1 = \frac{\left(h^{10}h^{11} - h^{11}h^{10}\right)}{\sqrt{1-kR^2/a^2}} = 0\,.$$

To conclude, we have

$$K^\mu = \left(-\sqrt{1-kR^2/a^2}, 0, 0, 0\right) \quad \text{(pseudo-Painlevé-Gullstrand coordinates).} \tag{A.7}$$

A.2.2 Comoving Coordinates

In comoving coordinates the FLRW line element is

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}\,dr^2 + R^2 d\Omega^2_{(2)} = h_{ab}\,dx^a dx^b + R^2 d\Omega^2_{(2)}\,, \tag{A.8}$$

where $R = a(t)r$ is the areal radius. The volume form on the 2-space (t, r) has components

$$\epsilon_{\alpha\beta} = \sqrt{|h|}\,(dt)_\alpha \wedge (dr)_\beta = \frac{a}{\sqrt{1-kr^2}}\left(\delta_{\alpha 0}\delta_{\beta 1} - \delta_{\alpha 1}\delta_{\beta 0}\right)$$

while

$$\epsilon^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\delta}\epsilon_{\gamma\delta} = \frac{a}{\sqrt{1-kr^2}}\,g^{\alpha\gamma}g^{\beta\delta}\left(\delta_{\gamma 0}\delta_{\delta 1} - \delta_{\gamma 1}\delta_{\delta 0}\right)$$
$$= \frac{a}{\sqrt{1-kr^2}}\left(g^{\alpha 0}g^{\beta 1} - g^{\alpha 1}g^{\beta 0}\right)\,.$$

The components of the Kodama vector in comoving coordinates are

$$\begin{aligned} K^\alpha &\equiv \epsilon^{\alpha\beta}\nabla_\beta R = \epsilon^{\alpha\beta}\left(\dot{a}r\delta_{\beta 0} + a\delta_{\beta 1}\right) \\ &= \frac{a}{\sqrt{1-kr^2}}\left(\dot{a}rh^{\alpha 0}h^{01} + ah^{\alpha 0}h^{11} - \dot{a}rh^{\alpha 1}h^{00} - ah^{\alpha 1}h^{01}\right) \\ &= \frac{a}{\sqrt{1-kr^2}}\left(ah^{\alpha 0}h^{11} - \dot{a}rh^{\alpha 1}h^{00}\right) . \end{aligned}$$

Now,

$$K^0 = \frac{a}{\sqrt{1-kr^2}}\,ah^{00}h^{11} = \frac{-a}{\sqrt{1-kr^2}}\,\frac{\left(1-kr^2\right)a}{a^2} = -\sqrt{1-kr^2}\,,$$

$$K^1 = \frac{-a}{\sqrt{1-kr^2}}\,\dot{a}rh^{11}h^{00} = \frac{\dot{a}ar}{\sqrt{1-kr^2}}\,\frac{\left(1-kr^2\right)}{a^2} = Hr\sqrt{1-kr^2}\,,$$

and the components of the Kodama vector are

$$K^\mu = \left(-\sqrt{1-kr^2}, Hr\sqrt{1-kr^2}, 0, 0\right) \quad \text{(comoving coordinates).} \tag{A.9}$$

The norm squared of K^a is

$$\begin{aligned} K^aK_a &= g_{00}(K^0)^2 + g_{11}(K^1)^2 = -(1-kr^2) + \frac{a^2}{1-kr^2}\,H^2r^2(1-kr^2) \\ &= -\left(1-\dot{a}^2r^2-kr^2\right) \doteq -\left(1-r^2/r_{\mathrm{AH}}^2\right)\ ; \end{aligned} \tag{A.10}$$

it vanishes at the apparent horizon

$$r_{\mathrm{AH}} = \frac{1}{\sqrt{\dot{a}^2+k}}\,. \tag{A.11}$$

Reference

1. Nielsen, A.B., Visser, M.: Production and decay of evolving horizons. Class. Quantum Grav. **23**, 4637 (2006)

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