

# Appendix

*I have been impressed with the urgency of doing. Knowing is not enough; we must apply. Being willing is not enough; we must do.*

—Leonardo da Vinci

## A.1 Painlevé-Gullstrand Coordinates for General Spherically Symmetric Metrics

Beginning from the metric given by Eq.(2.94) and following the notations of Ref. [1], we search for a new time coordinate  $\tau$  (Painlevé-Gullstrand time). The transformation  $t \rightarrow \tau(t, R)$  yields

$$d\tau = \frac{\partial\tau}{\partial t} dt + \frac{\partial\tau}{\partial R} dR$$

and

$$dt = \frac{1}{\partial\tau/\partial t} \left( d\tau - \frac{\partial\tau}{\partial R} dR \right),$$

which transforms the line element (2.94) into

$$\begin{aligned} ds^2 = & -\frac{e^{-2\phi} (1 - 2M/R)}{(\partial\tau/\partial t)^2} d\tau^2 + 2e^{-2\phi} \left( 1 - \frac{2M}{R} \right) \frac{\partial\tau/\partial R}{(\partial\tau/\partial t)^2} d\tau dR \\ & + \left[ -e^{-2\phi} \left( 1 - \frac{2M}{R} \right) \left( \frac{\partial\tau/\partial R}{\partial\tau/\partial t} \right)^2 + \frac{1}{1 - 2M/R} \right] dR^2 + R^2 d\Omega_{(2)}^2. \end{aligned} \tag{A.1}$$

We now impose that the new time coordinate  $\tau$  is such that  $g_{11} = 1$ , which implies that

$$\frac{\partial \tau}{\partial R} = \pm \frac{e^\phi}{1 - 2M/R} \sqrt{\frac{2M}{R}} \frac{\partial \tau}{\partial t}. \quad (\text{A.2})$$

Then, the metric component  $g_{01}$  in the new coordinates is

$$g_{01} = \pm \frac{e^{-\phi}}{\partial \tau / \partial t} \sqrt{\frac{2M}{R}} \quad (\text{A.3})$$

and the line element assumes the form (2.95).

## A.2 Kodama Vector in FLRW Space

Here we compute the components of the Kodama vector in FLRW space in pseudo-Painlevé-Gullstrand and in comoving coordinates.

### A.2.1 Pseudo-Painlevé-Gullstrand Coordinates

In these coordinates the 2-metric  $h_{ab}$  of Eq. (2.71) and its inverse are given by

$$(h_{ab}) = \begin{pmatrix} \frac{-(1-H^2R^2-kR^2/a^2)}{1-kR^2/a^2} & \frac{-HR}{1-kR^2/a^2} \\ \frac{-HR}{1-kR^2/a^2} & \frac{1}{1-kR^2/a^2} \end{pmatrix}, \quad (\text{A.4})$$

$$(h^{ab}) = \begin{pmatrix} -1 & -HR \\ -HR & (1 - H^2R^2 - kR^2/a^2) \end{pmatrix} \quad (\text{A.5})$$

by decomposing the metric (3.25). The volume form on the normal 2-space is

$$\epsilon_{ab} = \sqrt{|h|} (dt)_a \wedge (dR)_b = \frac{1}{\sqrt{1 - kR^2/a^2}} (\delta_{a0}\delta_{b1} - \delta_{a1}\delta_{b0}), \quad (\text{A.6})$$

while

$$\begin{aligned} \epsilon^{ab} &= g^{ac} g^{bd} \frac{(\delta_{c0}\delta_{d1} - \delta_{c1}\delta_{d0})}{\sqrt{1 - kR^2/a^2}} \\ &= \frac{(h^{a0}h^{b1} - h^{a1}h^{b0})}{\sqrt{1 - kR^2/a^2}}. \end{aligned}$$

The Kodama vector is

$$\begin{aligned} K^a &\equiv \epsilon^{ab} \nabla_b R = \frac{(h^{a0} h^{b1} - h^{a1} h^{b0})}{\sqrt{1 - kR^2/a^2}} \delta_{b1} \\ &= \frac{(h^{a0} h^{11} - h^{a1} h^{10})}{\sqrt{1 - kR^2/a^2}} \end{aligned}$$

and, therefore,

$$\begin{aligned} K^0 &= \frac{-1}{\sqrt{1 - kR^2/a^2}} (1 - H^2 R^2 - kR^2/a^2 + H^2 R^2) \\ &= \frac{-(1 - kR^2/a^2)}{\sqrt{1 - kR^2/a^2}} = -\sqrt{1 - kR^2/a^2}, \\ K^1 &= \frac{(h^{10} h^{11} - h^{11} h^{10})}{\sqrt{1 - kR^2/a^2}} = 0. \end{aligned}$$

To conclude, we have

$$K^\mu = \left( -\sqrt{1 - kR^2/a^2}, 0, 0, 0 \right) \quad (\text{pseudo-Painlevé-Gullstrand coordinates}). \quad (\text{A.7})$$

## A.2.2 Comoving Coordinates

In comoving coordinates the FLRW line element is

$$ds^2 = -dt^2 + \frac{a^2(t)}{1 - kr^2} dr^2 + R^2 d\Omega_{(2)}^2 = h_{ab} dx^a dx^b + R^2 d\Omega_{(2)}^2, \quad (\text{A.8})$$

where  $R = a(t)r$  is the areal radius. The volume form on the 2-space  $(t, r)$  has components

$$\epsilon_{\alpha\beta} = \sqrt{|h|} (dt)_\alpha \wedge (dr)_\beta = \frac{a}{\sqrt{1 - kr^2}} (\delta_{\alpha 0} \delta_{\beta 1} - \delta_{\alpha 1} \delta_{\beta 0})$$

while

$$\begin{aligned} \epsilon^{\alpha\beta} &= g^{\alpha\gamma} g^{\beta\delta} \epsilon_{\gamma\delta} = \frac{a}{\sqrt{1 - kr^2}} g^{\alpha\gamma} g^{\beta\delta} (\delta_{\gamma 0} \delta_{\delta 1} - \delta_{\gamma 1} \delta_{\delta 0}) \\ &= \frac{a}{\sqrt{1 - kr^2}} (g^{\alpha 0} g^{\beta 1} - g^{\alpha 1} g^{\beta 0}). \end{aligned}$$

The components of the Kodama vector in comoving coordinates are

$$\begin{aligned}
 K^\alpha &\equiv \epsilon^{\alpha\beta} \nabla_\beta R = \epsilon^{\alpha\beta} (\dot{a}r\delta_{\beta 0} + a\delta_{\beta 1}) \\
 &= \frac{a}{\sqrt{1-kr^2}} (\dot{a}rh^{\alpha 0}h^{01} + ah^{\alpha 0}h^{11} - \dot{a}rh^{\alpha 1}h^{00} - ah^{\alpha 1}h^{01}) \\
 &= \frac{a}{\sqrt{1-kr^2}} (ah^{\alpha 0}h^{11} - \dot{a}rh^{\alpha 1}h^{00}) .
 \end{aligned}$$

Now,

$$\begin{aligned}
 K^0 &= \frac{a}{\sqrt{1-kr^2}} ah^{00}h^{11} = \frac{-a}{\sqrt{1-kr^2}} \frac{(1-kr^2)a}{a^2} = -\sqrt{1-kr^2}, \\
 K^1 &= \frac{-a}{\sqrt{1-kr^2}} \dot{a}rh^{11}h^{00} = \frac{\dot{a}ar}{\sqrt{1-kr^2}} \frac{(1-kr^2)}{a^2} = Hr\sqrt{1-kr^2},
 \end{aligned}$$

and the components of the Kodama vector are

$$K^\mu = \left(-\sqrt{1-kr^2}, Hr\sqrt{1-kr^2}, 0, 0\right) \quad (\text{comoving coordinates}). \quad (\text{A.9})$$

The norm squared of  $K^a$  is

$$\begin{aligned}
 K^a K_a &= g_{00}(K^0)^2 + g_{11}(K^1)^2 = -(1-kr^2) + \frac{a^2}{1-kr^2} H^2 r^2 (1-kr^2) \\
 &= -(1 - \dot{a}^2 r^2 - kr^2) \doteq -(1 - r^2/r_{\text{AH}}^2); \quad (\text{A.10})
 \end{aligned}$$

it vanishes at the apparent horizon

$$r_{\text{AH}} = \frac{1}{\sqrt{\dot{a}^2 + k}}. \quad (\text{A.11})$$

## Reference

1. Nielsen, A.B., Visser, M.: Production and decay of evolving horizons. *Class. Quantum Grav.* **23**, 4637 (2006)

# Index

## Symbols

$f(\mathcal{B})$  gravity, 105, 151, 182, 183, 187, 188

## A

advanced time, 7, 8

anti-de Sitter, 108

anti-trapped surface, 30

apparent horizon, 2, 14, 37–40, 47, 49, 50, 53, 59, 63, 64, 66, 69–71, 78, 79, 81–86, 92–96, 98–101, 107, 108, 112–117, 123–127, 133, 134, 136–138, 142, 145, 149–153, 155–158, 167, 171–176, 180–182, 184, 186–189, 196

apparent horizon tube, 156

areal radius, 5–7, 41, 47, 49, 53, 60, 62, 64, 73, 78, 84, 87, 95, 107, 110, 112, 118, 121, 127, 133, 135, 141, 142, 147–150, 155, 170, 174, 178, 180, 181, 183, 195

areal volume, 71, 95

## B

binary system, 2, 35

Boyer-Linquist coordinates, 16, 18

Brans-Dicke parameter, 168, 169, 172, 174, 175, 180

Brans-Dicke theory, 38, 168, 169, 188

## C

Cauchy horizon, 1, 14, 39, 124

conformal anomaly, 143

conformal factor, 139–141, 143, 145, 146, 157

conformal Killing equation, 139

conformal Killing horizon, 37

conformal Killing vector, 139

conformal time, 140, 153, 155

conformal transformation, 139, 141, 143–145, 157, 176, 177

## D

de Sitter space, 1, 2, 61, 63, 77, 78, 81, 82, 87–92, 98, 101

deviation vector, 26, 27

dominant energy condition, 19

dragging of inertial frames, 17

dynamical horizon, 1, 2, 38, 40, 81

## E

Eddington-Finkelstein coordinates, 8, 11, 46

effective action, 3, 105

energy supply vector, 96

ergosphere, 17

event horizon, 1, 2, 5, 6, 8, 9, 12, 17, 33,

35–39, 43, 44, 49, 61, 70, 73–79, 82, 85,

88–92, 94, 99–101, 108, 122, 124–126,

142, 156, 168, 185

extremal horizon, 1, 14, 17, 50, 92, 115, 122,

124–126

## F

Fisher spacetime, 146

fluid-gravity duality, 108

future apparent horizon, 37

future inner trapping horizon, 39

future null infinity, 2, 35, 36

future outer trapping horizon, 39

**G**

Gauss-Bonnet gravity, 167, 168  
 generalized Raychaudhuri equation, 131  
 geodesic deviation equation, 26  
 geodesic equation, 25, 26, 28, 40  
 Gibbons-Hawking entropy, 91

**H**

Hawking radiation, 1, 2, 34, 39, 45, 59, 93,  
 106, 126, 156, 167, 186  
 Hawking temperature, 43, 93, 142, 143,  
 145, 189  
 Hawking-Hayward quasi-local energy, 1, 47,  
 50, 168  
 Hořava-Lifschitz gravity, 138, 186  
 Horndeski theory, 167, 186, 188  
 Hubble horizon, 78, 172  
 Hubble parameter, 19, 60, 61, 63, 107, 111,  
 113, 116, 127, 141, 155  
 hyperspherical coordinates, 61, 64, 72, 74,  
 76, 84

**I**

inflation, 2, 20, 59, 73, 78, 100, 157  
 isolated horizon, 40, 46  
 isotropic radius, 6, 109, 110, 117, 121,  
 128–130, 133, 140, 169, 173, 183

**J**

Jebsen-Birkhoff theorem, 168

**K**

Kerr-Schild coordinates, 12, 13  
 Kerr-Schild metric, 12, 13  
 Kerr-Schild transformation, 157  
 Killing equation, 36, 43, 44, 89, 90  
 Killing horizon, 36, 37, 40, 43, 44, 51, 89, 90,  
 126, 127, 139  
 Killing vector, 10, 17, 34, 36, 37, 40–44, 46,  
 51–53, 89–92, 126  
 Kodama vector, 37, 41, 66, 67, 92, 93, 96, 167,  
 168, 194–196  
 Kretschmann scalar, 17  
 Kruskal-Szekeres coordinates, 7, 8, 12, 14,  
 15, 61

**L**

Lemaître-Tolman-Bondi model, 106, 138,  
 156, 158

**M**

marginal surface, 30  
 marginally outer trapped tube, 30  
 marginally trapped surface, 35, 44  
 marginally trapped tube, 40  
 Misner-Sharp-Hernandez mass, 1, 46, 47, 50,  
 53, 63, 65, 66, 70, 71, 78, 92, 95, 98,  
 101, 108, 110, 112, 127, 134, 141, 145,  
 167, 168, 188

**N**

naked singularity, 14, 17, 108, 113, 117, 125,  
 136, 146, 150, 151, 156–158, 172, 175,  
 176, 181, 186  
 Nolan gauge, 63  
 Nolan interior solution, 117–119  
 normal surface, 29  
 null curvature condition, 38  
 null dominant energy condition, 19  
 null energy condition, 19, 30, 38, 50, 110

**P**

Painlevé-Gullstrand coordinates, 10–12, 46,  
 48, 49, 62–65, 67, 68, 70, 87, 90, 193  
 particle creation, 143, 144  
 particle horizon, 70, 72–76, 79, 85, 88, 100  
 past inner trapping horizon, 83  
 phantom energy, 19, 106, 132, 187  
 phantom field, 154  
 phantom fluid, 81, 100, 116, 117  
 phantom universe, 77, 116, 117, 132  
 positive curvature condition, 28

**Q**

quantum gravity, 40, 126, 138

**R**

Raychaudhuri equation, 28, 29  
 retarded time, 7, 8  
 Ricci tensor, 4  
 Riemann tensor, 4  
 Rindler horizon, 1, 3, 30, 32–34, 51, 52, 79  
 Rindler observer, 32, 52

**S**

S-curve, 151, 156, 158, 175, 176, 178, 182,  
 186, 188  
 slowly evolving horizon, 2, 41  
 spacetime singularity, 5, 16, 61, 110, 112, 114,  
 122, 123, 141, 147, 156, 171, 188

static limit, 17  
strong energy condition, 19, 77  
supergravity, 126, 167  
supernovae, 3  
Synge approach, 127

**T**

thermodynamics of spacetime, 3, 51, 100  
timelike membrane, 40  
Tolman-Oppenheimer-Volkoff equation, 119  
tortoise coordinate, 7, 15  
trapped surface, 2, 25, 30  
trapping horizon, 1, 2, 37–39, 44, 45, 50, 70,  
83, 88, 95, 105, 156, 158, 168, 188, 189  
trapping horizon tube, 188

**U**

uniform acceleration, 30, 31, 34, 52  
Unruh effect, 33, 34  
Unruh temperature, 33, 52  
untrapped surface, 30, 49

**V**

Vaidya spacetime, 1, 35, 38, 156

**W**

weak energy condition, 19, 81, 95,  
110, 116  
white hole, 8, 9, 12, 37, 39, 74, 86  
wormhole, 19, 37, 48, 115