Appendix

A.1 Specific Example Provided in Padilla et al. (2011)

Following the example provided in Padilla et al. (2011), we now take the following numerical values for the parameters $a_{m,n}$ and $b_{m,n}$: $a_{10} = 3\mu^2$, $a_{01} = -1\mu^2$, $b_{01} = 1/2\mu^2$, $a_{30} = 2\mu^2/H_0^4$, $a_{21} = -13\mu^2/H_0^4$, $a_{12} = 24\mu^2/H_0^4$, $a_{03} = -9\mu^2/H_0^4$ and $b_{03} = -18\mu^2/H_0^4$. Plugging these values into the expression for the sound speed yields:

$$cs_+^2(r) \sim 1 + 133.9\frac{m^2}{r^6}\frac{1}{H_0^4\mu^4} \tag{A.1}$$

$$cs_-^2(r) \sim 1 - 14933.9\frac{m^2}{r^6}\frac{1}{H_0^4\mu^4} \tag{A.2}$$

We thus have one mode that propagates sub-luminaly at infinity and the other mode propagating super-luminaly. It therefore seems that the example provided in Padilla et al. (2011) does not achieve the goal of avoiding superluminal propagation. At this level, this might be just due to a simple typo in the specific example provided in Padilla et al. (2011), but as we show in this thesis, there are no possible sets of quadratic and quartic bi-Galileon interactions that can prevent the propagation of one superluminal mode, even though the parameters at infinity are chosen such that bode modes propagate (sub-)luminally at far infinity.

A.2 Detailed Analysis of the Special Case in the Quartic Galileon: Dominant First Order Corrections

In this section of the appendix we will show that $\tau - \delta > 0$ when $\lambda_1^2 > \lambda_2^2$.

L. Heisenberg, *Theoretical and Observational Consistency of Massive Gravity*,
Springer Theses, DOI 10.1007/978-3-319-18935-2

We can write $\tau - \delta$ as

$$\tau - \delta = \frac{6b_{01}m}{\lambda_1^4 y_1}(1-\alpha) \times \Big[\frac{1}{8(3\alpha-1)^2}\left(a_{10}b_{01} - 2\frac{3\alpha-1}{5\alpha-1}a_{01}^2\right) + \frac{2}{(1-\alpha)(5\alpha-1)}\zeta^2 - \frac{1}{(5\alpha-1)(3\alpha-1)}\zeta a_{01}\Big]. \tag{A.3}$$

We also use the notation $\lambda_2^2 = \alpha\lambda_1^2$. Note that since $y_1, b_{01} > 0$ we must have $3\lambda_2^2 > \lambda_1^2$. Thus we have $1/3 < \alpha < 1$, the upper bound comes from our assumption that $\lambda_1^2 > \lambda_2^2$.

Now the first term in the brackets of the expression $\tau - \delta$ has the same sign as

$$a_{01}b_{01} - \epsilon a_{01}^2 , \tag{A.4}$$

with $0 < \epsilon < 1$. This is positive because in order to avoid ghost instabilities at large distances from the source $a_{10}b_{01} - a_{01}^2 > 0$ and $a_{10}, b_{01} > 0$, (see Eq. (4.47)). Meanwhile, the second term in the brackets is manifestly positive. Finally, the third term has the sign of $-\zeta a_{01}$.

So at this point our only hope of avoiding superluminalities is to consider a choice of parameters where $-\zeta a_{01} < 0$. Now we will proceed to show that $\tau - \delta > 0$ in this case as well.

Note that in the limit $\alpha \to 1$ with everything else fixed we have

$$\tau - \delta \longrightarrow \frac{3b_{01}m\zeta^2}{\lambda_1^4 y_1} > 0 . \tag{A.5}$$

Also in the limit $\alpha \to 1/3$ with everything else fixed we have

$$\tau - \delta \longrightarrow \frac{a_{10}b_{01}^2 m}{2\lambda_1^4 y_1}\frac{1}{(1-3\alpha)^2} > 0 . \tag{A.6}$$

Now consider the function

$$\sigma(\alpha) = 8(1-\alpha)(5\alpha-1)(3\alpha-1)^2\Big[\frac{1}{8(3\alpha-1)^2}\left(a_{10}b_{01} - 2\frac{3\alpha-1}{5\alpha-1}a_{01}^2\right) + \frac{2}{(1-\alpha)(5\alpha-1)}\zeta^2 - \frac{1}{(5\alpha-1)(3\alpha-1)}\zeta a_{01}\Big]. \tag{A.7}$$

We can write this function in the shortened notation as $\sigma(\alpha) = \sigma_0 + \sigma_1\,\alpha + \sigma_2\,\alpha^2$ with

$$\sigma_0 = 2a_{01}^2 - a_{10}b_{01} + 8a_{01}\zeta + 16\zeta^2 \tag{A.8}$$

$$\sigma_1 = -8a_{01}^2 + 6a_{10}b_{01} - 32a_{01}\zeta - 96\zeta^2 \tag{A.9}$$

$$\sigma_2 = 6a_{01}^2 - 5a_{10}b_{01} + 24a_{01}\zeta + 144\zeta^2\,. \tag{A.10}$$

The sign of $\sigma(\alpha)$ is the same as the sign of $\tau - \delta$ in the regime $1/3 < \alpha < 1$. Note that $\sigma(1/3), \sigma(1) > 0$ using the limits above.

Note that $\sigma_0, \sigma_1, \sigma_2$ do not have a definite sign, because $a_{01}\zeta, \zeta^2 > 0$, but $a_{01}^2 - a_{10}b_{01} < 0$. Therefore we need to investigate the behaviour of this function $\sigma(\alpha)$ in more detail.

Being a quadratic function $\sigma(\alpha)$ has a single critical point (either corresponding to a maximum or a minimum) $\alpha_{\rm crit}$. Given that $\sigma(1/3), \sigma(1) > 0$, we can avoid superluminalities if and only if $1/3 < \alpha_{\rm crit} < 1$ and simultaneously $\sigma(\alpha_{\rm crit}) < 0$.

Computing $\frac{d\,\sigma(\alpha)}{d\,\alpha} = 0$ yields for the critical point $\alpha_{\rm crit}$

$$\alpha_{\rm crit} = -\frac{\sigma_1}{2\,\sigma_2} = \frac{4a_{01}^2 - 3a_{10}b_{01} + 16a_{01}\zeta + 48\zeta^2}{\sigma_2}\,. \tag{A.11}$$

Plugging this back into the expression for $\sigma(\alpha_{\rm crit})$ gives the following expression

$$\sigma(\alpha_{\rm crit}) = 4(a_{10}b_{01} - a_{01}^2)\frac{a_{01}^2 + 8a_{01}\zeta + 16\zeta^2 - a_{10}b_{01}}{\sigma_2}\,. \tag{A.12}$$

It is useful to consider

$$1 - \alpha_{\rm crit} = 2\frac{a_{01}^2 + 4a_{01}\zeta + 48\zeta^2 - a_{10}b_{01}}{\sigma_2}\,. \tag{A.13}$$

If $\alpha_{\rm crit} < 1$ then this is positive. Similarly

$$\alpha_{\rm crit} - \frac{1}{3} = \frac{1}{3}\frac{6a_{01}^2 + 24a_{01}\zeta - 4a_{10}b_{01}}{\sigma_2}\,. \tag{A.14}$$

If $\alpha_{\rm crit} > 1/3$ then this is positive.[1]

We will now show that we cannot simultaneously satisfy all the criteria that we need to satisfy to avoid superluminalities. We consider four cases which will exhaust all possibilities:

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[1]When we write num of $\sigma(\alpha_{\rm crit})$, we mean $a_{01}^2 + 8a_{01}\zeta + 16\zeta^2 - a_{10}b_{01}$ by that, i.e. we are ignoring the uninteresting factor of $4(a_{10}b_{01} - a_{01}^2) > 0$.

Case 1: $\sigma_2 = 0$

In this case we have

$$\sigma(\alpha) = \sigma_0 + \sigma_1 \alpha$$

Since $\sigma(1/3), \sigma(1) > 0$ we know that $\sigma(\alpha) > 0$ in the whole interval $1/3 < \alpha < 1$.

Case 2: $\sigma_2 < 0$

Consider $\sigma(\alpha_{\text{crit}})$. If we assume that σ_2 is negative, then we can avoid superluminalities if and only if the numerator of $\sigma(\alpha_{\text{crit}})$ is positive.

However the condition that σ_2 is negative means that $a_{01}b_{01} > \frac{6}{5}a_{01}^2 + \frac{24}{5}a_{01}\zeta + \frac{144}{5}\zeta^2$, which implies that

$$\text{num of } \sigma(\alpha_{\text{crit}}) = a_{01}^2 + 8a_{01}\zeta + 16\zeta^2 - a_{10}b_{01} < -\frac{1}{5}(a_{01} - 8\zeta)^2 . \quad (A.15)$$

So we cannot avoid superluminalities in this case either.

Case 3: $\sigma_2 > 0, \zeta > 0$

Again we consider $\sigma(\alpha_{\text{crit}})$. We now assume that σ_2 is positive, so we need to check if numerator of $\sigma(\alpha_{\text{crit}})$ can be negative if we also assume that $\alpha_{\text{crit}} < 1$, i.e. $a_{10}b_{01} < a_{01}^2 + 4a_{01}\zeta + 48\zeta^2$, and also that $\alpha_{\text{crit}} > 1/3$, i.e. $a_{01}b_{01} < \frac{3}{2}a_{01}^2 + 6a_{01}\zeta$.

The inequality $\alpha_{\text{crit}} < 1$ tells us that

$$\text{num of } \sigma(\alpha_{\text{crit}}) = a_{01}^2 + 8a_{01}\zeta + 16\zeta^2 - a_{10}b_{01} > 4a_{01}\zeta - 32\zeta^2 = 4\zeta(a_{01} - 8\zeta) \quad (A.16)$$

and the inequality $\alpha_{\text{crit}} > 1/3$ tells us that

$$\text{num of } \sigma(\alpha_{\text{crit}}) = a_{01}^2 + 8a_{01}\zeta + 16\zeta^2 - a_{10}b_{01} > -\frac{1}{2}a_{01}^2 + 2a_{01}\zeta + 16\zeta^2 \quad (A.17)$$

Now let's take $\zeta > 0$. The first inequality then implies we need $a_{01} - 8\zeta < 0$ to avoid superluminalities. So we set $a_{01} = 8\zeta\epsilon$ for $0 < \epsilon < 1$ (if $\epsilon < 0$ then $-a_{01}\zeta > 0$). Then the second inequality becomes

$$\text{num} > 16\zeta^2(1 + \epsilon - 2\epsilon^2) = 16\zeta^2(1 - \epsilon)(1 + 2\epsilon) > 0 \quad (A.18)$$

So also in this case we are forced to have superluminalities.

Case 4: $\sigma_2 > 0, \zeta < 0$

Now we take $\zeta < 0$. The first inequality then implies we need $a_{01} - 8\zeta > 0$ to avoid superluminalties. However note that both a_{01} and ζ are negative.

So we set $a_{01} = 8\zeta\epsilon$ for $0 < \epsilon < 1$. Then the argument is exactly the same as above, and that concludes our set of possibilities. In conclusion there is no possible way to avoid superluminalities near the origin, even if one had been so lucky as to live in a theory with specifically tuned coefficients for which the first order corrections near the origin vanished. Our result is thus generic: superluminalities are always present near the origin if the field is to be trivial at infinity and stable both at small and large distances.

A.3 Dimensional Regularization

For the one-loop diagrams we required the dimensional regularization technique to obtain the quantum corrections. A recurrent integral which appears in our calculations is of the form

$$J_{\tilde{m},n} = \frac{1}{\tilde{m}^4} \int \frac{d^4k}{(2\pi)^4} \frac{k^{2n}}{\left(k^2 + \tilde{m}^2\right)^n} , \tag{A.19}$$

where $\tilde{m}$ is a placeholder for whichever mass appears in the propagator. By symmetry we have

$$\frac{1}{\tilde{m}^4} \int \frac{d^4k}{(2\pi)^4} \frac{k^{2(n-j)} k_{\alpha_1} \cdots k_{\alpha_{2j}}}{\left(k^2 + \tilde{m}^2\right)^n} = \frac{1}{2^j (j+1)!} \delta_{\alpha_1 \cdots \alpha_{2j}} J_{\tilde{m},n} , \tag{A.20}$$

with the generalized Kronecker symbol,

$$\delta_{\alpha_1 \cdots \alpha_{2j}} = \delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \cdots \alpha_{2j}} + \Big(\{\alpha_2\} \leftrightarrow \{\alpha_3, \ldots, \alpha_{2j}\} \Big) . \tag{A.21}$$

We also note that

$$J_{\tilde{m},n} = \frac{n(n+1)}{2} J_{\tilde{m},1} . \tag{A.22}$$

We do not need to express $J_{\tilde{m},1}$ explicitly in dimensional regularization, but can simply rely on these different relations to show how different diagrams repackage into a convenient form. It suffices to know that $J_{\tilde{m},1}$ contains the logarithmic divergence in $\tilde{m}$, which represents the running in renormalization techniques.

Reference

Padilla A, Saffin PM, Zhou S-Y (2011) Bi-galileon theory II: phenomenology. JHEP 1101:099. doi:10.1007/JHEP01(2011)099