

Acknowledgment

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Appendix

Proof of Lemma 4.1. Let $1 > \eta > 0$ and $\lambda > 0$, then $\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda$ increases with respect to $p > 0$. Further more, if $p = \lambda/(1-\eta)$, then $\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda = \lambda/(1-\eta)$. Therefore if $p \geq \lambda/(1-\eta)$, then $\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda \geq \lambda/(1-\eta) > 0$. On the other hand if $p \geq \lambda/(1-\eta)$, then $p - (\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda) = (\sqrt{(1-\eta)p} - \sqrt{\lambda})^2 \geq 0$, therefore $p \geq \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda$. \square

Proof of Lemma 4.2. Let $1 > \eta > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate the expression $\eta p - 2(\sqrt{1+\eta} - \sqrt{1-\eta})\sqrt{p\lambda}$ as $\eta x^2 - 2(\sqrt{1+\eta} - \sqrt{1-\eta})ax$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $\eta x^2 - 2(\sqrt{1+\eta} - \sqrt{1-\eta})ax = 0$ for x are 0 and $2(\sqrt{1+\eta} - \sqrt{1-\eta})a/\eta$. Therefore if $x > 2(\sqrt{1+\eta} - \sqrt{1-\eta})a/\eta$, or equivalently, $x^2 > 8(1 - \sqrt{1-\eta^2})a^2/\eta^2$, then $\eta x^2 - 2(\sqrt{1+\eta} - \sqrt{1-\eta})ax > 0$. Replacing x by \sqrt{p} and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 4.3. Let $1 > \eta > 0$ and $\lambda > 0$, then we have

$$\begin{aligned} (\sqrt{1+\eta} - \sqrt{1-\eta})^2 > 0 &\Leftrightarrow 1 - \sqrt{1+\eta}\sqrt{1-\eta} > 0 \\ &\Leftrightarrow 1 + \eta - \sqrt{1+\eta}\sqrt{1-\eta} > \eta \\ &\Leftrightarrow (1 + \eta) (\sqrt{1+\eta} - \sqrt{1-\eta})^2 > \eta^2 \\ &\Leftrightarrow (\sqrt{1+\eta} - \sqrt{1-\eta})^2 / \eta^2 > 1/(1 + \eta) \\ &\Leftrightarrow 8(1 - \sqrt{1-\eta^2})\lambda/\eta^2 > 4\lambda/(1 + \eta) \end{aligned}$$

Also we have

$$\begin{aligned}
(\sqrt{1+\eta} - \sqrt{1-\eta})^2 > 0 &\Leftrightarrow 0 > \sqrt{1+\eta}\sqrt{1-\eta} - 1 \\
&\Leftrightarrow \eta > \sqrt{1+\eta}\sqrt{1-\eta} - (1-\eta) \\
&\Leftrightarrow \eta^2 > (1-\eta) (\sqrt{1+\eta} - \sqrt{1-\eta})^2 \\
&\Leftrightarrow 1/(1-\eta) > (\sqrt{1+\eta} - \sqrt{1-\eta})^2 / \eta^2 \\
&\Leftrightarrow 4\lambda/(1-\eta) > 8(1 - \sqrt{1-\eta^2})\lambda/\eta^2
\end{aligned}$$

□

Proof of Lemma 4.4. Let $\eta > 0$ and $\lambda > 0$. If $p > 4\lambda/(1+\eta)$, then $2\sqrt{(1+\eta)p\lambda} - \lambda > 4\lambda - \lambda = 3\lambda > 0$. □

Proof of Lemma 4.5. Let $\eta > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate expression $p - 2\sqrt{(1+\eta)p\lambda} + \lambda$ as $x^2 - 2ax\sqrt{1+\eta} + a^2$ where $x > 0$ and $a > 0$. The solutions to the quadratic equation $x^2 - 2ax\sqrt{1+\eta} + a^2 = 0$ for x are $(\sqrt{1+\eta} - \sqrt{\eta})a$ and $(\sqrt{1+\eta} + \sqrt{\eta})a$. Therefore if $(\sqrt{1+\eta} + \sqrt{\eta})a > x > (\sqrt{1+\eta} - \sqrt{\eta})a$, or equivalently, $(1 + 2\eta + 2\sqrt{\eta(1+\eta)})a^2 > x^2 > (1 + 2\eta - 2\sqrt{\eta(1+\eta)})a^2$, then $0 > x^2 - 2ax\sqrt{1+\eta} + a^2$. Replacing x by \sqrt{p} and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 4.6. Let $\eta > 0$ and $\lambda > 0$. Note that

$$\begin{aligned}
\sqrt{1+\eta} + \sqrt{\eta} > \sqrt{1+\eta} &\Leftrightarrow 2/\sqrt{1+\eta} > 1/\sqrt{1+\eta} > 1/(\sqrt{1+\eta} + \sqrt{\eta}) \\
&\Leftrightarrow 2/(1+\eta) > \sqrt{1+\eta} - \sqrt{\eta} \\
&\Leftrightarrow 4/(1+\eta) > (\sqrt{1+\eta} - \sqrt{\eta})^2 \\
&\Leftrightarrow 4\lambda/(1+\eta) > (1 + 2\eta - 2\sqrt{\eta(1+\eta)})\lambda
\end{aligned}$$

□

Proof of Lemma 4.7. Let $4/5 > \eta > 0$ and $\lambda > 0$, then we have

$$\begin{aligned}
4\eta - 5\eta^2 > 0 &\Leftrightarrow 2\sqrt{\eta(1-\eta)} > \eta \\
&\Leftrightarrow 1 + 2\sqrt{\eta(1-\eta)} > 1 + \eta \\
&\Leftrightarrow (\sqrt{1-\eta} + \sqrt{\eta})^2 > (\sqrt{1+\eta})^2
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \sqrt{1-\eta} + \sqrt{\eta} > \sqrt{1+\eta} \\
&\Leftrightarrow \sqrt{1-\eta} > \sqrt{1+\eta} - \sqrt{\eta} \\
&\Leftrightarrow \sqrt{1-\eta} \left(\sqrt{1+\eta} + \sqrt{\eta} \right) > 1 \\
&\Leftrightarrow \left(1 + 2\eta + 2\sqrt{\eta(1+\eta)} \right) \lambda > \lambda/(1-\eta)
\end{aligned}$$

and we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 4.8. Let $4/5 > \eta > 0$ and $\lambda > 0$, then we have

$$\begin{aligned}
9(1-\eta) > 1+\eta &\Leftrightarrow 3\sqrt{1-\eta} > \sqrt{1+\eta} \\
&\Leftrightarrow 4\sqrt{1-\eta} > \sqrt{1+\eta} + \sqrt{1-\eta} \\
&\Leftrightarrow 2\sqrt{1-\eta} \left(\sqrt{1+\eta} - \sqrt{1-\eta} \right) > \eta \\
&\Leftrightarrow 4(1-\eta) \left(\sqrt{1+\eta} - \sqrt{1-\eta} \right)^2 > \eta^2 \\
&\Leftrightarrow 8 \left(1 - \sqrt{1-\eta^2} \right) \lambda/\eta^2 > \lambda/(1-\eta)
\end{aligned}$$

and we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 4.13. Let $4/5 > \eta > 0$ and $\lambda > 0$. First we prove (a) and (b) together. According to Lemma 4.8 part (a), $p_3 > p_2$, therefore $p_3 = \eta p_3 + (1-\eta)p_3 > \eta p_3 + (1-\eta)\sqrt{p_2 p_3} > \eta p_2 + (1-\eta)p_2 = p_2$. Next we prove (c). According to Lemma 4.8 part (a), $p_3 > p_2$, therefore $2\eta(\sqrt{p_3} - \sqrt{p_2})/\sqrt{p_2} > 0 \Rightarrow (1+2\eta)(\sqrt{p_3} - \sqrt{p_2})/\sqrt{p_2} p_3 > p_3$. Finally we prove (d). According to Lemma 4.3, we have $4p_2 > p_3$, therefore $2\sqrt{p_2} > \sqrt{p_3} \Leftrightarrow \sqrt{p_2} > \sqrt{p_3} - \sqrt{p_2} \Leftrightarrow 1 > (\sqrt{p_3} - \sqrt{p_2})/\sqrt{p_2}$. Therefore $(1+2\eta)p_3 > (1+2\eta)(\sqrt{p_3} - \sqrt{p_2})/\sqrt{p_2} p_3$. \square

Proof of Lemma 4.14. Let $\eta > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{r}$ and $a \equiv \sqrt{\lambda}$ and the expression $r - 2\sqrt{(1+2\eta)r\lambda/(1+\eta)} + \lambda$ can be restated as $x^2 - 2ax\sqrt{(1+2\eta)/(1+\eta)} + a^2$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $x^2 - 2ax\sqrt{(1+2\eta)/(1+\eta)} + a^2 = 0$ for x are $(\sqrt{1+2\eta} - \sqrt{\eta})a/\sqrt{1+\eta}$ and $(\sqrt{1+2\eta} + \sqrt{\eta})a/\sqrt{1+\eta}$. Thus if $(\sqrt{1+2\eta} + \sqrt{\eta})a/\sqrt{1+\eta} > x > (\sqrt{1+2\eta} - \sqrt{\eta})a/\sqrt{1+\eta}$, or equivalently, if $\left(1 + 3\eta + 2\sqrt{\eta(1+2\eta)}\right)a/(1+\eta) > x^2 > \left(1 + 3\eta - 2\sqrt{\eta(1+2\eta)}\right)a/(1+\eta)$, then $0 > x^2 - 2ax\sqrt{(1+2\eta)/(1+\eta)} + a^2$. Replacing x by \sqrt{r} and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 4.15. Let $1 > \eta > 0$ and $\lambda > 0$, and we have $\sqrt{1+\eta} > \sqrt{1-\eta}$ and $\sqrt{1+2\eta} > \sqrt{\eta}$, therefore we have $2\sqrt{1+\eta} > \sqrt{1+\eta} +$

$\sqrt{1-\eta}$ and $2\sqrt{1+2\eta} > \sqrt{1+2\eta} + \sqrt{\eta}$. By multiplying these inequalities $4\sqrt{1+2\eta}\sqrt{1+\eta} > (\sqrt{1+2\eta} + \sqrt{\eta})(\sqrt{1+\eta} + \sqrt{1-\eta})$. By multiplying $(\sqrt{1+\eta} - \sqrt{1-\eta})$ and dividing $2\eta\sqrt{1+\eta}$ on both sides we have $2\sqrt{1+2\eta}(\sqrt{1+\eta} - \sqrt{1-\eta})/\eta > (\sqrt{1+2\eta} + \sqrt{\eta})$. By squaring both sides $(1+2\eta)p_3 > (1+3\eta+2\sqrt{\eta(1+2\eta)})\lambda/(1+\eta)$. \square

Proof of Lemma 4.16. Note that $f(x)$ is continuous and differentiable over interval $[\sqrt{p_2}, \sqrt{p_3}]$:

$$\begin{aligned} \frac{df(x)}{dx} &= -2\eta x - \sqrt{p_2} \left((1-2\eta) - \frac{r}{x^2} \right) \text{ and } \frac{d^2f(x)}{dx^2} = -2\eta - \frac{2r\sqrt{p_2}}{x^3} < 0 \\ \frac{df(x)}{dx} \Big|_{x=\sqrt{p_2}} &= -\sqrt{p_2} + \frac{r}{\sqrt{p_2}} = \frac{1}{\sqrt{p_2}} (r - p_2) \\ \frac{df(x)}{dx} \Big|_{x=\sqrt{p_3}} &= -\sqrt{p_2} - 2\eta(\sqrt{p_3} - \sqrt{p_2}) + \frac{r\sqrt{p_2}}{p_3} = \frac{\sqrt{p_2}}{p_3} (r - r_2) \end{aligned}$$

If $r \in (0, p_2]$, then $f(x)$ is decreasing over $[\sqrt{p_2}, \sqrt{p_3}]$, therefore $x^* = \sqrt{p_2}$. If $r \geq r_2$, then $f(x)$ is increasing over $[\sqrt{p_2}, \sqrt{p_3}]$, therefore $x^* = \sqrt{p_3}$. If $r \in (p_2, r_2)$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{p_2}$ and decreasing in the neighborhood below $x = \sqrt{p_3}$. Also note that $d^2f(x)/dx^2 < 0$, therefore $x^* \in (\sqrt{p_2}, \sqrt{p_3})$ that satisfies the first order condition

$$\frac{df(x)}{dx} \Big|_{x=x^*} = 0 \Rightarrow 2\eta (x^*)^3 + (1-2\eta)\sqrt{p_2} (x^*)^2 - r\sqrt{p_2} = 0 \quad (\text{A.1})$$

which is a cubic equation. According to the general formula for roots of cubic equation, $x^* = \sqrt{p_{cu}} \in (\sqrt{p_2}, \sqrt{p_3})$. \square

Proof of Lemma 4.17. Note that $f(x)$ is continuous and differentiable for $x \geq \sqrt{p_3}$:

$$\begin{aligned} \frac{df(x)}{dx} &= -\sqrt{p_1} \left(1 + 2\eta - \frac{r}{x^2} \right) \text{ and } \frac{d^2f(x)}{dx^2} = -\frac{2r\sqrt{p_1}}{x^3} < 0 \\ \frac{df(x)}{dx} \Big|_{x=\sqrt{p_3}} &= \frac{\sqrt{p_1}}{p_3} (r - r_3) \text{ and } \lim_{x \rightarrow +\infty} \frac{df(x)}{dx} = -(1+2\eta)\sqrt{p_1} < 0 \end{aligned}$$

If $r \in (0, r_3]$, then $f(x)$ is decreasing for $x \geq \sqrt{p_3}$, therefore $x^* = \sqrt{p_3}$. If $r > r_3$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{p_3}$ and decreasing when x approaches $+\infty$. Also note that $d^2f(x)/dx^2 < 0$, therefore $x^* > \sqrt{p_3}$ that satisfies first order condition $df(x)/dx|_{x=x^*} = 0 \Rightarrow x^* = \sqrt{r/(1+2\eta)}$. \square

Proof of Lemma 4.22. Let $\eta > 1/3$ and $\lambda > 0$. Note that

$$\begin{aligned} 3\eta > 1 &\Leftrightarrow 2\sqrt{\eta} > \sqrt{1+\eta} \\ &\Leftrightarrow \sqrt{1+\eta} > 2\left(\sqrt{1+\eta} - \sqrt{\eta}\right) \\ &\Leftrightarrow (1+\eta)\left(\sqrt{1+\eta} + \sqrt{\eta}\right)^2 > 4 \\ &\Leftrightarrow \left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)\lambda > 4\lambda/(1+\eta) \end{aligned}$$

□

Proof of Lemma 4.25. Note that $f(x)$ is continuous and differentiable for $x \geq \sqrt{p_4}$:

$$\begin{aligned} \frac{df(x)}{dx} &= -\sqrt{p_1}\left(1+2\eta - \frac{r}{x^2}\right) \text{ and } \frac{d^2f(x)}{dx^2} = -\frac{2r\sqrt{p_1}}{x^3} < 0 \\ \frac{df(x)}{dx} \Big|_{x=\sqrt{p_4}} &= \frac{\sqrt{p_1}}{p_4}(r-r_4) \text{ and } \lim_{x \rightarrow +\infty} \frac{df(x)}{dx} = -(1+2\eta)\sqrt{p_1} < 0 \end{aligned}$$

If $r \in (0, r_4]$, then $f(x)$ is decreasing for $x \geq \sqrt{p_4}$, therefore $x^* = \sqrt{p_4}$. If $r > r_4$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{p_4}$ and decreasing when x approaches $+\infty$. Note that $d^2f(x)/dx^2 < 0$, therefore $x^* > \sqrt{p_4}$ satisfies first order condition $df(x)/dx|_{x=x^*} = 0 \Rightarrow x^* = \sqrt{r/(1+2\eta)}$. □

Proof of Lemma 4.26. Let $\eta > 0$ and $\lambda > 0$, then we have

$$\begin{aligned} \sqrt{\eta}\left(\sqrt{1+2\eta} - \sqrt{1+\eta}\right) + \eta > 0 &\Leftrightarrow 0 > -\sqrt{\eta}\sqrt{1+2\eta} + \sqrt{\eta}\sqrt{1+\eta} - \eta \\ &\Leftrightarrow \sqrt{1+2\eta}\sqrt{1+\eta} > \left(\sqrt{1+2\eta} + \sqrt{\eta}\right)\left(\sqrt{1+\eta} - \sqrt{\eta}\right) \\ &\Leftrightarrow \left(\sqrt{1+\eta} + \sqrt{\eta}\right)\sqrt{1+2\eta} > \left(\sqrt{1+2\eta} + \sqrt{\eta}\right)/\sqrt{1+\eta} \end{aligned}$$

by squaring both sides we have $(1+2\eta)p_4 > \left(1+3\eta+2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$. □

Proof of Lemma 5.2. Let $1 > \bar{\eta} > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate the expression $\bar{\eta}p/2 + 2\sqrt{(1-\bar{\eta})p\lambda} - \lambda$ as $\bar{\eta}x^2/2 + 2ax\sqrt{(1-\bar{\eta})} - a^2$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $\bar{\eta}x^2/2 + 2ax\sqrt{(1-\bar{\eta})} - a^2 = 0$ for x are $0 > -2\left(\sqrt{1-\bar{\eta}/2} + \sqrt{1-\bar{\eta}}\right)a/\bar{\eta}$ and $2\left(\sqrt{1-\bar{\eta}/2} - \sqrt{1-\bar{\eta}}\right)a/\bar{\eta} > 0$. Therefore if $2\left(\sqrt{1-\bar{\eta}/2} - \sqrt{1-\bar{\eta}}\right)a/\bar{\eta} > x > 0$, or equivalently, $4\left(\sqrt{1-\bar{\eta}/2} - \sqrt{1-\bar{\eta}}\right)^2 a^2/\bar{\eta}^2 > x^2 > 0$, then $0 >$

$\bar{\eta}x^2/2 + 2ax\sqrt{(1-\bar{\eta})} - a^2$. Replacing x by \sqrt{p} and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 5.3. Let $1 > \bar{\eta} > 0$ and $\lambda > 0$, then we have

$$\begin{aligned}
 0 > \bar{\eta}^2 - 2\bar{\eta} &\Leftrightarrow \bar{\eta}^2 - 4\bar{\eta} + 4 > 2\bar{\eta}^2 - 6\bar{\eta} + 4 \\
 &\Leftrightarrow (2 - \bar{\eta})^2 > 4(1 - \bar{\eta}/2)(1 - \bar{\eta}) \\
 &\Leftrightarrow 2 - \bar{\eta} > 2\sqrt{1 - \bar{\eta}/2}\sqrt{1 - \bar{\eta}} \\
 &\Leftrightarrow \bar{\eta} > 2\sqrt{1 - \bar{\eta}/2}\sqrt{1 - \bar{\eta}} - 2(1 - \bar{\eta}) \\
 &\Leftrightarrow 1/\sqrt{1 - \bar{\eta}} > 2\left(\sqrt{1 - \bar{\eta}/2} - \sqrt{1 - \bar{\eta}}\right)/\bar{\eta} \\
 &\Leftrightarrow \lambda/(1 - \bar{\eta}) > 4\left(\sqrt{1 - \bar{\eta}/2} - \sqrt{1 - \bar{\eta}}\right)^2 \lambda/\bar{\eta}^2
 \end{aligned}$$

\square

Proof of Lemma 5.4. Let $2 > \bar{\eta} > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate the expression $(1 - \bar{\eta}/2)p - 2\sqrt{p\lambda} + \lambda$ as $(1 - \bar{\eta}/2)x^2 - 2ax + a^2$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $(1 - \bar{\eta}/2)x^2 - 2ax + a^2 = 0$ for x are $\sqrt{2}a/(\sqrt{2} + \sqrt{\bar{\eta}}) > 0$ and $\sqrt{2}a/(\sqrt{2} - \sqrt{\bar{\eta}}) > 0$. Therefore if $\sqrt{2}a/(\sqrt{2} - \sqrt{\bar{\eta}}) > x > \sqrt{2}a/(\sqrt{2} + \sqrt{\bar{\eta}})$, or equivalently, $2a^2/(2 + \bar{\eta} - 2\sqrt{2\bar{\eta}}) > x^2 > 2a^2/(2 + \bar{\eta} + 2\sqrt{2\bar{\eta}})$, then $0 > (1 - \bar{\eta}/2)x^2 - 2ax + a^2$. Replacing x by \sqrt{p} and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 5.5. Let $\bar{\eta} > 0$ and $\lambda > 0$, then $1 + \sqrt{2\bar{\eta}} > 0 \Leftrightarrow 2 + \sqrt{2\bar{\eta}} > 1 \Leftrightarrow \sqrt{2} > 1/(\sqrt{2} + \sqrt{\bar{\eta}}) \Leftrightarrow 4\lambda > 2\lambda/(2 + \bar{\eta} + 2\sqrt{2\bar{\eta}})$. \square

Proof of Lemma 5.6. Let $8/9 > \bar{\eta} > 0$ and $\lambda > 0$, then we have

$$\begin{aligned}
 8\bar{\eta} - 9\bar{\eta}^2 > 0 &\Leftrightarrow 2\sqrt{2\bar{\eta}} > 3\bar{\eta} \\
 &\Leftrightarrow 2 - 2\bar{\eta} > 2 + \bar{\eta} - 2\sqrt{2\bar{\eta}} \\
 &\Leftrightarrow \sqrt{2}\sqrt{1 - \bar{\eta}} > \sqrt{2} - \sqrt{\bar{\eta}} \\
 &\Leftrightarrow \sqrt{2}/(\sqrt{2} - \sqrt{\bar{\eta}}) > 1/\sqrt{1 - \bar{\eta}} \\
 &\Leftrightarrow 2\lambda/(2 + \bar{\eta} - 2\sqrt{2\bar{\eta}}) > \lambda/(1 - \bar{\eta})
 \end{aligned}$$

and we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 5.7. Let $1 > \bar{\eta} > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate expression $\bar{\eta}p/2 - 2\left(1 - \sqrt{1 - \bar{\eta}}\right)\sqrt{p\lambda}$ as $\bar{\eta}x^2/2 - 2\left(1 - \sqrt{1 - \bar{\eta}}\right)ax$ with $x > 0$ and $a > 0$. The solution to the quadratic equation $\bar{\eta}x^2/2 - 2\left(1 - \sqrt{1 - \bar{\eta}}\right)ax = 0$ for x are 0 and $4\left(1 - \sqrt{1 - \bar{\eta}}\right)a/\bar{\eta} > 0$. Therefore if $4\left(1 - \sqrt{1 - \bar{\eta}}\right)a/\bar{\eta} > x > 0$, or equivalently, $16\left(2 - \bar{\eta} - 2\sqrt{1 - \bar{\eta}}\right)a^2/\bar{\eta}^2 > x^2 > 0$, then $0 > \bar{\eta}x^2/2 - 2\left(1 - \sqrt{1 - \bar{\eta}}\right)ax$. Replacing x by \sqrt{p} and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 5.8. Let $8/9 > \bar{\eta} > 0$ and $\lambda > 0$, then we have

$$\begin{aligned} 0 > 9\bar{\eta}^2 - 8\bar{\eta} &\Leftrightarrow 16(1 - \bar{\eta}) > 9\bar{\eta}^2 - 24\bar{\eta} + 16 \\ &\Leftrightarrow 4\sqrt{1 - \bar{\eta}} > 4 - 3\bar{\eta} \\ &\Leftrightarrow 4\left(1 - \sqrt{1 - \bar{\eta}}\right)/\bar{\eta} > 1/\sqrt{1 - \bar{\eta}} \\ &\Leftrightarrow 16\left(2 - \bar{\eta} - 2\sqrt{1 - \bar{\eta}}\right)^2\lambda/\bar{\eta}^2 > \lambda/(1 - \bar{\eta}) \end{aligned}$$

and we obtain (a). The proofs for (b) and (c) are similar. \square

Proof of Lemma 5.9. Let $1 > \bar{\eta} > 0$ and $\lambda > 0$, then $1 > \sqrt{1 - \bar{\eta}} \Leftrightarrow \bar{\eta} > 2\sqrt{1 - \bar{\eta}} - 2 + 2\bar{\eta} \Leftrightarrow 4\lambda/(1 - \bar{\eta}) > 16\left(2 - \bar{\eta} - 2\sqrt{1 - \bar{\eta}}\right)\lambda/\bar{\eta}^2$. Also note that $1 > \sqrt{1 - \bar{\eta}} \Leftrightarrow 2 - \bar{\eta} - 2\sqrt{1 - \bar{\eta}} > 0 \Leftrightarrow 2\left(1 - \sqrt{1 - \bar{\eta}}\right) > \bar{\eta} \Leftrightarrow 16\left(2 - \bar{\eta} - 2\sqrt{1 - \bar{\eta}}\right)\lambda/\bar{\eta}^2 > 4\lambda$. \square

Proof of Lemma 5.14. Let $8/9 > \bar{\eta} > 0$ and $\lambda > 0$. According to Lemma 5.8 and 5.9, $4\bar{p}_1 > \bar{p}_2 > \bar{p}_1$, therefore:

$$\begin{aligned} 2\sqrt{\bar{p}_1} > \sqrt{\bar{p}_2} &\Leftrightarrow \sqrt{\bar{p}_1} > \sqrt{\bar{p}_2} - \sqrt{\bar{p}_1} > 0 \\ &\Leftrightarrow 1 > \left(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1}\right)/\sqrt{\bar{p}_1} \\ &\Leftrightarrow \bar{\eta} > \bar{\eta}\left(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1}\right)/\sqrt{\bar{p}_1} \\ &\Leftrightarrow 1 > (1 - \bar{\eta}) + \bar{\eta}\left(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1}\right)/\sqrt{\bar{p}_1} \\ &\Leftrightarrow \bar{p}_2 > (1 - \bar{\eta})\bar{p}_2 + \bar{\eta}\bar{p}_2\left(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1}\right)/\sqrt{\bar{p}_1} \end{aligned}$$

Since $\bar{\eta}\bar{p}_2 + (1 - \bar{\eta})\sqrt{\bar{p}_1\bar{p}_2} - \bar{\eta}\bar{p}_2\sqrt{\bar{p}_2}/2\left(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1}\right) = (1 - \bar{\eta})\sqrt{\bar{p}_1\bar{p}_2} + \bar{\eta}\bar{p}_2\left(\sqrt{\bar{p}_2} - \sqrt{4\bar{p}_1}\right)/2\left(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1}\right)$. Since $4\bar{p}_1 > \bar{p}_2 > \bar{p}_1$, therefore we have $(1 - \bar{\eta})\bar{p}_2 > (1 - \bar{\eta})\sqrt{\bar{p}_1\bar{p}_2}$ and $\bar{\eta}\bar{p}_2\left(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1}\right)/\sqrt{\bar{p}_1} > 0 >$

$\bar{\eta}\bar{p}_2(\sqrt{\bar{p}_2} - \sqrt{4\bar{p}_1})/2(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1})$. Thus $(1 - \bar{\eta})\bar{p}_2 + \bar{\eta}\bar{p}_2(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1})/\sqrt{\bar{p}_1} > \bar{\eta}\bar{p}_2 + (1 - \bar{\eta})\sqrt{\bar{p}_1\bar{p}_2} - \bar{\eta}\bar{p}_2\sqrt{\bar{p}_2}/2(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1})$. Therefore we obtain (a). According to Lemma 5.8, $\bar{p}_2 > \bar{p}_1$, therefore $(1 - \bar{\eta})(\bar{p}_2 - \bar{p}_1) + \bar{\eta}\bar{p}_2(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1})/\sqrt{\bar{p}_1} > 0 \Leftrightarrow (1 - \bar{\eta})\bar{p}_2 + \bar{\eta}\bar{p}_2(\sqrt{\bar{p}_2} - \sqrt{\bar{p}_1})/\sqrt{\bar{p}_1} > (1 - \bar{\eta})\bar{p}_1 = \lambda$. Therefore we obtain (b). \square

Proof of Lemma 5.15. Note that $f(x)$ is continuous and differentiable over interval $[\sqrt{\bar{p}_1}, \sqrt{\bar{p}_2}]$:

$$\begin{aligned} \frac{df(x)}{dx} &= -\bar{\eta}x - \sqrt{\bar{p}_1} \left((1 - 2\bar{\eta}) - \frac{r}{x^2} \right) \text{ and } \frac{d^2f(x)}{dx^2} = -\bar{\eta} - \frac{2r\sqrt{\bar{p}_1}}{x^3} < 0 \\ \frac{df(x)}{dx} \Big|_{x=\sqrt{\bar{p}_1}} &= -(1 - \bar{\eta})\sqrt{\bar{p}_1} + \frac{r}{\sqrt{\bar{p}_1}} = \frac{1}{\sqrt{\bar{p}_1}}(r - \lambda) \\ \frac{df(x)}{dx} \Big|_{x=\sqrt{\bar{p}_2}} &= -\bar{\eta}\sqrt{\bar{p}_2} - (1 - 2\bar{\eta})\sqrt{\bar{p}_1} + \frac{r\sqrt{\bar{p}_1}}{\bar{p}_2} = \frac{\sqrt{\bar{p}_1}}{\bar{p}_2}(r - \bar{r}_2) \end{aligned}$$

If $r \in (0, \lambda]$, then $f(x)$ is decreasing over $[\sqrt{\bar{p}_1}, \sqrt{\bar{p}_2}]$, therefore $x^* = \sqrt{\bar{p}_1}$. If $r \geq \bar{r}_2$, then $f(x)$ is increasing over $[\sqrt{\bar{p}_1}, \sqrt{\bar{p}_2}]$, therefore $x^* = \sqrt{\bar{p}_2}$. If $r \in (\lambda, \bar{r}_2)$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{\bar{p}_1}$ and is decreasing in the neighborhood below $x = \sqrt{\bar{p}_2}$. Also note that $d^2f(x)/dx^2 < 0$, therefore $x^* \in (\sqrt{\bar{p}_1}, \sqrt{\bar{p}_2})$ that satisfies the first order condition

$$\frac{df(x)}{dx} \Big|_{x=x^*} = 0 \Rightarrow \bar{\eta}(x^*)^3 + (1 - 2\bar{\eta})\sqrt{\bar{p}_1}(x^*)^2 - r\sqrt{\bar{p}_1} = 0 \quad (\text{A.2})$$

which is a cubic equation. According to the general formula for roots of cubic equation, $x^* = \sqrt{\bar{p}_{cu}} \in (\sqrt{\bar{p}_1}, \sqrt{\bar{p}_2})$. \square

Proof of Lemma 5.16. Note that $f(x)$ is continuous and differentiable for $x \geq \sqrt{\bar{p}_2}$:

$$\begin{aligned} \frac{df(x)}{dx} &= -\sqrt{\lambda} \left(1 - \frac{r}{x^2} \right) \text{ and } \frac{d^2f(x)}{dx^2} = -\frac{2r\sqrt{\lambda}}{x^3} < 0 \\ \frac{df(x)}{dx} \Big|_{x=\sqrt{\bar{p}_2}} &= \frac{\sqrt{\lambda}}{\bar{p}_2}(r - \bar{p}_2) \text{ and } \frac{df(x)}{dx} \Big|_{x \rightarrow +\infty} = -\sqrt{\lambda} < 0 \end{aligned}$$

If $r \in (0, \bar{p}_2]$, then $f(x)$ is decreasing for $r \geq \sqrt{\bar{p}_2}$, therefore $x^* = \sqrt{\bar{p}_2}$. If $r > \sqrt{\bar{p}_2}$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{\bar{p}_2}$. Since $f(x)$ is decreasing as $x \rightarrow +\infty$ and since $f(x)$ is concave ($d^2f(x)/dx^2 < 0$), therefore $x^* > \sqrt{\bar{p}_2}$ is solved from first order condition $df(x)/dx|_{x=x^*} = 0 \Rightarrow x^* = \sqrt{r}$. \square

Proof of Lemma 5.18. Let $2 > \bar{\eta} \geq 8/9$ and $\lambda > 0$, then we have

$$\begin{aligned} 2\bar{\eta} \geq 16/9 &\Leftrightarrow \sqrt{2\bar{\eta}} \geq 4/3 > 1 \\ &\Leftrightarrow 1 > 2 - \sqrt{2\bar{\eta}} \\ &\Leftrightarrow \sqrt{2}/(\sqrt{2} - \sqrt{\bar{\eta}}) > 2 \\ &\Leftrightarrow 2\lambda/(2 + \bar{\eta} - 2\sqrt{2\bar{\eta}}) > 4\lambda \end{aligned}$$

□

Proof of Lemma 5.21. Note that $f(x)$ is continuous and differentiable for $x \geq \sqrt{\bar{p}_3}$:

$$\begin{aligned} \frac{df(x)}{dx} &= -\sqrt{\lambda} \left(1 - \frac{r}{x^2}\right) \text{ and } \frac{d^2f(x)}{dx^2} = -\frac{2r\sqrt{\lambda}}{x^3} < 0 \\ \frac{df(x)}{dx} \Big|_{x=\sqrt{\bar{p}_3}} &= \frac{\sqrt{\lambda}}{\bar{p}_3} (r - \bar{p}_3) \text{ and } \frac{df(x)}{dx} \Big|_{x \rightarrow +\infty} = -\sqrt{\lambda} < 0 \end{aligned}$$

If $r \in (0, \bar{p}_3]$, then $f(x)$ is decreasing for $x \geq \sqrt{\bar{p}_3}$, therefore $x^* = \sqrt{\bar{p}_3}$. If $r > \sqrt{\bar{p}_3}$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{\bar{p}_3}$. Since $f(x)$ is decreasing as $x \rightarrow +\infty$ and since $f(x)$ is concave ($d^2f(x)/dx^2 < 0$), therefore $x^* > \sqrt{\bar{p}_3}$ is solved from first order condition $df(x)/dx|_{x=x^*} = 0 \Rightarrow x^* = \sqrt{r}$. □

Proof of Lemma 5.23. Let $\bar{\eta} > 2$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate the expression $(1 - \bar{\eta}/2)p - 2\sqrt{p\lambda} + \lambda$ as $(1 - \bar{\eta}/2)x^2 - 2ax + a^2$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $(1 - \bar{\eta}/2)x^2 - 2ax + a^2 = 0$ for x are $\sqrt{2}a/(\sqrt{2} + \sqrt{\bar{\eta}}) > 0$ and $0 > \sqrt{2}a/(\sqrt{2} - \sqrt{\bar{\eta}})$. Therefore if $\sqrt{2}a/(\sqrt{2} + \sqrt{\bar{\eta}}) > x > 0$, or equivalently, $2a^2/(2 + \bar{\eta} + 2\sqrt{2\bar{\eta}}) > x^2 > 0$, then $(1 - \bar{\eta}/2)x^2 - 2ax + a^2 > 0$. Replacing x by \sqrt{p} and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □

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