Commentaries and Remarks

1. Formula (1.1.10) is a special case of the operator identity of the form

$$AS - SA = \Pi_1 \Pi_2^*, \tag{1}$$

which is a generalization of the well-known notion of a node (see M.S. Livšic [53] and then M.S. Brodskii [10]). The identities of the form (1) are used for solving problems in numerous domains (system theory [90], factorization problems [90], interpolation theory [28, 93], the inverse spectral problem [90, 94] and theory of nonlinear integrable equations [91, 94]). There are close ties between all these problems and between corresponding results.

2. Apart from the regularization method, there was in fact no approach to the study of equations of the first kind. The operator identities method permits us to construct a theory of equations of the form

$$Sf = \varphi \tag{2}$$

without a requirement of invertibility of the operator S [83, 84, 86].

This approach is presented in all details in Chapter 2 for operators S with a difference kernel and in Chapter 5 for operators with a W-difference kernel.

3. The method of inverting operators with a difference kernel considered in this book was first introduced in connection with the problem of reducing Volterra operators to their simplest form [76]. In this book the inversion method was set forth independently of Volterra operators. This permitted us to omit the requirement on S to be positive.

4. As it was shown in Chapter 1 and Chapter 2, a bounded in $L^p(0, \omega)$ $(p \geq 1)$ operator S with a difference kernel admits a representation

$$Sf = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\omega s(x-t) f(t)\, \mathrm{d}t, \quad f \in L^p(0, \omega). \tag{3}$$

The representation of the operator S in the form (3) allows us to consider uniformly different classes of the operators S and of the corresponding operator equations

$$Sf = \varphi. \tag{4}$$

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In order to demonstrate this, we give a list of the main special cases of formula (3) and also indicate the chapters, where these special cases are considered:

$$Sf = \sum_{j=N}^{M} \mu_j f(x - x_j) + \int_0^{\omega} K(x-t) f(t)\,\mathrm{d}t, \tag{5}$$

where $f(x) = 0\ x\overline{\in}[0, \omega]$, $K(x) \in L(-\omega, \omega)$ (Chapter 1);

$$Sf = \mu f(x) + \int_0^{\omega} K(x-t) f(t)\,\mathrm{d}t, \tag{6}$$

where $K(x) \in L(-\omega, \omega)$, $\mu \neq 0$ (Chapters 1–3, 9);

$$Sf = \int_0^{\omega} K(x-t) f(t)\,\mathrm{d}t, \tag{7}$$

where $K(x) \in L(-\omega, \omega)$ (Chapters 1–4, 8, 9);

$$Sf = \fint_0^{\omega} \left(\frac{1}{x-t} + K(x-t) \right) f(t)\,\mathrm{d}t, \tag{8}$$

where $K(x) \in L(-\omega, \omega)$ (Chapter 3);

$$Sf = \frac{1}{\Gamma(\mathrm{i}\alpha + 1)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^{x} (x-t)^{\mathrm{i}\alpha} f(t)\,\mathrm{d}t \tag{9}$$

(Chapters 1, 3).

5. Explicit solutions of equations. Explicit solutions of the equation

$$Sf = \int_0^{\omega} k(x-t) f(t)\,\mathrm{d}t = \varphi(x) \tag{10}$$

are constructed in the cases

$$k(x) = \frac{1 - \beta\,\mathrm{sgn}\,(x)}{|x|^{\alpha-1}}, \quad -1 \le \beta \le 1, \quad 0 < \alpha < 2 \quad \text{(Chapters 3 and 9)}, \tag{11}$$

$$k(x) = \ln \frac{b}{2|x|}, \quad b > 0 \quad \text{(Chapters 3 and 9)}, \tag{12}$$

$$k(x) = \ln \frac{\sin(b/2)}{2\sin(|x|/2)}, \quad b > 0 \quad \text{(Chapter 9)}, \tag{13}$$

$$k(x) = \ln \frac{\sinh(b/2)}{2\sinh(|x|/2)}, \quad b > 0 \quad \text{(Chapter 9)}. \tag{14}$$

6. Convolution form, Chapter 7. Convolution form representation of the generator L was obtained, first, for stable processes [92]. Later we proved that the convolution form representation of the generator L is valid for a broad class of Lévy processes [97]. In Chapter 7 of this book we prove a general fact:

If X_t is a Lévy process, then the corresponding generator L can be represented in the convolution form.

7. Quasi-potential operator $\boldsymbol{B}$, Chapter 8. M. Kac and H. Pollard [33] showed that the study of stable Lévy processes is connected with the solution of integro-differential equations (see (8.3.26)). M. Kac solved such equations for the case of Cauchy symmetric processes [30]. Later H. Widom solved such equations for all symmetric stable Lévy processes [111]. We developed the results of M. Kac and H. Widom further and proved their analogs for a wide class of Lévy processes [97]. Here we construct the mentioned above integro-differential equations for all Lévy processes. The solution of these equations is given in terms of the quasi-potential operator B (which is obtained by inversion of the corresponding integro-differential operator). In Chapter 8 we prove the following general fact:

Let X_t be a Lévy process. Then the corresponding quasi-potential operator B is a bounded operator in the Banach space of continuous functions.

8. Wiener processes, Chapter 8. The Wiener process is an important special case of the stable processes (namely, the case where $\alpha = 2$). Our approach was applied to this case in the book [97].

9. Proposition 8.3.7, Chapter 8. Consider the stochastic process

$$Y_t = \int_0^t V(X_\tau)\,\mathrm{d}\tau. \tag{15}$$

Denote by $R_t(y)$ the distribution function of Y_t. Then we have

$$E\left[\mathrm{e}^{-uY_t}\right] = \int_0^t \mathrm{e}^{-uy}\,\mathrm{d}_y R_t(y), \tag{16}$$

where $R_t(+0) - R_t(0) = p(t, \Delta)$. It follows from (16) that

$$\lim_{u\to\infty} E\left[\mathrm{e}^{-uY_t}\right] = p(t, \Delta). \tag{17}$$

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Glossary of Important Notations

$\vec{A}$	bounded triangular operator, see (11.1.1)
$A,\ A^*$	operators given by formulas (6) or (2.1.2)
$\big(A, \gamma, \nu(\,\mathrm{d}x)\big)$	Lévy–Khinchine triplet, see (7.1.3)
$B(x, \lambda)$	solution of the equation $Sf = \mathrm{e}^{\mathrm{i}x\lambda}$, see (1.3.9)
$B_\gamma(x, \lambda)$	solution of the equation $Sf = \mathrm{e}^{\mathrm{i}x\lambda}$, see (2.1.6)
$\mathbb{C}$	complex plane
$\mathbb{C}_+$	open upper half-plane $\{z :\ \mathrm{Im}\,(z) > 0\}$
$\mathbb{C}_-$	open lower half-plane $\{z :\ \mathrm{Im}\,(z) < 0\}$
C_0	space of the continuous functions $f(x)$, which satisfy the condition $\lim f(x) = 0$ $(\lvert x\rvert \to \infty)$ and have the norm defined by $\lVert f \rVert = \sup_x \lvert f(x)\rvert$
C_0^n	set of the functions $f(x) \in C_0$ such that $f^{(k)}(x) \in C_0$ $(1 \le k \le n)$
$C(a)$	set of functions $f(x) \in C_0$ which satisfy (7.2.1)
C_Δ	special set of functions, see (8.3.16)
col	column, col $\begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$
$\mathfrak{D}$	set of generalized functions of the form (2.3.1)
$D(A, B)$	connected set of complex numbers, which contains $z = \infty$ but contains neither points of spectrum of A nor of B.
$D(T)$	domain of definition of the operator T
D_V	domain of definition of the operator V, see (9.2.2)
D_Δ	space of the continuous functions $g(x)$ on the domain Δ
D_Δ^0	subspace of D_Δ, see (8.4.1)
deg	order of a polynomial
diag	diagonal matrix
dim	dimension of a space
e_x	characteristic function, see (1.1.2)
e^z	exponential function, $\mathrm{e}^z = \exp(z)$
$E[\cdot]$	mathematical expectation
$F_0(x, t)$	distribution function, see (8.3.1)

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$\operatorname{gmul}\lambda_k$	geometric multiplicity of the eigenvalue λ_k, i.e., a number of linearly independent eigenfunctions, corresponding to this eigenvalue
i	complex unity, $\mathrm{i}^2 = -1$
I	identity operator
I_n	$n \times n$ identity matrix
Im	imaginary part of either complex number or matrix
$\operatorname{ind}\lambda_k$	index of the eigenvalue, i.e., the dimension of the largest Jordan block associated with this eigenvalue
$\mathcal{J}^{\mathrm{i}\alpha}$	fractional integral operator of purely imaginary order $\mathrm{i}\alpha$, see (1.0.4) and Section 11.4
$\mathcal{J}(\tau,\eta)$	total intensity of radiation, see (3.8.1)
$K_F(\mu_j)$	multiplicity of the root μ_j of $F(\mu)$, see Section 4.2 point 3
$K_G(\lambda_j)$	multiplicity of the root λ_j of $G(\lambda)$, see Section 4.2 point 3
$\ker A$	kernel of an operator A, that is, the subspace, which A maps to zero
$\mathcal{L}_m$	solution of the equation $S\mathcal{L}_m = x^{m-1}$, see (1.2.1)
$L(0,\omega)$	space $L^1(0,\omega)$
$L^p(0,\omega)$	normed space of functions f with the norm $\|f\|_p = \left(\int_0^\omega \lvert f(x)\rvert^p \,\mathrm{d}x\right)^{1/p}$
$L^p_n(0,\omega)$	normed space of vector functions f $(f(x) \in \mathbb{C}^n)$ with the norm $\|f\|_p = \left(\int_0^\omega \left(f^*(x)f(x)\right)^{p/2} \mathrm{d}x\right)^{1/p}$
L_Δ	truncated generator, see (8.3.25)
M	function, $M(x) = s(x)$ $(0 \le x \le \omega)$, see (7)
$\operatorname{mes}\Delta$	the summarized length of the non-intersecting segments $[a_k, b_k]$ such that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, i.e., $\operatorname{mes}\Delta = \sum_{k=1}^n \left(b_k - a_k\right)$
N_1	solution of the equation $SN_1 = M$, see (10)
N_2	solution of the equation $SN_2 = \mathbb{1}$, see (10)
$p(t,\Delta)$	probability that a sample of X_τ remains inside Δ for $0 \le \tau \le t$, see (8.1.2)
$P(X=0)$	probability of the event $X = 0$
P_t	transition operator, see (7.1.5)
$P_t(x_0,\Delta)$	probability $P(X_t \in \Delta)$ when $P(X_0 = x_0) = 1$, see (7.1.5)
P_Δ	projector, $P_\Delta f(x) = \begin{cases} f(x) & \text{for} \quad x \in \Delta, \\ 0 & \text{for} \quad x \notin \Delta. \end{cases}$
$\mathbb{R}$	real axis

R_V	range of the operator V, see (9.2.2)
range (A)	range of an operator A
Re	real part of either complex number or matrix
S^*	operator adjoint to S (or its extension), see, e.g., Proposition 2.1.1
sgn	function, $\operatorname{sgn} u = \begin{cases} 1, & u > 0, \\ 0, & u = 0, \\ -1, & u < 0 \end{cases}$
$w_A(z)$	transfer matrix function, see (5.1.4)
$W_p^{(l)}$	set of functions $\varphi(x)$ such that $\varphi^{(l)}(x) \in L^p(0, \omega)$, see Section 2.2
$\Gamma(z)$	gamma function
$\delta(x)$	delta function
κ	number of the negative eigenvalues of the operator $S - \nu I$ (counted with their multiplicities)
$\nu(\,\mathrm{d}x)$	Lévy measure satisfying (7.1.4)
$\rho(\lambda, \mu)$	function, see (12), (2.1.17)
φ^*	function $\varphi^* = U\varphi$, see (4.1.9)
$\Psi(x, s, u)$	Laplace transform of $F(x, t, u)$, see (8.3.10)
$\Psi_\infty(x, s)$	function, see (8.3.21)
$\{H_1, H_2\}$	class of linear bounded operators acting from the Banach space H_1 into the Banach space H_2
$\langle \cdot, \cdot \rangle$	scalar product, i.e., $\langle f, g \rangle = \int_0^\omega g(x)^* f(x)\,\mathrm{d}x$, see (1.1.3), (1.4.2), (5.2.32)
$\langle \cdot, \cdot \rangle_\Delta$	scalar product in $L^2(\Delta)$, that is, in L^2 on the set of segments Δ
$\langle \cdot, \cdot \rangle_H$	scalar product in the Hilbert space H
$\lfloor K \rfloor$	integer part of K
$\mathbb{1}$	function, which identically equals 1
$\mathbb{1}_{\lvert x \rvert < 1}$	function of x, which equals 1 when $\lvert x \rvert < 1$ and equals 0 when $\lvert x \rvert > 1$
$\displaystyle\fint_0^\omega$	Cauchy integral, principal value

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