Appendix A
m Is a Bimodule Morphism

This appendix presents a detailed demonstration that for any associative algebra $\mathcal{A}$ with unit 1, the multiplication map $m = m_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is an $\mathcal{A}$-bimodule map. This is the sort of elementary result that the reader should be able to produce with only a hint at most, but we present it in detail to show the reader what is entailed in a rather complete, thorough proof. Other results of this sort will not always be written out in such great detail.

The first step is to define canonical $\mathcal{A}$-bimodule structures on $\mathcal{A}$ and on $\mathcal{A} \otimes \mathcal{A}$. We start with $\mathcal{A}$. The left action of $\mathcal{A}$ on $\mathcal{A}$ is simply the multiplication $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$. That this is a left action follows from $a(bx) = (ab)x$ and $1x = x$ for all $a, b, x \in \mathcal{A}$, namely, the associativity of multiplication in $\mathcal{A}$ and the fact that 1 is a left unit. Also, m is linear, an essential though trivial observation that we will not repeat hereafter. This left action gives a canonical left $\mathcal{A}$-module structure to $\mathcal{A}$. (Canonical here really means functorial, but going into that would involve too many trivial details.) The right action of $\mathcal{A}$ on $\mathcal{A}$ is also the multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. That this is a right action follows from $(xa)b = x(ab)$ and $x1 = x$ for all $a, b, x \in \mathcal{A}$. We are now using the fact that 1 is a right unit for the multiplication. This right action gives a canonical right $\mathcal{A}$-module structure to $\mathcal{A}$. But we have not yet proved enough to conclude that these two lateral $\mathcal{A}$-module structures give an $\mathcal{A}$-bimodule structure. We also have to show that they are compatible, namely, that $a(xb) = (ax)b$ for all $a, b, x \in \mathcal{A}$. But this is just the associativity again, and so we have a canonical $\mathcal{A}$-bimodule structure on $\mathcal{A}$.

We next define the $\mathcal{A}$-bimodule structure on $\mathcal{A} \otimes \mathcal{A}$. The left action of $\mathcal{A}$ on $\mathcal{A} \otimes \mathcal{A}$ is given by left multiplication on the first factor: $a(x \otimes y) := ax \otimes y$ for all $a, x, y \in \mathcal{A}$. The diagram for this is

$$\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{m \otimes id} \mathcal{A} \otimes \mathcal{A}. \tag{A.1}$$

S.B. Sontz, *Principal Bundles*, Universitext, DOI 10.1007/978-3-319-15829-7

The right action of $\mathcal{A}$ on $\mathcal{A} \otimes \mathcal{A}$ is given by right multiplication on the second factor: $(x \otimes y)a := x \otimes ya$ for all $a, x, y \in \mathcal{A}$. Its diagram is

$$\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{id \otimes m} \mathcal{A} \otimes \mathcal{A}. \tag{A.2}$$

Exercise A.1. *Prove:*

- $m \otimes id$ *gives* $\mathcal{A} \otimes \mathcal{A}$ *the structure of a left* $\mathcal{A}$*-module.*
- $id \otimes m$ *gives* $\mathcal{A} \otimes \mathcal{A}$ *the structure of a right* $\mathcal{A}$*-module.*
- $m \otimes id$ *and* $id \otimes m$ *give* $\mathcal{A} \otimes \mathcal{A}$ *the structure of an* $\mathcal{A}$*-bimodule.*

Proposition A.1. *The multiplication map* $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ *is an* $\mathcal{A}$*-bimodule map.*

Proof. First, we show that m commutes with the left actions of $\mathcal{A}$ on $\mathcal{A} \otimes \mathcal{A}$ and on $\mathcal{A}$, respectively. This comes down to the following diagram chase:

$$\begin{array}{ccc} a \otimes x \otimes y & \xrightarrow{l.a.} & ax \otimes y \\ {\scriptstyle id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ a \otimes xy & \xrightarrow{l.a.} & a(xy) = (ax)y, \end{array} \tag{A.3}$$

where the horizontal arrows are the appropriate left actions. The proof that m commutes with the right actions of $\mathcal{A}$ on $\mathcal{A} \otimes \mathcal{A}$ and on $\mathcal{A}$ is equally straightforward.
■

Corollary A.1. $\mathcal{A}^2 := \mathrm{Ker}\, m$ *is an* $\mathcal{A}$*-sub-bimodule of* $\mathcal{A} \otimes \mathcal{A}$.

This corollary is an essential ingredient in the discussion in the main text of universal first-order differential calculi (FODCs).

Appendix B
Hopf Algebras, an Overview

To make this presentation more self-contained, we present a quick introduction to Hopf algebras. The idea is to derive—or at least mention—some basic facts that are used throughout the text. For further details, there are an amazing number of tomes dedicated to this topic. Among these, Kassel's book [45] seems to be the most leisurely. Other standard references are [1] and [78].

Unfortunately, this overview is long on definitions and short on examples. a typical failing of expositions of this sort. Anyway, the reader has been forewarned.

B.1 Basic Properties and Identities

Definition B.1. *An* algebra *with unit over the field of complex numbers* $\mathbb{C}$ *is a triple* $\langle \mathcal{A}, \mu, 1_{\mathcal{A}} \rangle$, *where* $\mathcal{A}$ *is a vector space over* $\mathbb{C}$ *together with a multiplication, that is, a bilinear map*

$$\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

and with a unit element $1_{\mathcal{A}} \in \mathcal{A}$. *We denote* $\mu(a, b)$, *the multiplication of elements* $a, b \in \mathcal{A}$, *simply as* ab. *And when context permits the ambiguity in notation, we write* 1 *instead of* $1_{\mathcal{A}}$. *These satisfy*

1. *(Associativity)* $a(bc) = (ab)c$ *for all* $a, b, c \in \mathcal{A}$.
2. *(Unit property)* $1_{\mathcal{A}} a = a 1_{\mathcal{A}} = a$ *for all* $a \in \mathcal{A}$.

Remarks. The vector space $\mathcal{A}$ need not be finite dimensional. The bilinear multiplication induces a unique linear map, also denoted by μ:

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

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The tensor product here is the *algebraic* tensor product. The unit element also induces a unique linear map

$$\eta : \mathbb{C} \to \mathcal{A}$$

defined by $\eta(\lambda) := \lambda \, 1_{\mathcal{A}}$ for all $\lambda \in \mathbb{C}$. We often simply say that $\mathcal{A}$ is an algebra without explicitly mentioning the multiplication and unit. But at the other notational extreme, we sometimes write the multiplication and unit as $\mu_{\mathcal{A}}$ and $\eta_{\mathcal{A}}$, respectively, in order to emphasize that these structures pertain to a particular vector space $\mathcal{A}$.

This definition can be written equivalently in terms of the vector space $\mathcal{A}$ and the two linear maps μ and η. In that case, the two defining properties are translated into the requirement that these two diagrams commute:

$$\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{id \otimes \mu} & \mathcal{A} \otimes \mathcal{A} \\
\mu \otimes id \big\downarrow & & \big\downarrow \mu \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mu} & \mathcal{A},
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{A} \otimes \mathbb{C} & \xrightarrow{id \otimes \eta} & \mathcal{A} \otimes \mathcal{A} & \xleftarrow{\eta \otimes id} & \mathbb{C} \otimes \mathcal{A} \\
\cong \big\downarrow & & \big\downarrow \mu & & \big\downarrow \cong \\
\mathcal{A} & \xrightarrow{id} & \mathcal{A} & \xleftarrow{id} & \mathcal{A}.
\end{array}$$

The vertical arrow on the left of the last diagram is the canonical isomorphism $\mathcal{A} \otimes \mathbb{C} \to \mathcal{A}$ given by $a \otimes \lambda \mapsto \lambda \, a$. A similar comment holds for the vertical arrow on the right.

Definition B.2. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras. A* morphism *f from $\mathcal{A}_1$ to $\mathcal{A}_2$, denoted $f : \mathcal{A}_1 \to \mathcal{A}_2$, is a linear map f from the vector space $\mathcal{A}_1$ to the vector space $\mathcal{A}_2$ satisfying preservation of multiplication, namely, the commutativity of the diagram*

$$\begin{array}{ccc}
\mathcal{A}_1 \otimes \mathcal{A}_1 & \xrightarrow{f \otimes f} & \mathcal{A}_2 \otimes \mathcal{A}_2 \\
\mu_{\mathcal{A}_1} \big\downarrow & & \big\downarrow \mu_{\mathcal{A}_2} \\
\mathcal{A}_1 & \xrightarrow{f} & \mathcal{A}_2,
\end{array}$$

and preservation of unit, namely, the commutativity of

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{id} & \mathbb{C} \\
\eta_{\mathcal{A}_1} \big\downarrow & & \big\downarrow \eta_{\mathcal{A}_2} \\
\mathcal{A}_1 & \xrightarrow{f} & \mathcal{A}_2.
\end{array}$$

These diagrams can also be written as $f(ab) = f(a)f(b)$ *for all* $a, b \in \mathcal{A}_1$ *and* $f(1) = 1$.

Corollary B.1. *The collection whose objects are all the algebras over* $\mathbb{C}$ *and whose arrows are all the morphisms between algebras is a category.*

We leave to the reader the definition of the composition of morphisms as well as the remaining details in the proof of the result.

So all the usual terminology of categories can be applied here. In particular, we have monomorphisms, epimorphisms, and isomorphisms of algebras.

The dual structure is defined next. This is done from the outset using tensor products and diagrams.

Definition B.3. *A* co-algebra *over the field of complex numbers* $\mathbb{C}$ *is a triple* $\langle \mathcal{C}, \phi, \varepsilon \rangle$, *where* $\mathcal{C}$ *is a vector space over* $\mathbb{C}$ *together with a co-multiplication, which is a linear map*

$$\phi : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C},$$

and with a co-unit, a linear map,

$$\varepsilon : \mathcal{C} \to \mathbb{C}$$

such that the following two diagrams commute. First, we require that the co-commutativity diagram be commutative:

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} & \xleftarrow{id \otimes \phi} & \mathcal{C} \otimes \mathcal{C} \\ {\scriptstyle \phi \otimes id} \uparrow & & \uparrow {\scriptstyle \phi} \\ \mathcal{C} \otimes \mathcal{C} & \xleftarrow{\phi} & \mathcal{C}. \end{array}$$

Notice that this diagram is dual to the commutativity diagram for algebras in the sense that all arrows have been reversed and so multiplication has been replaced by co-multiplication. This condition is equivalent to the formula

$$(\phi \otimes id)\phi = (id \otimes \phi)\phi. \tag{B.1}$$

Second, we require that the co-unit diagram be commutative:

$$\begin{array}{ccccc} \mathcal{C} \otimes \mathbb{C} & \xleftarrow{id \otimes \varepsilon} & \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\varepsilon \otimes id} & \mathbb{C} \otimes \mathcal{A} \\ \cong \uparrow & & \uparrow {\scriptstyle \phi} & & \uparrow \cong \\ \mathcal{C} & \xleftarrow{id} & \mathcal{C} & \xrightarrow{id} & \mathcal{C}. \end{array}$$

This is equivalent to

$$(\varepsilon \otimes id)\phi \cong id, \tag{B.2}$$

$$(id \otimes \varepsilon)\phi \cong id. \tag{B.3}$$

Continuing as in the case of algebras, we now define morphisms. (For the curious reader, the concept of a co-morphism, namely, the dual of a morphism, is identical with that of a morphism.)

Definition B.4. *Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are co-algebras. A* morphism *f from $\mathcal{C}_1$ to $\mathcal{C}_2$, denoted $f : \mathcal{C}_1 \to \mathcal{C}_2$, is a linear map f from the vector space $\mathcal{C}_1$ to the vector space $\mathcal{C}_2$ satisfying preservation of co-multiplication, namely, the commutativity of*

$$\begin{array}{ccc} \mathcal{C}_1 \otimes \mathcal{C}_1 & \xrightarrow{f \otimes f} & \mathcal{C}_2 \otimes \mathcal{C}_2 \\ \phi_{\mathcal{C}_1} \uparrow & & \uparrow \phi_{\mathcal{C}_2} \\ \mathcal{C}_1 & \xrightarrow{f} & \mathcal{C}_2, \end{array}$$

and preservation of co-unit, namely, the commutativity of

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{id} & \mathbb{C} \\ \varepsilon_{\mathcal{C}_1} \uparrow & & \uparrow \varepsilon_{\mathcal{C}_2} \\ \mathcal{C}_1 & \xrightarrow{f} & \mathcal{C}_2. \end{array}$$

Corollary B.2. *The collection whose objects are all the co-algebras over $\mathbb{C}$ and whose arrows are all the morphisms between co-algebras is a category.*

Next, we put these two structures on one and the same vector space.

Definition B.5. *A* bi-algebra *over the complex numbers $\mathbb{C}$ is $\langle \mathcal{B}, \mu, \eta, \phi, \varepsilon \rangle$ such that $\langle \mathcal{B}, \mu, \eta \rangle$ is an algebra, $\langle \mathcal{B}, \phi, \varepsilon \rangle$ is a co-algebra and these two structures are compatible in the sense that ϕ and ε are morphisms of algebras.*

The condition on ε means that

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \tag{B.4}$$

$$\varepsilon(1) = 1. \tag{B.5}$$

The condition on ϕ means that

$$\phi(ab) = \phi(a)\phi(b), \tag{B.6}$$

$$\phi(1) = 1 \otimes 1. \tag{B.7}$$

We will come back and explain later in more detail the condition on ϕ in terms of Sweedler's notation.

Exercise B.1. *Define morphism of bi-algebras and define the corresponding category.*

After the definitions of these three mathematical structures, we are finally ready to define the structure of interest in this appendix. First, we recall that the definition of the *twist map* $\sigma_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is given by $\sigma_{\mathcal{A}}(a \otimes b) := b \otimes a$.

Definition B.6. *A* Hopf algebra *is* $\langle \mathcal{A}, \mu, \eta, \phi, \varepsilon, \kappa \rangle$*, where* $\langle \mathcal{A}, \mu, \eta, \phi, \varepsilon \rangle$ *is a bi-algebra and* $\kappa : \mathcal{A} \to \mathcal{A}$ *is a* co-inverse *(or* antipode*) for the bi-algebra. This means that* κ *is an* antimultiplicative morphism*, namely,*

$$\kappa\mu = \mu(\kappa \otimes \kappa)\sigma_{\mathcal{A}}, \tag{B.8}$$

$$\kappa(1) = 1 \quad \text{or} \quad \kappa\eta = \eta, \tag{B.9}$$

and that κ *is an* anti-co-multiplicative morphism*, namely,*

$$\phi\kappa = \sigma_{\mathcal{A}}(\kappa \otimes \kappa)\phi \tag{B.10}$$

$$\varepsilon\kappa = \varepsilon, \tag{B.11}$$

and that κ *makes these diagrams commute:*

$$\begin{array}{ccccccc} \mathcal{A} & \xrightarrow{\phi} & \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\kappa \otimes id} & \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mu} & \mathcal{A} \\ \downarrow = & & & & & & \downarrow = \\ \mathcal{A} & & \xrightarrow{\varepsilon} & \mathbb{C} & \xrightarrow{\eta} & & \mathcal{A}, \end{array}$$

$$\begin{array}{ccccccc} \mathcal{A} & \xrightarrow{\phi} & \mathcal{A} \otimes \mathcal{A} & \xrightarrow{id \otimes \kappa} & \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mu} & \mathcal{A} \\ \downarrow = & & & & & & \downarrow = \\ \mathcal{A} & & \xrightarrow{\varepsilon} & \mathbb{C} & \xrightarrow{\eta} & & \mathcal{A}. \end{array}$$

The formulas equivalent to these diagrams are

$$\mu(\kappa \otimes id)\phi = \eta\varepsilon \tag{B.12}$$

$$\mu(id \otimes \kappa)\phi = \eta\varepsilon \tag{B.13}$$

The experts will realize that this definition is not the most efficient since the identities (B.12) and (B.13) suffice to prove the other identities in the definition. See [45], Theorem III.3.4, for the details. In the same theorem in [45] it is also shown that $\kappa^2 = id_{\mathcal{A}}$ provided that either the multiplication of $\mathcal{A}$ is commutative or the co-multiplication of $\mathcal{A}$ is co-commutative.

Another formula we use is

$$\phi(\kappa(a^{(1)}))(a^{(2)} \otimes 1) = 1 \otimes \kappa(a), \tag{B.14}$$

where we use Sweedler's notation $\phi(a) = a^{(1)} \otimes a^{(2)}$ as discussed below. To prove (B.14), we note that by (B.10) we have

$$\begin{aligned}\phi\kappa(a^1) &= \sigma_{\mathcal{A}}(\kappa \otimes \kappa)\phi(a^1)\\ &= \sigma_{\mathcal{A}}(\kappa \otimes \kappa)(a^{11} \otimes a^{12})\\ &= \kappa(a^{12}) \otimes \kappa(a^{11}).\end{aligned}$$

Then we complete the calculation as follows:

$$\begin{aligned}\phi\kappa(a^1)(a^2 \otimes 1) &= (\kappa(a^{12}) \otimes \kappa(a^{11}))(a^2 \otimes 1)\\ &= (\kappa(a^{21}) \otimes \kappa(a^1))(a^{22} \otimes 1)\\ &= \kappa(a^{21})\, a^{22} \otimes \kappa(a^1)\\ &= \eta\, \varepsilon(a^2) \otimes \kappa(a^1)\\ &= \varepsilon(a^2)\, 1 \otimes \kappa(a^1)\\ &= 1 \otimes \kappa(\varepsilon(a^2)\, a^1)\\ &= 1 \otimes \kappa(a),\end{aligned}$$

where we used the co-associativity (B.1), (B.12), and (B.2). So that finishes the proof of (B.14).

One more identity that we use is

$$(\eta\varepsilon \otimes id)\phi(a) = 1 \otimes a. \tag{B.15}$$

Again using Sweedler's notation introduced below, this is so because

$$\begin{aligned}(\eta\varepsilon \otimes id)\phi(a) &= (\eta\varepsilon \otimes id)(a^{(1)} \otimes a^{(2)})\\ &= \eta\varepsilon(a^{(1)}) \otimes a^{(2)}\\ &= 1 \otimes \eta\varepsilon(a^{(1)})a^{(2)}\\ &= 1 \otimes \varepsilon(a^{(1)})a^{(2)}1\\ &= 1 \otimes a.\end{aligned}$$

Exercise B.2. *Define morphisms of Hopf algebras and the category of Hopf algebras over* $\mathbb{C}$.

The co-inverse κ is not necessarily bijective though that is true in many examples. In general, a given bi-algebra may have no co-inverse. But if a co-inverse does exist, then it is known to be unique. For example, see [45] for these and other details.

There are a number of different types of convolution associated with Hopf algebras. The definition of the *convolution* of a linear functional $g : \mathcal{A} \to \mathbb{C}$ and an element $a \in \mathcal{A}$ is defined to be

$$g * a := (id \otimes g)\phi(a) = a^{(1)} \otimes g(a^{(2)}) \in \mathcal{A} \otimes \mathbb{C}$$

using Sweedler's notation. Or alternatively, using $\mathcal{A} \otimes \mathbb{C} \cong \mathcal{A}$,

$$g * a := g(a^{(2)})a^{(1)} \in \mathcal{A}.$$

Using this definition, we see that

$$\varepsilon(g * a) = \varepsilon(g(a^{(2)})a^{(1)}) = g(a^{(2)})\varepsilon(a^{(1)}) = g(\varepsilon(a^{(1)})a^{(2)}) = g(a). \tag{B.16}$$

Similarly, but with the opposite order, we define the *convolution* of an element $b \in \mathcal{A}$ and a linear functional $f : \mathcal{A} \to \mathbb{C}$ by

$$b * f := (f \otimes id)\phi(b) = f(b^{(1)})b^{(2)}.$$

As in (B.16), it immediately follows that

$$\varepsilon(b * f) = f(b). \tag{B.17}$$

Moreover, we have the associativity identity

$$f * (c * g) = (f * c) * g \tag{B.18}$$

for all linear functionals $f, g : \mathcal{A} \to \mathbb{C}$ and all $a \in \mathcal{A}$. To see this, we first expand the left side, getting

$$f * (c * g) = f * (g(c^{(1)})c^{(2)}) = g(c^{(1)})(f * c^{(2)}) = g(c^{(1)})f(c^{(22)})c^{(21)}.$$

On the other hand, the right side gives us

$$(f * c) * g = f(c^{(2)})c^{(1)} * g = f(c^{(2)})g(c^{(11)})c^{(12)}.$$

Then the co-associativity of the co-multiplication proves that the two sides of (B.18) are equal.

We next quote some identities for the bijections r and s that are given in [86]. These follow readily from the definitions of these maps, and so the proofs will be left to the reader. In the following, $q \in \mathcal{A} \otimes \mathcal{A}$ and $a, b \in \mathcal{A}$.

$$r\big((a \otimes 1)q\big) = (a \otimes 1)r(q) \tag{B.19}$$

$$r\big(q(1 \otimes b)\big) = r(q)\phi(b) \tag{B.20}$$

$$s\big((a \otimes 1)q\big) = (1 \otimes a)s(q) \tag{B.21}$$

$$s\big(q(1 \otimes b)\big) = s(q)\phi(b) \tag{B.22}$$

$$\mu r^{-1} = id \otimes \varepsilon \tag{B.23}$$

$$\mu s^{-1} = \varepsilon \otimes id \tag{B.24}$$

$$(\varepsilon \otimes id)r^{-1}(a \otimes b) = \varepsilon(a)b \tag{B.25}$$

$$(\varepsilon \otimes id)s^{-1}(a \otimes b) = \varepsilon(b)a \tag{B.26}$$

Next, we consider the dual space $\mathcal{A}'$ of a Hopf algebra $\mathcal{A}$, where

$$\mathcal{A}' := \{f : \mathcal{A} \to \mathbb{C} \mid f \text{ is linear}\}.$$

We would like to use the pullbacks of the operations in $\mathcal{A}$ to define a Hopf algebra structure on $\mathcal{A}'$. But in general this does not quite work out. Here are some details on this topic. First, we take $f \in \mathcal{A}'$ and define

$$\phi^o(f) := \mu_{\mathcal{A}}^* f = f \circ \mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C},$$

where $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes the linear map induced by the bilinear multiplication map of $\mathcal{A}$, and $\mu_{\mathcal{A}}^*$ is its pullback. So the composition $f \circ \mu_{\mathcal{A}}$ is linear; that is, $\phi^o(f) \in (\mathcal{A} \otimes \mathcal{A})'$. We would like to say that ϕ^o is a co-multiplication on $\mathcal{A}'$, but such a co-multiplication acts as a map

$$\mathcal{A}' \to \mathcal{A}' \otimes \mathcal{A}'.$$

And while $\mathcal{A}' \otimes \mathcal{A}' \subset (\mathcal{A} \otimes \mathcal{A})'$ this inclusion is often proper. Therefore we have no guarantee in general that the range of ϕ^o is contained in the smaller space $\mathcal{A}' \otimes \mathcal{A}'$. This may help motivate in part the following definition.

Definition B.7. $\mathcal{A}^o := \{f \in \mathcal{A}' \mid \phi^o(f) \in \mathcal{A}' \otimes \mathcal{A}'\}$.

Then the main result is that $\mathcal{A}^o$ can be made into a Hopf algebra in a natural way. We have already defined the co-multiplication ϕ^o. Here are the definitions of the remaining operators. The multiplication μ^o in $\mathcal{A}^o$ is

$$\mu^o(f \otimes g) := (f \otimes g)\phi : \mathcal{A} \to \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}.$$

To define the unit η^o, we first note that the co-unit $\varepsilon : \mathcal{A} \to \mathbb{C}$ is an element in $\mathcal{A}^o$. We define $1_{\mathcal{A}^o} := \varepsilon$. Then we define the unit $\eta^o : \mathbb{C} \to \mathcal{A}^o$ by

$$\eta^o(\lambda) := \lambda 1_{\mathcal{A}^o} \quad \text{for } \lambda \in \mathbb{C}.$$

The co-unit $\varepsilon^o : \mathcal{A}^o \to \mathbb{C}$ is defined for $f \in \mathcal{A}^o$ by

$$\varepsilon^o(f) := f(1_{\mathcal{A}}) \in \mathbb{C}.$$

The co-inverse $\kappa^o : \mathcal{A}^o \to \mathcal{A}^o$ is defined for $f \in \mathcal{A}^o$ by the composition

$$\kappa^o(f) := f \circ \kappa : \mathcal{A} \to \mathbb{C}.$$

Notice that each of these five operations on $\mathcal{A}^o$ is the pullback of its "dual" operation on $\mathcal{A}$.

Exercise B.3. *Show that $\langle \mathcal{A}^o, \mu^o, \eta^o, \phi^o, \varepsilon^o, \kappa^o \rangle$ is a Hopf algebra.*

We now briefly discuss dual pairings of Hopf algebras.

Definition B.8. *Let $\langle \mathcal{A}_j, \mu_j, \eta_j, \phi_j, \varepsilon_j, \kappa_j \rangle$ be Hopf algebras for $j = 1, 2$. Then a bilinear mapping $\mathcal{A}_1 \times \mathcal{A}_2 \to \mathbb{C}$, denoted by $(f, a) \mapsto \langle f, a \rangle$, is said to be a* dual pairing *if for all $f, g \in \mathcal{A}_1$ and $a, b \in \mathcal{A}_2$, we have*

$$\begin{aligned}
\langle \phi_1(f), a \otimes b \rangle &= \langle f, ab \rangle = \langle f, \mu_2(a \otimes b) \rangle, \\
\langle \mu_1(f \otimes g), a \rangle &= \langle fg, a \rangle = \langle f \otimes g, \phi_2(a) \rangle, \\
\varepsilon_1(f) &= \langle f, 1_{\mathcal{A}_2} \rangle, \\
\varepsilon_2(a) &= \langle 1_{\mathcal{A}_1}, a \rangle, \\
\langle \kappa_1(f), a \rangle &= \langle f, \kappa_2(a) \rangle.
\end{aligned}$$

For example, the map $\mathcal{A}^o \times \mathcal{A} \ni (f, a) \mapsto f(a)$ is a dual pairing of Hopf algebras.

Exercise B.4. *Show that the last condition on the co-inverses is a consequence of the previous four conditions.*

We say that the pairing is *nondegenerate* if the usual conditions hold, namely, (a) $\langle f, a \rangle = 0$ for all $f \in \mathcal{A}_1$ implies $a = 0$ and (b) $\langle f, a \rangle = 0$ for all $a \in \mathcal{A}_2$ implies $f = 0$.

B.2 Sweedler's Notation

To introduce Sweedler's notation (see [78]), we take the primary example of a co-multiplication map

$$\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}.$$

For any $a \in \mathcal{A}$, we can always write

$$\phi(a) = \sum_{k \in K} b_k \otimes c_k,$$

where the elements $b_k, c_k \in \mathcal{A}$ and k runs over some finite index set K. This is how one writes the co-product of a in standard notation. This is rather bulky, so we look for a more compact, manageable notation. One standard shortcut is to omit the index set K from the notation, and so one writes

$$\phi(a) = \sum_{k} b_k \otimes c_k.$$

The reader now has to understand that k runs over some unnamed finite index set. The Einstein convention carries this one step further by dropping the summation sign. So in that convention one writes

$$\phi(a) = b_k \otimes c_k.$$

Now it is the presence of the repeated index k that indicates a summation should be taken with respect to it. But Sweedler's notation is different from Einstein's. The idea is that both b_k and c_k depend on $a \in \mathcal{A}$, and so the notation should reflect this fact, even though b_k and c_k are not *uniquely* determined by a. By the way, the index set K is also not uniquely determined. So in Sweedler's notation we write $a^{(1)}$ instead of b_k and $a^{(2)}$ instead of c_k. Then we have Sweedler's notation for the co-product,

$$\phi(a) = a^{(1)} \otimes a^{(2)}.$$

The 1 in the notation $a^{(1)}$ indicates that it is the first factor (actually, a finite set of factors since there is an implicit summation here), while the 2 in $a^{(2)}$ indicates that it is the second factor. The summation is implicit, as in the case of Einstein's notation, but now the way we know this is by the repetition of the symbol a on the right side. Notice that the symbol a appears on both sides of this equation. This "balancing" property is characteristic of Sweedler's notation. The super-indices are placed in parentheses so that they are not confused with powers of a. Some authors put these as subindices and so they write

$$\phi(a) = a_{(1)} \otimes a_{(2)}.$$

This form of Sweedler's notation is not used in this book. Even other forms of Sweedler's notation maintain the summation sign, but we will also not use any of that. In [45] there is even another version of Sweedler's notation that uses primes instead of superindices.

However, when doing rough calculations, the reader may prefer not to write the parentheses in the superindices since that can slow down the mental processes needed to do the calculation. But they must be placed in any document meant for public circulation since otherwise people will easily confuse them for powers.

Why is Sweedler's notation more convenient? In the above very simple example of the co-multiplication, we gain very little, if anything at all. But in large complicated expressions with multiple occurrences of the co-product ϕ this notation provides enormous benefits. To get a hint of this, let's consider a "double" co-product

$$(\phi \otimes id_{\mathcal{A}})\,\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}.$$

Calculating the action of this on an element $a \in \mathcal{A}$, we obtain

$$\begin{aligned}(\phi \otimes id_{\mathcal{A}})\,\phi &= (\phi \otimes id_{\mathcal{A}})(a^{(1)} \otimes a^{(2)})\\ &= \phi(a^{(1)}) \otimes a^{(2)}\\ &= a^{(11)} \otimes a^{(12)} \otimes a^{(2)},\end{aligned}$$

where there is an iterated use of Sweedler's notation in the last equality but again in a compact form. More specifically, we write

$$\phi(a^{(1)}) = a^{(11)} \otimes a^{(12)} \tag{B.27}$$

instead of the bulkier, but totally correct form

$$\phi(a^{(1)}) = (a^{(1)})^{(1)} \otimes (a^{(1)})^{(2)},$$

which nobody wants to use! I call the notation of the right side of (B.27) the *iterated Sweedler's notation*, but I doubt that this is standard terminology. This form of Sweedler's notation is quite natural and easy to use for anyone with a recursive bent of mind.

Now let's consider the other "double" co-product

$$(id_{\mathcal{A}} \otimes \phi)\,\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}.$$

Calculating the action of this on an element $a \in \mathcal{A}$, we obtain

$$\begin{aligned}(id_{\mathcal{A}} \otimes \phi)\,\phi &= (id_{\mathcal{A}} \otimes \phi)(a^{(1)} \otimes a^{(2)})\\ &= a^{(1)} \otimes \phi(a^{(2)})\\ &= a^{(1)} \otimes a^{(21)} \otimes a^{(22)},\end{aligned}$$

where we now are using the iterated Sweedler's notation to write out $\phi(a^{(2)})$.

Now, we generally work with co-multiplications that are co-associative, which means that

$$(\phi \otimes id_{\mathcal{A}})\,\phi = (id_{\mathcal{A}} \otimes \phi)\,\phi.$$

In that case, of course, we have the identity

$$a^{(11)} \otimes a^{(12)} \otimes a^{(2)} = a^{(1)} \otimes a^{(21)} \otimes a^{(22)}$$

in the iterated Sweedler's notation. But be careful! We do not necessarily have $a^{(11)} = a^{(1)}$ and so forth. Each side of the equation here has an implicit double summation, and the index sets on the two sides need not be equal. And even if they are equal, this "element-by-element" type of equality will not hold in general.

However, in this case there is yet another variant of Sweedler's notation that is used in this book. We write

$$a^{(1)} \otimes a^{(2)} \otimes a^{(3)} := a^{(11)} \otimes a^{(12)} \otimes a^{(2)} = a^{(1)} \otimes a^{(21)} \otimes a^{(22)}.$$

I call $a^{(1)} \otimes a^{(2)} \otimes a^{(3)}$ the *single-digit Sweedler's notation* although, again, I doubt that this is standard terminology. Also, it is a bit of a misnomer when one comes to an expression with $a^{(k)}$ and $k \geq 10$ in it, but that does not happen in usual calculations. Of course, the 22 in $a^{(22)}$ is not twenty-two! As the reader can see, the meaning of a simple expression such as $a^{(1)}$ depends on the context of the formula where it is being used. Its correct interpretation, including the number of implicit summations involved, depends on the other occurrences of the symbol a in Sweedler's notation in the same expression. As a quick thought exercise, the reader might review the various uses of $a^{(1)}$ and $a^{(2)}$ in this discussion and identify their exact meanings.

In this book I use both the iterated and the single-digit versions of Sweedler's notations, always without summation signs. (But some authors include the summation sign in their versions of Sweedler's notation!) And sometimes I use both notations in the same expression, for example, in

$$(id \otimes \phi \otimes id)(a^{(1)} \otimes a^{(2)} \otimes a^{(3)}) = a^{(1)} \otimes a^{(21)} \otimes a^{(22)} \otimes a^{(3)}.$$

This is actually a dangerous practice since ambiguities can easily arise that can confuse one rather thoroughly. At least, that is my experience! But, as the saying goes, it is okay if you know what you are doing.

Of course, I could also use here only the single-digit version of Sweedler's notation by writing

$$(id \otimes \phi \otimes id)(a^{(1)} \otimes a^{(2)} \otimes a^{(3)}) = a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)}.$$

But then the meanings of the $a^{(k)}$'s are quite different for each side of the equation. And there is no "audit trail" as there is with the iterated notation. What I mean is that by the co-associativity of ϕ, we also have

$$(\phi \otimes id \otimes id)(a^{(1)} \otimes a^{(2)} \otimes a^{(3)}) = a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)}$$

$$\text{and} \quad (id \otimes id \otimes \phi)(a^{(1)} \otimes a^{(2)} \otimes a^{(3)}) = a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)},$$

so that one cannot tell by just examining the expression on the right side, namely, $a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)}$, how it was produced. In fact, the single-digit Sweedler's notation only makes sense if co-associativity holds. Notice that there is an algorithm that produced the superindices when going from the expression on the left side to that on the right. This algorithm erases the factor $a^{(k)}$ on which ϕ acts, replaces it with the factor $a^{(k)} \otimes a^{(k+1)}$, and also replaces each factor $a^{(l)}$ with $l > k$ in the original expression with $a^{(l+1)}$.

The single-digit Sweedler's notation has an algorithm as well for reducing the number of factors. Consider

$$(id \otimes id \otimes \varepsilon \otimes id)(a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)}) = a^{(1)} \otimes a^{(2)} \otimes \varepsilon(a^{(3)}) \otimes a^{(4)}$$
$$= a^{(1)} \otimes \varepsilon(a^{(3)})a^{(2)} \otimes 1_{\mathcal{A}} \otimes a^{(4)}.$$

Now we can apply the identity (B.3) to $\varepsilon(a^{(3)})a^{(2)}$ and this will yield the element denoted by $a^{(2)}$, but with respect to another single-digit Sweedler's notation, namely, the one with three factors and with the digits 1, 2, and 3. So we end up with

$$(id \otimes id \otimes \varepsilon \otimes id)(a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)}) = a^{(1)} \otimes a^{(2)} \otimes 1_{\mathcal{A}} \otimes a^{(3)}.$$

So now the algorithm replaces the occurrences of $a^{(k)}$ and $a^{(k+1)}$ with an occurrence of $a^{(k)}$ and also replaces each factor $a^{(l)}$ with $l > k + 1$ in the original expression with $a^{(l-1)}$. After a while one gets the hang of these algorithms, and the single-digit Sweedler's notation becomes rather convenient. As is usual in these sorts of matters, a bit of practice is quite essential.

Another virtue of the single-digit Sweedler's notation is that each side of an equation satisfies a consistency condition. If the number of occurrences of $a^{(k)}$ is n on one side of the equation, then necessarily the index k takes each of the values in $\{1, 2, \ldots, n\}$ (or in $\{0, 1, \ldots, n - 1\}$) exactly once on that side. Thus it helps explain how it happens that the symbols $a^{(k)}$ can have different meanings on the two sides of one equation, since the number of occurrences often depends on which side you are considering. Notice that the algorithms for introducing and removing occurrences of $a^{(k)}$ are compatible with this consistency check in the sense that they transform an expression satisfying the condition into another expression also satisfying the condition. This consistency check can help you both in finding correct expressions as well as in correcting silly errors. Of course, all errors are silly.

We now come back to a detail not fully explained earlier, namely, what does it mean to say that the co-multiplication ϕ is an algebra morphism? Quickly, this means that $\phi(ab) = \phi(a)\phi(b)$ as well as $\phi(1) = 1 \otimes 1$. To be more explicit, we write these in Sweedler's notation as

$$(ab)^{(1)} \otimes (ab)^{(2)} = a^{(1)}b^{(1)} \otimes a^{(2)}b^{(2)},$$
$$1^{(1)} \otimes 1^{(2)} = 1 \otimes 1.$$

So far we have only considered Sweedler's notation for a co-multiplication. The reader is invited to think about using the same notation for any linear map $V_1 \to V_2 \otimes V_3$, where each V_j is a vector space. In particular, we can and will use both versions of Sweedler's notation for right and left co-actions. The only difference will be that the superindices (in both the iterated and the single-digit notations) will often start from 0 instead of 1. I try to use the superindices $1, 2, \ldots$ for the factors that are repeated and the superindex 0 for the one factor that occurs exactly once. (However, excuse me, kind reader, if I sometimes forget to do this.) For example, if $\Phi : V \to V \otimes \mathcal{A}$ is a right co-action of $\mathcal{A}$ on the vector space V, then the notation is

$$\Phi(v) = v^{(0)} \otimes v^{(1)}.$$

And for the iterated co-action $(\Phi \otimes id_{\mathcal{A}})\Phi : V \to V \otimes \mathcal{A} \otimes \mathcal{A}$, the notation is

$$(\Phi \otimes id_{\mathcal{A}})\Phi(v) = \Phi(v^{(0)}) \otimes v^{(1)} = v^{(0)} \otimes v^{(1)} \otimes v^{(2)}$$

in the single-digit Sweedler's notation. So here $v^{(0)} \in V$, the factor that always occurs exactly once, while for $j \geq 1$ we have that $v^{(j)} \in \mathcal{A}$, the (possibly) repeated factor. Notice that, as before, the exact meaning of an expression such as $v^{(0)}$ or $v^{(1)}$ depends on the context. Using the identity $(\Phi \otimes id_{\mathcal{A}})\Phi = (id_V \otimes \phi)\Phi$ for a co-action, we also can write

$$\begin{aligned}(\Phi \otimes id_{\mathcal{A}})\Phi(v) &= (id_V \otimes \phi)\Phi(v) \\ &= (id_V \otimes \phi)(v^{(0)} \otimes v^{(1)}) \\ &= v^{(0)} \otimes v^{(1)} \otimes v^{(2)},\end{aligned}$$

where we have used the single-digit Sweedler's notation for ϕ. This is not distinguishable with the previous expansion of this expression. And again we do not have an audit trail. Also, this notation requires that the co-action identity holds.

Using the iterated Sweedler's notation instead, we have

$$(\Phi \otimes id_{\mathcal{A}})\Phi(v) = \Phi(v^{(0)}) \otimes v^{(1)} = v^{(00)} \otimes v^{(01)} \otimes v^{(1)}$$

and

$$(id_V \otimes \phi)\Phi(v) = v^{(0)} \otimes v^{(11)} \otimes v^{(12)}.$$

So in this version of Sweedler's notation, the co-action identity becomes

$$v^{(00)} \otimes v^{(01)} \otimes v^{(1)} = v^{(0)} \otimes v^{(11)} \otimes v^{(12)}.$$

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