Appendix A
Discussion of the Exercises

This appendix contains comments and hints for most of the problems. Hints are not given in the text itself. It is highly recommended to ponder a problem long and well before availing yourself of this material, which is not claimed to be complete or crystal clear.

Chapter 2

- Exercise 2.1: The condition in the exercise is necessary and sufficient because of the definition of atlas.
- Exercise 2.2: Throw into the given atlas all of the charts compatible with it. Then one proves that the resulting set of charts is (i) a smooth atlas, (ii) maximal, and (iii) unique with these two properties.
- Exercise 2.3: In all cases one has to exhibit a set of charts and show that it is an atlas. In the first two examples, the set of charts is given. In addition, for S^n, one has to show the two atlases are compatible with each other.
  For the real projective space, use the images in the quotient space of the first atlas on S^n. This gives the standard differential structure on this topological space. And ponder why this does not work for the second atlas on S^n. A nontrivial fact here is that the minimal number of charts for defining this particular differential structure on $\mathbb{R}P^n$ is $n + 1$. You are not expected to prove this now.
  For the complex projective space, imitate the case of the real projective space. Recall that the complex plane $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$. And so $\mathbb{C}^n$ is homeomorphic to $\mathbb{R}^{2n}$.
  For the last example, you have to know what a Riemann surface is. Of course! If you do, use that knowledge. (I will leave you to do that on your own, since we will not use this result later on in this book.) If not, think about acquiring that knowledge.

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- Exercise 2.4: The statement in the exercise is a very minor modification of the definition of smooth function. In fact, the modification is so minor the reader may miss it. The exercise does away with reference to the point $x \in M$. In other words, smoothness is a local structure (i.e., depending on the open sets in the spaces) and not a pointwise structure. That's it. So think about why $x \in M$ is an unnecessary detail.
- Exercise 2.5: This is indeed tricky because the manifold $M = \mathbb{R}$ being discussed here is the same as the model space of the manifold. But as the model space, it is the real line $\mathbb{R}$, as learned in introductory calculus. And M is also the real line as learned in introductory calculus, but the identity map from M to $\mathbb{R}$ is not the diffeomorphism that gives the identification. Just work with the definitions of all of these concepts (carefully, of course) and you will see what is happening.
- Exercise 2.6: First, show that if two manifolds are diffeomorphic, then they have the same dimension and are homeomorphic. To do this, you can reduce the problem to a linear problem by taking the derivative of the diffeomorphism to get an isomorphism of vector spaces. This uses material in the following section. For the first part, I only know of proofs that use some tool from algebraic topology. The idea is that there exist invariants (the simplest being homotopy groups and homology groups) that show that S^n and S^m are not homeomorphic for $n \neq m$.
  For the second part, use that the one-point compactification of $\mathbb{R}^n$ is (or more correctly, is homeomorphic to) S^n. This readily implies that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.
  For the third part, think about compactness of the spaces.
- Exercise 2.7: A vector space of dimension zero has *one* point, namely, the zero vector. For example, if the domain space has dimension zero, then the increment vector h can only be zero. And when the codomain space has dimension zero, then the function is constant and you should know from elementary calculus how to take its derivative.
- Exercise 2.8: Use Definition (2.4) very carefully to calculate $T(g \circ f)$ and $T(1_U)$. If you have never worked with commuting diagrams, take your time and be patient with yourself. The advantage of simple diagrams like $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ is that you have only to paste them together in the obvious way

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

  and not the other way, to get a more complicated diagram.
- Exercise 2.9: The three facts are labeled R, S, and T to stand for the three properties of an equivalence relation: reflexive, symmetric, and transitive. So use each fact to prove the corresponding property. And the facts themselves are proved by applying the definitions of V_α, $V_{\alpha\beta}$, and $\psi_{\alpha\beta}$. But watch out! Some care must be given to the exact domains of the functions.

- Exercise 2.10: Map X to M by first sending each $x \in V_\alpha$ to

$$\phi_\alpha^{-1}(x) \in U_\alpha \subset M.$$

  This defines a mapping from $\bigsqcup_\alpha V_\alpha$ to M. Then show that this passes to the quotient of the domain by the equivalence relation $\sim$ to give a map $X \to M$. Then this map is the desired diffeomorphism.
- Exercise 2.11: This follows in analogy to the solution to Exercise 2.9, the only difference being the definitions of the transition functions and their domains. So work with these new definitions but using the pattern established in Exercise 2.9.
- Exercise 2.12: The first part of the problem parallels Exercise 2.10. However, notice that the natural charts $(\tilde{U}_\alpha, \tilde{\phi}_\alpha)$ have codomains of a special form (open times Euclidean space) and that the transition functions also have a special form, namely, $T\psi_{\beta\alpha}$, which we know is of class C^∞.
  The dimension of TM is the dimension of the codomains of its charts.
  WARNING: One has to show that TM is a Hausdorff space. At that point, one has to use that the given manifold M is itself a Hausdorff space.
  The tangent bundle map is a new structure that is not analogous to something previously done. One defines τ_M above each U_α to make the square on the right of the diagram commute; that is, locally it is projection onto the first factor. One shows that this definition is compatible on the intersections $U_\alpha \cap U_\beta$; this justifies the "pasting" together and gives the commutativity of the left square. As with any map, τ_M is smooth if and only if it is locally smooth near every point in its domain. But locally it is projection on the first factor.
- Exercise 2.13: Pick one natural chart $(\tilde{U}_\alpha, \tilde{\phi}_\alpha)$ above a coordinate chart (U_α, ϕ_α) of M. Take a point $x \in U_\alpha$ and consider the fiber $T_x M = \tau_M^{-1}(x)$ over it. Then $\phi_\alpha^{-1} : T_x M \to \{\phi_\alpha(x)\} \times \mathbb{R}^m$ is a bijection. Since the codomain here is a vector space of dimension m over the reals, we can use this bijection to make $T_x M$ into a vector space of dimension m over the reals. If $x \in U_\beta$, then we get a possibly different vector space structure on $T_x M$. Then one shows that these two vector space structures are actually identical. However, the isomorphisms ϕ_α^{-1} and ϕ_β^{-1} are in general not equal.
- Exercise 2.14: You are on your own for this one.
- Exercise 2.15: The essential point of this problem is to see whether the resulting space $(V_1 \sqcup V_2)/\sim$ is Hausdorff. In part (i), the points $0 \in V_1$ and $0 \in V_2$ are not identified under the equivalence relation $\sim$, but any neighborhood of one of them intersects all neighborhoods of the other. In part (ii), these points $0 \in V_1$ and $0 \in V_2$ also give distinct points in $(V_1 \sqcup V_2)/\sim$, but these points can be separated with open sets, which is the Hausdorff property. The space $(V_1 \sqcup V_2)/\sim$ is then a well-known differential manifold. If you have studied stereographic coordinates, then this should be something you already know.
- Exercise 2.16: This is the usual list of wishes for gifts from Santa Claus. But you have to be Santa's helper and make the gifts yourself. The first gift comes from calculating Tf in two overlapping charts and showing that the results agree in the overlap. To check that Tf is smooth, it suffices to check that it is smooth locally.

The commutative diagram holds locally and so must hold globally as well. If $T_x f$ is linear in one choice of charts U and V with $f(U) \subset V$, then it is linear in all such pairs of charts. Why? Because the linear structure of the fiber does not depend on the choice of chart, as shown in Exercise 2.13.

- Exercise 2.17: More playing with definitions and knowing that these properties hold locally. See Exercise 2.8.
- Exercise 2.18: The atlas on $\mathbb{R}^{k+1}$ consisting of one global chart will not work. You need something more refined to take into consideration the sphere. Think of the case $k = 2$ and how it relates to coordinates on the Earth. Yes, the ancient idea is locally correct. The surface of the Earth locally is like a plane sitting as a slice in three-dimensional space.
  The curve $t \mapsto (t, rt) \in \mathbb{R}^2$, where $t \in \mathbb{R}$ passes down to give a curve in the quotient space $\mathbb{T}^2$. The image of this curve in $\mathbb{T}^2$ is closed if and only if r is rational. If r is irrational, then the image of this curve is a dense subset of $\mathbb{T}^2$.
- Exercise 2.19: For every $k \geq 0$, evaluate $g^{(k)}(x)$ for $x \neq 0$ by using elementary calculus. You do not need an exact formula, just enough to be able to calculate $g^{(k)}(0)$ from its definition as a limit. (Warning: Do not use that the kth derivative is continuous at 0 before proving that this is so.) Prove that this limit exists and evaluate it. You can use everything you know from calculus.
- Exercise 2.20: Calculate the derivatives $f^{(k)}(a)$ for all k and construct the Taylor series of f centered at a. Then show that the defining property of a real analytic function does not hold near $x = a$. The same method works for a'.

Chapter 3

- Exercise 3.1: With two charts, there are four functions in its cocycle. Using the notation (U_j, ϕ_j) for the charts with $j = 1, 2$, the functions in the cocycle are $g_{11}, g_{12}, g_{21}, g_{22}$. But g_{11}, g_{22} are identity functions, while $g_{12} = g_{21}^{-1}$. So the cocycle is determined completely by the function g_{21}, say.
  With one chart, there is only one function in the cocycle, and it is the identity function on its domain.
- Exercise 3.2: This commutative diagram is the key to this construction:

$$
\begin{array}{ccccc}
E \supset & \pi^{-1}(U_\alpha) & \xrightarrow{\tilde{\phi}_\alpha} & \phi_\alpha(U_\alpha) \times \mathbb{R}^l & \\
\downarrow \pi & \downarrow \pi & & \downarrow \pi_1 & . \\
M \supset & U_\alpha & \xrightarrow{\phi_\alpha} & \phi_\alpha(U_\alpha) &
\end{array}
$$

  The elements of the atlas $\{(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)\}_{\alpha \in A}$ are the natural charts on the total space E.
  The construction of the tangent bundle is both a special case and a motivating example for this exercise.

- Exercise 3.3: The manifold M may have billions and billions of charts in a given atlas, and the cocycle is defined for each pair of these charts. But the map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is the constant function $g_{\alpha\beta}(x) = e$ for all $x \in U_\alpha \cap U_\beta$. Of course, this *is* a cocycle, as the reader should check.
- Exercise 3.4: Show that the composition $\rho \circ t_{\beta\alpha}$ satisfies the cocycle condition (3.3), given that $t_{\beta\alpha}$ satisfies that condition.
- Exercise 3.5: Careful here! The linear maps A and $A \otimes A$ are smooth, since all linear maps are smooth. But that is not the issue here, but rather (a) the proof that the mapping $A \mapsto A \otimes A$ maps *invertible* linear maps A to *invertible* linear maps and (b) that it is a smooth mapping. For part (a), one uses that $A^{-1} \otimes A^{-1}$ is the inverse of $A \otimes A$, provided that A is invertible.
  For part (b), it seems to be easier to show that $T \mapsto T \otimes T$ is a smooth map from $End(V)$ to $End(V \otimes V)$, where $T \in End(V)$ or equivalently, for $T : V \to V$ linear. Of course, the mapping $T \mapsto T \otimes T$ is not linear. And to do that, one can take a basis $\{x_i\}$ of V and write T in that basis and write $T \otimes T$ in the basis $\{x_i \otimes x_j\}$. The latter matrix is called the *Kronecker product* of the matrix of T with itself. When it is written this way, one can see that all the entries of the matrix for $T \otimes T$ are very elementary (and smooth!) functions of the entries of the matrix for T.
- Exercise 3.6: This is the generalization of Exercise 3.5 to k factors instead of 2 factors. One way to prove this is to generalize the proof of Exercise 3.5. Another way is to use induction on $k \geq 2$. Note that the cases $k = 0$ and $k = 1$ are trivial.
- Exercise 3.7: It is a question of showing that the matrix entries of $(A^{-1})^*$ are smooth functions of the matrix entries of A. This is a two-step proof. First, one identifies the matrix elements of A^{-1} in terms of those of A. But there is a formula from linear algebra that does exactly that! It is known as Cramer's rule. One has to show that this implies that entries in A^{-1} are smooth functions of the entries of A. Then one identifies the matrix entries of $(A^{-1})^*$ in terms of those of A^{-1} and ultimately in terms of those of A. But this second step is a very easy application of linear algebra.
- Exercise 3.8: To say that "M is something that transforms as M does" seems to be the height of what is known as a *circular definition*. It is not, because it is not a definition but rather a description. As an example, a contravariant vector transforms in a very certain way under coordinate changes, a way that is given by a formula. That is a property of contravariant vectors. Moreover, that certain way of transforming is then known as the way that a contravariant vector transforms. It is a (trivial) consequence of the definitions that a contravariant vector is something that transforms as a contravariant vector does. In fact, in this case this property is even characteristic; that is, it identifies exactly the class of contravariant vectors.
  Keeping all this in mind, it is not so difficult to write out rigorously the *description* of a covariant vector in terms of its transformation property. The reader might also read again the remarks after (2.12) about the similar expression that "a vector is something that transforms as a vector."

- Exercise 3.9: This is a rather personal exercise in creating your own comprehension of this material.
- Exercise 3.10: The calculations are completely straightforward to show that ρ_s and ρ_{ps} are representations. Also, the fact that they are not equivalent is rather transparent. The difficult bit is understanding that ρ_{ps} is important in physics. One might think, as some Nobel laureates thought, that only ρ_s enters in the theory of physically measurable quantities. That is not a mathematical result, of course. The basis for using any of these representations has to come from experiment. When the dust settled in the 1950s, the conclusion was that ρ_{ps} describes certain (but far from all!) measurable quantities associated with the *weak interaction*.

  The upshot is that whatever your sources of intuition might be and whatever conclusions they have led you to, one has to get used to thinking about pseudoscalars. Because there they are in nature! So again this is a challenge for creating your own understanding of this matter.
- Exercise 3.11: Let s_1 and s_2 be smooth sections. So $s_1(x)$ as well as $s_2(x)$ lie in the same fiber E_x for every $x \in M$. But we already know that E_x is a vector space. So we define the sum and scalar multiplication in terms of these vector space operations on E_x. For example, $s_1 + s_2$ is defined pointwise by $(s_1 + s_2)(x) := s_1(x) + s_2(x)$. The tricky bit is to show that these vector space operations on smooth sections then yield *smooth* sections. (That they give sections is easy enough.) As usual, this is shown locally in a system of coordinates. Then it is straightforward to show that the set of smooth sections is a vector space over $\mathbb{R}$.

  To show that the space of sections is in general infinite dimensional, one can consider first the case of a trivial vector bundle. Every vector bundle is locally trivial, and sections with support inside of a coordinate neighborhood, where the bundle is trivial, are easy enough to construct using a smooth approximate characteristic function.

  Notice that in the text when we say "section," we automatically mean a smooth map. Here we include the adjective "smooth" to make things more explicit.
- Exercise 3.12:

  For the first statement, we take two vector bundle maps, which are commutative square diagrams by definition, with the codomain of one vector bundle map being the domain of the second vector bundle map. Then we paste together these two squares on their common vertical edge. Composing the horizontal arrows (above and below) converts this 1×2 rectangle into a 1×1 square. And that square (which does commute) is the composition sought for.

  For the second statement, one is given a collection of objects (vector bundles) and morphisms (vector bundle maps). So one only has to show that the axioms of a category are satisfied.

  Behind the third statement are simply two basic facts from calculus:

  1. The chain rule.
  2. The derivative of the identity is the identity.

However, the definition of the functor on smooth maps $f : M \to N$ was not given in the statement of the problem. So you have to give the definition of the vector bundle map from the bundle $\tau_M : T(M) \to M$ to the vector bundle $\tau_N : T(N) \to N$. The hint is that you have already seen this vector bundle map. Then, when all has been defined, one proves that the defining properties of a functor are satisfied.
The fourth statement is similar to the second statement. This exercise consists of showing that the given objects and morphisms satisfy the axioms of a category.

- Exercise 3.13: Think about the set of equivalences of the explicitly trivial bundle with itself.
- Exercise 3.14: Warning: The approach of this exercise is generally considered not to be elegant, even though it is rigorous. What "elegant" is supposed to mean is anyone's guess.
  However, "does not depend on the choice of the two bases" means that if we choose other bases $\{e'_\alpha \mid \alpha \in A\}$ of V and $\{f'_\beta \mid \beta \in B\}$ of W, then the space constructed with basis $e'_\alpha \otimes f'_\beta$ for $\alpha \in A$, $\beta \in B$, say $(V \otimes W)'$, is isomorphic to the space $V \otimes W$ by a *uniquely* determined isomorphism. The construction of this unique isomorphism is the heart of the problem.
- Exercise 3.15: The first statement can be proved from the construction in Exercise 3.14. Or you can prove it directly.
  For the second statement, given a basis $\{e_\alpha \mid \alpha \in A\}$ of V, one defines the *dual basis* $\{e^*_\alpha : V \to \mathbb{R} \mid \alpha \in A\}$ of V^* by $e^*_\alpha(e_\beta) := \delta_{\alpha\beta}$ (the Kronecker delta) on the basis and then extending linearly to V. Here the number of elements in A is n. The thing to prove here is that the so-called dual basis is in fact a basis for the dual space V^*.
  The space $\mathrm{Hom}(V, W)$ is isomorphic to the vector space of all $m \times n$ matrices with real entries. This latter vector space has a well-known basis whose elements E_{ij} are the matrices with all but the (i, j) entry equal to zero and the (i, j) entry equal to 1. Here $1 \le i \le m$ and $1 \le j \le n$. Since there are nm such matrices E_{ij}, this establishes the dimension of $\mathrm{Hom}(V, W)$.
  The above remarks show that the dimension of $V^* \otimes W$ is nm. So it is isomorphic to $\mathrm{Hom}(V, W)$ by a dimension argument. But we want something more explicit, and that is what $\eta_{V,W}$ is all about. One way to define $\eta_{V,W}$ is to map the basis element $e^*_i \otimes f_j$ of $V^* \otimes W$ to the basis element E_{ij} of $\mathrm{Hom}(V, W)$. If we look at it this way, we do not see the underlying structure of $\eta_{V,W}$. Hence, for any decomposable element $e^* \otimes f \in V^* \otimes W$, meaning that $e^* \in V^*$ and $f \in W$, we define

$$\eta_{V,W}(e^* \otimes f) := E_{e^* f} : V \to W,$$

  where $E_{e^* f}(x) := e^*(x) f$ for $x \in V$. For physicists who are used to Dirac notation, we note that if V has an inner product (and therefore a corresponding isomorphism $V^* \cong V$), then $E_{e^* f} = |e\rangle\langle f|$, where $e \in V$ is the element corresponding to $e^* \in V^*$ under the isomorphism.

The naturality of the isomorphism $\eta_{V,W}$ follows by checking that the defining property of naturality holds. See any text containing the basics of category theory (e.g., the classical text [35]) for the exact definition. Since the concept of naturality is not emphasized in this book, we leave this detail to the interested reader.
Recall that the rank of a linear map is the dimension of its range. Given this viewpoint, one realizes immediately that the rank of $E_{e^* f}$ is 1 if and only if both e^* and f are nonzero. Otherwise, its rank is 0. Clearly, there are linear maps in $\mathrm{Hom}(V, W)$ with rank ≥ 2 if both V and W have dimension ≥ 2.
- Exercise 3.16: Just use Definition (3.7).

Chapter 4

- Exercise 4.1: This is less than solving a differential equation. We can consider the equations $\gamma(0) = x$ and $\gamma'(0) = v$ as initial conditions for the unknown curve γ. But there is no differential equation that $\gamma'(t)$ has to satisfy. Thinking of writing $\gamma(t)$ as a Taylor series

$$\begin{aligned}\gamma(t) &= \gamma(0) + \gamma'(0)\, t + \sum_{k=2}^{\infty} \frac{1}{k!}\gamma^{(k)}(0)\, t^k \\ &= x + vt + \sum_{k=2}^{\infty} \frac{1}{k!}\gamma^{(k)}(0)\, t^k,\end{aligned}$$

  we see that even for such real analytic functions, there is an infinite-dimensional space of solutions γ. The simplest function of this form is $\gamma(t) = x + vt$.
  There are even more C^∞-functions γ satisfying these two conditions.
  But what about the definition of the domain J of γ?
- Exercise 4.2: For part (a), by hypothesis the matrix $(A^{-1})^t$ exists. Now multiply it on the left and on the right by the matrix A^t to see whether it really is the inverse of A^t.
  For part (c), the matrices under consideration satisfy $A^t A = AA^t = I$, the identity matrix. These are the $n \times n$ *orthogonal matrices*. The set of all of these matrices is typically denoted by $O(n)$. Moreover, $O(n)$ is a group under matrix multiplication. It is also a differential manifold embedded in $\mathbb{R}^{n^2}$. It has a dimension equal to n^2 minus the number of independent constraints in the relations $A^t A = AA^t = I$. Finally, these two structures on $O(n)$ are compatible, which means that the matrix multiplication operation and the inverse operation ($A \mapsto A^{-1}$) are smooth maps. For the time being, we leave these details for the further consideration of the interested reader.

- Exercise 4.3: The condition $df(y) = w$ expands to

$$\frac{\partial f}{\partial x_j}(y) = w_j$$

for all $j = 1, \dots, n$, where each $w_j \in \mathbb{R}$ is a component of w. One solution (far from being unique) is given for $x \in U$ by

$$f(x) = w_1 x_1 + w_2 x_2 + \cdots + w_n x_n.$$

Can you find some of the many, many more solutions?
- Exercise 4.4:

(a) Define df locally in a chart by the usual formula from multivariable calculus and show that what you get does not depend on the choice of the chart.
(b) $d\omega = 0$ is true if and only if it is true locally in each chart. Then use the previous part to calculate $d\omega$ locally.
(c) First, let's consider the question of uniqueness. But clearly, if $\omega = df$, then we also have $\omega = d(f + c)$ for any constant $c \in \mathbb{R}$. Since M is connected, this gives all possible functions $g : M \to \mathbb{R}$ such that $\omega = dg$.
The problem of existence reduces to solving the differential equation $df = \omega$ for the unknown f in terms of the given ω. Note that $d\omega = 0$ is a necessary condition for solving this equation. One says that $d\omega = 0$ are the *integrability conditions*. The meat of the matter is that these are also sufficient for solving $df = \omega$. How do we see this? By integration! We define for any $y \in M$

$$f(y) = \int_0^y \omega = \int_0^y \big(\omega_1(x)\, dx_1 + \cdots + \omega_n(x)\, dx_n\big),$$

where we integrate over any simple curve in M that starts at the origin $0 \in M$ and ends at $y \in M$. The condition $d\omega = 0$ implies that this integral does not depend on the choice of the particular simple curve. Then by a standard argument as given in a course on multivariable calculus, one proves that $df = \omega$.
(d) A convex set C is star-shaped with respect to every point $p \in C$. To get a star-shaped set that is not convex, think about the usual iconic image for a star in Western culture, be it a starfish, a star fruit, or a piñata representing a star.
(e) This generalizes part (c), but the proof is essentially the same except that now the integral starts at any star center $p \in S$ instead of at 0. The function f so obtained is again unique up to an additive constant.
(f) The classical example in calculus of several variables starts by taking $U = \mathbb{R}^2 \setminus \{0\}$. Then one can define $d\theta$ at every point in this open set, where θ is the polar angular coordinate, even though the function θ can only be smoothly defined on a cut plane, such as $\mathbb{R}^2$ minus the closed positive x semiaxis. One way to convince yourself that the domain of this 1-form is U is to write it in Cartesian coordinates.

Chapter 5

- Exercise 5.1: If they are linearly independent, find a basis $v_1, v_2, \dots, v_n$ with $v_1 = v$ and $v_2 = w$.
  The converse is to show if they are linearly dependent (some linear combination of them is 0), then the wedge product is 0.
- Exercise 5.2: Much as in Exercise 5.1. Linearly dependent means one of the vectors is a linear combination of the rest.
- Exercise 5.3: Definition (5.4) can be written as

$$\left(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_j}\right) \wedge \left(e_{\beta_1} \wedge \cdots \wedge e_{\beta_k}\right) := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_j} \wedge e_{\beta_1} \wedge \cdots \wedge e_{\beta_k},$$

  where the right side here is the previously defined wedge product of $j+k$ vectors. So the point is whether we can "extract" the j vectors $e_{\alpha_1}, \cdots, e_{\alpha_j}$ from w_1 when we only know that $w_1 = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_j}$; that is, w_1 is some wedge product of j basis vectors in strictly increasing order.
  And the answer is yes, we can. Now that you understand what the problem is and what the solution is, prove that it is so!
- Exercise 5.4: Expand each vector v_j in the given basis and compute away.
- Exercise 5.5: The preservation of products is an immediate consequence of Exercise 5.4 in the case $k = 2$. So it only remains to show that $\Lambda^* T$ maps the identity to the identity.
- Exercise 5.6: Λ^k is a functor from the category of vector spaces to itself. Λ^* is a functor from the category of vector spaces to the category of associative, graded algebras with identity.
- Exercise 5.7: $r = \det T$.
- Exercise 5.8: For $k = 0$ and $k = n$, the dimension of $\Lambda^k V$ is 1 and so every element in that space is trivially decomposable. For $k = 1$, we have $\Lambda^k V = V$, so again every vector in it is decomposable. The case $k = n - 1$ might not be so obvious, though there seems to be some sort of duality going on here that reduces this case to the $k = 1$ case.
  For the case $n = 4$ and $k = 2$, play around with 2-forms in four variables. The example you find will also work for any $n \geq 4$ and $k = 2$. Your example must use all four variables since any expression for a 2-form in fewer variables is decomposable by the above remarks.
- Exercise 5.9: I don't think this will be used later on. So you are on your own.
- Exercise 5.10: The cases $k = 0$ and $k = 1$ are rather straightforward. So let's consider the case $k \geq 2$. Then what happens to the decomposable element $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ under the interchange $v_1 \leftrightarrow v_2$? Could it be equal to $-v_1 \otimes v_2 \otimes \cdots \otimes v_k$? That is to say, can we have the following?

$$v_2 \otimes v_1 \otimes \cdots \otimes v_k = -v_1 \otimes v_2 \otimes \cdots \otimes v_k$$

  Well, this does hold if some $v_j = 0$, and so $v_1 \otimes v_2 \otimes \cdots \otimes v_k = 0$.

Now show that it cannot happen if all $v_j \neq 0$.

- Exercise 5.11: Each eigenvalue has multiplicity 3.
- Exercise 5.12: Just calculate using the definitions.
- Exercise 5.13: I would rather that you think long and hard about this important result. If necessary (but only after really thinking long and hard about it), you may consult the literature.
- Exercise 5.14: More than half the battle is to realize that (5.7) says something that can be proved. And therefore needs to be proved! To prove it, just use the definition of d to calculate $d(x_k)$.
- Exercise 5.15: Use Exercise 5.14.
- Exercise 5.16: This formula holds if and only if it holds locally. So prove it in a chart. When you do that, the natural thing to do is prove the formula for the choices $X = \partial/\partial x_j$ and $Y = \partial/\partial x_k$. And then ones thinks that both sides of the formula are bilinear and so the formula holds for all X and Y. But then one begins to wonder why the third term on the right side of the formula (5.9) is there. After all, $[\partial/\partial x_j, \partial/\partial x_k] = 0$, which you used when checking (5.9). So it seems that (5.9) is true without the third term.

  But there is a slight, but essential, misunderstanding in this reasoning. If $X = \sum_j a_j \partial/\partial x_j$ and $Y = \sum_k b_k \partial/\partial x_k$, then using bilinearity over the reals, we see that (5.9) holds and that the third term is zero, *provided* that the a_j and b_k are real numbers. But general vector fields have the above form when a_j and b_k are smooth functions. So we need to check that (5.9) is bilinear over the ring $C^\infty(V)$, where $V \subset M$ is the open set of the chart. Actually, (5.9) is bilinear over $C^\infty(M)$. The upshot is that you now have a new exercise to show this more general bilinearity. And when you do that exercise, you will see why that third term on the right side has to be there.

  A recurring theme in Helgason's book [21] is that tensor fields are multilinear over the ring of C^∞-functions.

Chapter 6

The exercises in this chapter are meant to be its most difficult part. However, the underlying idea behind them is that they are (or come directly from) local properties of smooth functions, which is to say, calculus of several variables. Do not worry if you cannot do all of them during a first reading of this material. But during that first pass, one should understand the meaning and possible importance of each exercise.

- Exercise 6.1: If necessary, review the theory of the existence, uniqueness, and smoothness of solutions of ordinary differential equations. There you will find the proofs of these two theorems for the case when $M = U$, an open subset of $\mathbb{R}^n$. That is really all that is behind these two theorems. The extension to an arbitrary manifold M is a bother, but it is one of those chores in life that must be attended to. Yes, this is a hard problem, but the idea behind it is simply to use the previously available theory of ordinary differential equations. The idea, of course, is to use that theory in coordinate charts and "paste" the results together.

- Exercise 6.2:
  C^∞ is a local property and the theory of ordinary differential equations gives us that σ_t is locally C^∞.
  The three properties of the family of maps σ_t are condensed into this brief phrase: σ_t is a *group of diffeomorphisms* of M. The first property follows from the initial condition imposed on γ_p, while the second property follows from the uniqueness of the integral curve with a given initial condition. The third property is a consequence of the previous two properties.
- Exercise 6.3: Write the definition of $\mathcal{L}_X$ in local coordinates.
- Exercise 6.4: Again, local coordinates do the trick. This and the previous exercise are actually solved in the subsequent text by using local coordinates. However, the equation

$$f\big(p + tY(p) + o(t)\big) = f(p) + tDf(p) \cdot Y(p) + o(t)$$

  is blithely used there without justification. So you should justify it.
- Exercise 6.5: Yes, Z is given by a vector field Y. The "tricky" bits are to find that vector field Y and then to prove that $Z = Y$. For the first step, note that Taylor's theorem says that for any given point $y \in U$, we can write

$$f(x) = f(y) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(y)(x_j - y_j) + r(x - y),$$

  where the remainder term $r(x-y)$ is $o(x-y)$. Applying the given linear operator Z to the right side, remembering that $y \in U$ is constant while $x \in U$ is the variable, gives

$$\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(y) Z(x_j) + Z(r(\cdot - y)).$$

  Note that Z sends constants to zero because it satisfies Leibniz's rule. So $Y_j := Z(x_j - y_j) = Z(x_j) \in C^\infty(U)$ and $Z(f(y)) = 0$. This leads one to suspect that the vector field

$$Y := \sum_{j=1}^{n} Y_j \frac{\partial}{\partial x_j}$$

  on U is what we are looking for. Of course, we already know that $Y : C^\infty(U) \to C^\infty(U)$ is a linear map that satisfies Leibniz's rule. This quickly leads to

$Zx_k = Yx_k$ for each coordinate function x_k and then to any polynomial in the coordinate functions. Is that enough to show $Zf = Yf$ for all $f \in C^\infty(U)$?

- Exercise 6.6: The point is that the action of a vector field Y on a function f does not depend on the coordinate system used. So the same is true of the operator $\mathcal{L}_Y$ acting on f.
- Exercise 6.7: It is a question of identifying the vector space in which $(T\sigma_t)^{-1}\left[X(\gamma_p(t))\right] - X(p)$ lies. By definition, $X(p) \in T_p(M)$, a finite-dimensional Euclidean space. Check that $(T\sigma_t)^{-1}\left[X(\gamma_p(t))\right]$ also lies in $T_p(M)$. [It must! Otherwise, the difference $(T\sigma_t)^{-1}\left[X(\gamma_p(t))\right] - X(p)$ makes no sense.] To study the limit, one does what one often does: Look at the expression in local coordinates.
- Exercise 6.8: We know that $\mathcal{L}_Y X(p) \in T_p(M)$ by Exercise 6.7. It remains to show that $p \mapsto \mathcal{L}_Y X(p)$ is a smooth map $M \to T(M)$. How? Local coordinates.
- Exercise 6.9: The point is that σ_t always makes sense for t in some small neighborhood of 0.
- Exercise 6.10: First off, what could go wrong with a vector field on an open subset $U \subset \mathrm{R}^n$? In that case, an integral curve could arrive in finite time at the boundary of U and then there would be no way to extend it. Or, if U is unbounded, it could go to infinity in finite time, and then there is no way to extend it to later times. But these pathologies cannot occur in a compact manifold M. Why not? If the domain of the integral curve is (a, b) for some $b \in \mathbb{R}$, then let $c_n \in (a, b)$ be any sequence with $c_n \to b$. Then the images of this sequence must have a converging subsequence in M by compactness. And this allows one to extend the integral curve at least a little bit beyond b by solving the ODE with new initial condition at $x = b$. And that is a contradiction. So we must have $b = +\infty$. Similarly, $a = -\infty$. That's the basic idea, but without all of the details.
- Exercise 6.11: Straightforward calculations.
- Exercise 6.12: The definition of $\mathcal{L}_Y\omega$ is the tricky bit even though the idea is simple enough. One wants to define $\mathcal{L}_Y\,\omega(p)$ by forming the difference of two vectors in $T_p^*(M)$, dividing by t and taking the limit as $t \to 0$. This is in analogy with the definition of $\mathcal{L}_Y\, X(p)$. Then the identity becomes an easy calculation using the definitions. This identity is a sort of acid test for verifying that you defined $\mathcal{L}_Y\,\omega(p)$ correctly, because if that is not defined correctly, there is no way you will be able to prove the identity!
  By the way, the reader should realize that this identity is a form of Leibniz's rule.

Chapter 7

- Exercise 7.1: This problem lists some of the basic properties that follow immediately from the definitions.
- Exercise 7.2: This boils down to showing two things: (a) $T(L_e) = id_{T(G)}$ and (b) $T(L_{gh}) = T(L_g)T(L_h)$.

- Exercise 7.3: A rather easy argument.
- Exercise 7.4: Prove that $(T(L_g))^{-1} = T(L_{g^{-1}})$. The rest of the problem is quite immediate.
- Exercise 7.5: This problem looks easier than it actually is. While $T(L_g)$ acts on vector fields, such as X, Y, and $[X, Y]$, it has not been defined so far on expressions such as XY and YX, which in general are second-order differential operators.
- Exercise 7.6: Immediate from definitions.
- Exercise 7.7: The verifications are rather straightforward, though the Jacobi identity may give you pause. There is a triple-cross product identity that helps out. Or just do all the gory details.
  Play around with the standard basis of $\mathbb{R}^3$ to see how associativity fails.
- Exercise 7.8: We have to show that the multiplication map and the inverse map are smooth functions for the group $GL(n, \mathbb{R})$. Now the differential structure on $GL(n, \mathbb{R})$ comes from its being identified with an open subset of the Euclidean space $\mathbb{R}^{n^2}$. So the coordinate functions $A \mapsto A_{ij}$ are smooth on $GL(n, \mathbb{R})$, where A_{ij} is the (i, j) matrix entry of A. Here $1 \le i, j \le n$.
  For $A, B \in GL(n, \mathbb{R})$, we note that the matrix product AB has entries that are bilinear, and hence smooth, in the coordinates of A and B. For A^{-1}, we note that its matrix entries are given by Cramer's rule, which explicitly exhibits them as smooth functions of the matrix entries of A.
- Exercise 7.9: I will let you play with this without any help from me. If you get stuck, see any standard text on Lie groups.
- Exercise 7.10: Suppose that $b < +\infty$ and argue by contradiction. Let $g = \phi(b/2)$. Show that $\psi(t) := g\,\phi(t)$ defines an integral curve of X for $t \in (a, b)$ by using a calculation similar to (7.1). But $\psi(0) = g$. So $\psi(t) = \phi(t + b/2)$ for $t \in (a, b/2)$ by the uniqueness of integral curves passing through g. Pasting ψ onto the "right end" of ϕ gives an integral curve of X passing through e with domain $(a, 3b/2)$. This contradicts the maximality of b. So we must have $b = +\infty$. Similarly, one shows that $a = -\infty$.
- Exercise 7.11: This is shown using the chain rule.
- Exercise 7.12: This is a standard exercise in analysis. We actually show that the series converges absolutely, which by a standard result shows that the series itself converges. To converge absolutely means that the corresponding series with norms on each term converges as a series of (nonnegative) real numbers; that is,

$$\sum_{j=0}^{\infty} \Big|\Big|\frac{1}{j!}X^j\Big|\Big|_{op} \le 1 + ||X||_{op} + \frac{1}{2!}||X||_{op}^2 + \frac{1}{3!}||X||_{op}^3 + \cdots$$
$$= \exp(||X||_{op}) < \infty.$$

  Here, for the sake of convenience, we used the operator norm $||\cdot||_{op}$ since $||M^k||_{op} \le ||M||_{op}^k$ holds for every matrix M and every integer $k \ge 0$.

- Exercise 7.13: This is the derivative of a power series. Another standard result in analysis says that a power series defines a C^∞-function within its (open) disk of convergence and that its first derivative is given by differentiating the original power series term by term.
  To prove that $\exp(tX)$ is a one-parameter subgroup, you need to know how to multiply power series.

Chapter 8

- Exercise 8.1: No comments.
- Exercise 8.2: That Σ is a subgroup and that it is a closed subset of $GL(n)$ are easy enough.
  You may wish to write $T \in \Sigma$ in block matrix form with respect to a basis of $\mathbb{R}^n$ that contains a subset that is a basis of W_0. (Such bases of $\mathbb{R}^n$ do exist. Can you prove that?) The number of nonzero elements in such a block matrix will be the dimension of Σ.
- Exercise 8.3: dim $G(n,k) = \dim GL(n) - \dim \Sigma = n^2 - \dim \Sigma$.
- Exercise 8.4: No comments.
- Exercise 8.5: Put the standard Euclidean inner product and norm on the vector space $\mathbb{R}^n$. Then find the stabilizer of the action of $O(n)$ on $G(n,k)$ in terms of block matrices.
  Specifically, the group $O(k) \times O(n-k)$ is realized as a closed subgroup of $O(n)$ by considering it as the $n \times n$ matrices in block form with two nonzero blocks along the diagonal, the first block being a $k \times k$ orthogonal matrix and the second block being an $(n-k) \times (n-k)$ orthogonal matrix.
  The dimension calculation now is

  $$\dim G(n,k) = \dim O(n) - \big(\dim O(k) + \dim O(n-k)\big).$$

  The conclusion that $G(n,k)$ is compact is now trivial since it is being exhibited as a quotient of the compact space $O(n)$. Before when it was seen as the quotient of the noncompact space $GL(n)$, this was not obvious at all.

Chapter 9

- Exercise 9.1: Yes, this is a hard problem. But the idea behind the proof should by now be a familiar technique for the reader. One is given a “kit” plus the “instructions” for putting the kit together. And this is the essence of the proof of Theorem 9.1. The point is that it is a good exercise for you to take the idea and implement it as a detailed proof.

- Exercise 9.2: Yes, they are actually charts according to the definition. But no, it does not really matter. The point is that they are standard model spaces for describing certain spaces locally. The fact that the model spaces are open subsets in a Euclidean space is not so important. Just as long as the model spaces are well-known objects, we can easily use them to describe the assembled "kit space" in familiar terms.
- Exercise 9.3: This is another example of the "kit plus instructions" technique. Here are a few more comments:
  Replace the model fiber space G in the construction of a principal bundle by F. When the space F has some extra structure (say, it is a ring), then it comes down to showing that the natural definition of that structure on the fibers of the bundle in terms of a given local trivialization does not depend on the choice of local trivialization.
  Also think of the special case, which we have seen with vector bundles, where $G = GL(n, \mathbb{R})$ acts on $F = \mathbb{R}^n$ from the left by the standard ("canonical") action. Here the extra structure on the fiber is that of a vector space over $\mathbb{R}$.
  Another special case was given in Chapter 8 in the construction of the Grassmannian k-planes $G(TM, k)$ over M.
  These two special cases should give you the pattern for doing this more general problem where the fiber is modeled on an abstract space F.
- Exercise 9.4:

  Part (a) is more or less immediate from definitions.
  Part (b) is decidedly trickier. Yes, one can use a local trivialization to put a group structure on fibers. But no, this structure depends on the local trivialization and so is not a part of the intrinsic structure of a principal bundle.
  Part (c) also follows from definitions. We will come back to this fact in the next exercise.

- Exercise 9.5: The element g in part (c) of Exercise 9.4 is given its own notation and consequently realized as being a particular case of a general concept: an affine operation. The proofs of parts (a)–(f) are algebraic in nature and rather easy. The technique for the proof that the affine operation is smooth is hardly a surprise; one proves that it is locally smooth using local coordinates.
- Exercise 9.6: In general, the right multiplication does not preserve the affine operation.
- Exercise 9.7: It is the trivial principal bundle $\pi_1 : M \times G \to M$, where π_1 is the projection onto the first factor. Of course, you should check that the given definition for $g_{\beta\alpha}$ does actually define a cocycle. Then it is a question of constructing the associated principal bundle.
- Exercise 9.8:
  For $\mathrm{Aut}(A)$, the product is the composition of morphisms ("arrows") in the category and the identity element is the identity morphism, usually denoted as 1_A.
  The objects in the category of all principal bundles with structure Lie group G is obvious enough. However, the definition of the set of "arrows" or "morphisms" between two such objects could be less than obvious, at least to some

readers. If E_1 and E_2 are the total spaces for two principal bundles over the manifolds M_1 and M_2, respectively, but with the same structure Lie group G, then one reasonable definition for a morphism would be a G-invariant map $F : E_1 \to E_2$. Then F induces a smooth map $f : M_1 \to M_2$, making the obvious diagram commute. The product of two such G-invariant maps is obvious enough. Specifically, if $F : E_1 \to E_2$ and $H : E_2 \to E_3$ are morphisms of principal bundles, all of which have the same structure group G, then their composition is the usual composition of maps $H \circ F : E_1 \to E_3$.
Showing that every principal bundle is locally trivial is a question of understanding well all of the definitions involved.

- Exercise 9.9: The commutativity of the diagrams follows directly from the definitions of the various arrows in them.
  $\Psi(s)$ is smooth if and only if its projections onto the two factors of $M \times G$ are smooth.
  $\Psi(s)$ is easily checked to be G-invariant. It is a trivialization because it has an inverse map $M \times G \to P$.
- Exercise 9.10: Just use all the definitions.
- Exercise 9.11: A *sequence* in a category is a linear diagram of objects and arrows, such as

$$\cdots \to A_{-2} \to A_{-1} \to A_0 \to A_1 \to \cdots . \tag{A.1}$$

A sequence can have either a finite or an infinite number of objects, denoted here by A_j. The index set for j is any interval of integers, whether finite or infinite.
Furthermore, suppose that each arrow f in the category has a kernel object, denoted by $\mathrm{Ker} f$, and an image object, denoted by $\mathrm{Im} f$. This is certainly true for the category of vector spaces with arrows being linear maps. Then any such sequence (A.1) is said to be *exact* if, for every object A_j in the sequence that is both the domain of one arrow and the codomain of another arrow, say

$$\cdots \to A_{j-1} \xrightarrow{f_{j-1}} A_j \xrightarrow{f_j} A_{j+1} \to \cdots ,$$

satisfies $\mathrm{Ker} f_j = \mathrm{Im} f_{j-1}$.
In a category with a zero object, denoted 0, a *short sequence* is a sequence with five objects, the first and last of which are the zero object. So a short sequence in the category of vector spaces has the form

$$0 \to V_1 \xrightarrow{i} V_2 \xrightarrow{j} V_3 \to 0,$$

where V_1, V_2, and V_3 are vector spaces and 0 denotes the zero vector space. This sequence is also exact if the kernel of the outgoing arrow is equal to the image of the incoming arrow at each of the objects V_1, V_2, and V_3. In the notation for the above short sequence, this means that i is a monomorphism (one-to-one), j is an epimorphism (onto), and $\mathrm{Ker}\, j = \mathrm{Im}\, i$.

In practice, it is often fairly easy to show that i is a monomorphism, that j is an epimorphism, and that the composition $ji = 0$, which is equivalent to $\operatorname{Im} i \subset \operatorname{Ker} j$. But often to prove the opposite inclusion $\operatorname{Ker} j \subset \operatorname{Im} i$ requires much more work.

- Exercise 9.12: The smoothness of $A^\sharp : P \to TP$ is proved locally.
  The linearity of $A \to A^\sharp$ is not that difficult, but proving that this is a Lie algebra morphism requires that one dig deeper into the definitions.

Chapter 10

- Exercise 10.1: What is the kernel of $T\pi$? See Exercise 9.11.
- Exercise 10.2: Most of this follows quickly enough from definitions. For example,

$$\operatorname{Ker} \omega_p = \operatorname{Ker} (T_e \iota_p)^{-1} \circ \operatorname{pr}_p = \operatorname{Ker} \operatorname{pr}_p$$

  since $(T_e \iota_p)^{-1}$ is an isomorphism. Finally, $\operatorname{Ker} \operatorname{pr}_p = \operatorname{Hor}(T_p P)$ by the definition of the projection.
  The smoothness of ω is proved, as expected, locally.
  The definition of ω is that it is the linear map ω_p on the fiber $T_p P$.
- Exercise 10.3: The map $\zeta : TN \to \mathbb{R}$ determines the map $N \to T^*N$ that sends $x \in N$ to the linear map $\zeta_x : T_x N \to \mathbb{R}$, that is, $\zeta_x \in T_x^* N$. So $\sigma : x \mapsto \zeta_x$ is the section of T^*N corresponding to ζ.
  Conversely, given a section $\sigma : N \to T^*N$, we have that $\sigma_x \in T_x^* N$ for all $x \in N$. But this means that $\sigma_x : T_x N \to \mathbb{R}$ linearly. Then we define $\zeta : TN \to \mathbb{R}$ by letting its restriction to each fiber $T_x N$ be given by σ_x.
  So far, this is all algebraic. One also has to show that ζ is smooth if and only if σ is smooth.
- Exercise 10.4: Since $\Omega^1(N; W)$ consists of maps into the vector space W, one uses the sum and scalar multiplication in W to define the sum and scalar multiplication for the maps in $\Omega^1(N; W)$. The only subtle bit (but not too subtle) is that one has to show closure of these newly defined operations; that is, the resulting map actually lies in $\Omega^1(N; W)$.
  If either N or W is a one-point set, there is not much inside $\Omega^1(N; W)$. Otherwise, look at what happens locally in N to study the dimension of $\Omega^1(N; W)$.
- Exercise 10.5: It is illegal in a mathematically sophisticated nation to call something in mathematics a pullback if it is not a contravariant functor. It is a question of defining pullback as a composition on the right. Here we have defined $f^*(\zeta) = \zeta \circ f_*$. So there is no suspense. This will be a contravariant functor. It really has nothing to do with the particular structures in play in this problem.

The only particular property here is that f^* is linear, something not too difficult to prove.

- Exercise 10.6: Hint: Construct the inverse map.
- Exercise 10.7: The diagrams are proved to be commutative as usual by chasing an arbitrary element through them. Then it all comes down to the definitions of the arrows.
- Exercise 10.8: No comment.
- Exercise 10.9: First, show that $g^{-1}dg$ is the Maurer–Cartan form of $GL(n)$. Then use the first part of Exercise 10.7 for the other groups.
- Exercise 10.10: This is a straightforward application of linear algebra techniques.

Chapter 11

- Exercise 11.1: This follows from the fact that $d\omega$ is a $\mathfrak{g}$-valued 2-form.
- Exercise 11.2: Take a chart and define a horizontal vector field X there with $X(p) = w$. Then multiply X by any compactly supported smooth approximate characteristic function defined in the local coordinates of the chart. Then extend by zero to the rest of M to define W. This construction shows that W is not unique.
  For the second statement, take the derivative of the right side of (11.3).

Chapter 12

- Exercise 12.1: Nothing fancy here. Just show that gauge equivalence is a reflexive, symmetric, and transitive relation.
- Exercise 12.2: No further comments.
- Exercise 12.3: A straightforward check.
- Exercise 12.4: You just have to check that the two transformations do the same thing to the Schrödinger equation.
- Exercise 12.5: No further comments are needed.

Chapter 13

- Exercise 13.1: This depends on your particular needs.
- Exercise 13.2: This is the generalization of Exercise 12.1 to this context. But the proof is a bit trickier since more identities have to be used. Maybe the proof of symmetry will be the trickiest part for you.
- Exercise 13.3: These elementary results have elementary proofs. Just use the definitions.

- Exercise 13.4: This is a slight modification of Part 2 of Exercise 13.3. The hypothesis here is a bit different and so is the conclusion.
- Exercise 13.5: In this context, “formal” means take the derivative of $e^{-iHt/\hbar}$ with respect to t as if H were a real number. Continuing to pretend that H is a real number, one “sees” that the operators H and $e^{-iHt/\hbar}$ commute.

Chapter 14

- Exercise 14.1: This is a quick interlude because the proof can be written on one line.
- Exercise 14.2: Either use Exercise 14.1 or prove it in its own one-line proof.
- Exercise 14.3: As promised, no comments.
- Exercise 14.4: Use the definitions of $A^{\sharp P}$ and of $A^{\sharp G}$.
- Exercise 14.5: Work on this now to see if you can understand what is going on here. That is the best strategy. But if you can’t figure it out, don’t worry. The continuation of the text will make this point amply clear.
- Exercise 14.6: Reflect away! If you cannot recollect this fact, consult a (good!) introductory text on differential equations.
- Exercise 14.7: We have enough information to apply the “kit-plus-instructions” method. The notation here is also different from that used in Theorem 9.1.
- Exercises 14.8 and 14.9: These are straightforward verifications using the definitions.

Chapter 15

- Exercise 15.1: This is exactly what happens in the more well-known theory of electrostatics. Since the equations for E in electrostatics are the same as the equations for B in magnetostatics, the same arguments apply.
- Exercise 15.2: This is simply a matter of writing down the 3×3 matrix Ψ_* and then calculating its determinant.
- Exercise 15.3: Use the chain rule to write the vector fields

$$\frac{\partial}{\partial r}, \qquad \frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial \phi} \tag{A.2}$$

in terms of the vector fields

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z}.$$

Using those identities and the fact that, by definition, these last three vector fields form an orthonormal basis in the standard metric on $\mathbb{R}^3$, one can then calculate the lengths of the vector fields in (A.2).
As an extra exercise, one can calculate the inner product among all pairs of vector fields in (A.2). And this calculation should help you understand why one says that (A.2) is an *orthogonal basis*, but not an orthonormal basis.

- Exercise 15.4: The first part is a straightforward computation. The only delicate part is establishing the domain on which that computation makes sense.
  For the second part, it suffices to show that A^N is differentiable on V_N. Why? But you already showed that when you proved the first part, right?
- Exercise 15.5: So it is a question of finding a point in V_N that can be connected to every other point in V_N by a line segment lying entirely in V_N. Such a point (and there are many) can be found by drawing a picture of V_N, or merely a two-dimensional slice of V_N, such as the plane containing both the x-axis and the z-axis.
  Generically, for a pair of points $p_1, p_2 \in V_N$, the straight-line segment between them in $\mathbb{R}^3$ will lie in V_N, that is, will not intersect the removed semiaxis Z^-. But for some pairs that line segment does intersect the removed semiaxis Z^-. Consequently, V_N in not convex. Again, drawing some sketches may help you see that this is true.
- Exercise 15.6: Use Stokes' theorem.
- Exercise 15.7: This exercise is a combination of Exercises 15.4 and 15.5 though now for V_S instead of V_N.

Chapter 16

- Exercise 16.1: This is a check of your understanding of matrix algebra for matrices whose entries come from a noncommutative ring. We assume that you have studied the case where the matrix entries come from a field. The results in this problem are straightforward analogs of the field case. So are their proofs.
- Exercise 16.2: Obvious is in the mind of the beholder.
- Exercise 16.3: Grind away using the definitions.
- Exercise 16.4: Show that it is SD.

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