Appendices

A.1

The Principle of Mathematical Induction

The Principle of Induction represents a useful *rule for proving* properties that hold for any integer number n, or possibly from a certain integer $n_0 \in \mathbb{N}$ onwards.

Theorem A.1.1 (Principle of Mathematical Induction) *Let $n_0 \geq 0$ be an integer and denote by $P(n)$ a predicate defined for every integer $n \geq n_0$. Suppose the following conditions hold:*

i) $P(n_0)$ is true;
ii) for any $n \geq n_0$, if $P(n)$ is true then $P(n+1)$ is true.

Then $P(n)$ is true for all integers $n \geq n_0$.

Proof. The proof relies on the fact that each non-empty subset of $\mathbb{N}$ admits a minimum element; this property, which is self-evident, may be deduced from the axioms that define the set $\mathbb{N}$.

Let us proceed by contradiction, and assume there is an integer $n \geq n_0$ such that $P(n)$ is false. This is the same as saying that the set

$$F = \{n \in \mathbb{N} : n \geq n_0 \text{ and } P(n) \text{ is false}\}$$

is not empty. Define $\bar{n} = \min F$. As $P(\bar{n})$ is false, condition *i)* prevents $\bar{n}$ from being equal to n_0, so $\bar{n} > n_0$. Therefore $\bar{n} - 1 \geq n_0$, and $P(\bar{n} - 1)$ is true by definition of the minimum. But applying *ii)* with $n = \bar{n} - 1$ implies that $P(\bar{n})$ is true, that is, $\bar{n} \notin F$. This contradicts the fact that $\bar{n}$ is the minimum of F. □

In practice, the Principle of Induction is employed as follows: one checks first that $P(n_0)$ is true; then one assumes that $P(n)$ is true for a generic n, and proves that $P(n+1)$ is true as well.

As a first application of the Principle of Induction, let us prove **Bernoulli's inequality**: For all $r \geq -1$,

$$(1+r)^n \geq 1 + nr\,, \qquad \forall n \geq 0\,.$$

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In this case, the predicate $P(n)$ is given by "$(1+r)^n \geq 1+nr$". For $n=0$ we have $(1+r)^0 = 1 = 1+0r$, hence $P(0)$ holds.

Assume the inequality is true for a given n and let us show it holds for $n+1$. Observing that $1+r \geq 0$, we have

$$\begin{aligned}(1+r)^{n+1} &= (1+r)(1+r)^n \geq (1+r)(1+nr)\\ &= 1+r+nr+nr^2 = 1+(n+1)r+nr^2\\ &\geq 1+(n+1)r\,,\end{aligned}$$

and thus the result.

Recall that this inequality has been already established in Example 5.18 with another proof, which however is restricted to the case $r > 0$.

The Principle of Induction allows us to prove various results given in previous chapters. Hereafter, we repeat their statements and add the corresponding proofs.

▶ Proof of the Newton binomial expansion, p. 20

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\,, \qquad n \geq 0\,.$$

Proof. For $n=0$ we have

$$(a+b)^0 = 1 \quad \text{and} \quad \sum_{k=0}^{0} \binom{0}{0} a^0 b^0 = a^0 b^0 = 1\,,$$

so the relation holds.

Let us now suppose that the formula is true for a generic n and verify that it remains true for the successive integer; the claim is thus

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k\,.$$

We expand the $n+1$-term

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b)\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}\,;\end{aligned}$$

by putting $k+1=h$ in the second sum we obtain

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} = \sum_{h=1}^{n+1} \binom{n}{h-1} a^{n+1-h} b^h$$

and, going back to the original variable k, since h is merely a symbol, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} = \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^{k}.$$

Therefore

$$\begin{aligned}(a+b)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^{k} \\ &= \binom{n}{0} a^{n+1} b^{0} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n+1-k} b^{k} + \binom{n}{n} a^{0} b^{n+1}.\end{aligned}$$

Using (1.12), with n replaced by $n+1$, and recalling that

$$\binom{n}{0} = 1 = \binom{n+1}{0} \quad \text{e} \quad \binom{n}{n} = 1 = \binom{n+1}{n+1}$$

we eventually find

$$\begin{aligned}(a+b)^{n+1} &= \binom{n+1}{0} a^{n+1} b^{0} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^{k} + \binom{n+1}{n+1} a^{0} b^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^{k},\end{aligned}$$

i.e., the claim. □

▶ Proof of the Theorem of existence of zeroes, p. 109

Theorem 4.23 (Existence of zeroes) *Let f be a continuous map on a closed, bounded interval $[a,b]$. If $f(a)f(b) < 0$, i.e., if the images of the end-points under f have different signs, f admits a zero within the open interval (a,b).*
If moreover f is strictly monotone on $[a,b]$, the zero is unique.

Proof. We refer to the proof given on p. 109. Therein, it is enough to justify the existence of two sequences $\{a_n\}$ and $\{b_n\}$, finite or infinite, that fulfill the predicate $P(n)$:

$$\begin{aligned}&[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \\ &f(a_n) < 0 < f(b_n) \quad \text{and} \quad b_n - a_n = \frac{b_0 - a_0}{2^n}.\end{aligned}$$

When $n = 0$, by assumption $f(a_0) = f(a) < 0 < f(b) = f(b_0)$, so trivially we have $b_0 - a_0 = \dfrac{b_0 - a_0}{2^0}$.

Assume the above relations hold up to a certain n. Let $c_n = \dfrac{a_n + b_n}{2}$ be the mid-point of the interval $[a_n, b_n]$. If $f(c_n) = 0$, the construction of the sequences terminates, since a zero of the function is found. If $f(c_n) \neq 0$, let us verify $P(n+1)$. If $f(c_n) > 0$, we set $a_{n+1} = a_n$ and $b_{n+1} = c_n$, whereas if $f(c_n) < 0$, we set $a_{n+1} = c_n$ and $b_{n+1} = b_n$. The interval $[a_{n+1}, b_{n+1}]$ is a sub-interval of $[a_n, b_n]$, and

$$f(a_{n+1}) < 0 < f(b_{n+1}) \quad \text{and} \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2^{n+1}} \,. \qquad \square$$

▶ Proof of inequality (5.16), p. 140

Let us begin by establishing the following general property.

Property A.1.2 *Let $\{b_m\}_{m \geq 0}$ be a sequence with non-negative terms. Assume there exists a number $r > 0$ for which the following inequalities hold:*

$$b_{m+1} \leq r\, b_m \,, \qquad \forall m \geq 0 \,.$$

Then one has

$$b_m \leq r^m b_0 \,, \qquad \forall m \geq 0 \,.$$

Proof. We apply the Principle of Induction. For $m = 0$, the inequality is trivially true, since $b_0 \leq r^0 b_0 = b_0$.

Let us assume the inequality to be true for m and let us check it for $m + 1$. Using the assumption, one has

$$b_{m+1} \leq r\, b_m \leq r\, r^m b_0 = r^{m+1} b_0 \,. \qquad \square$$

If all terms of the sequence $\{b_m\}_{m \geq 0}$ are strictly positive, a similar statement holds with strict inequalities, i.e., with $\leq$ replaced by $<$.

Next, consider inequality (5.16). In order to derive the implication

$$a_{n+1} < r\, a_n \qquad \Rightarrow \qquad a_{n+1} < r^{n - n_\varepsilon} a_{n_\varepsilon + 1} \,,$$

let us set $b_m = a_{m + n_\varepsilon + 1}$ and observe that the assumption

$$a_{n+1} < r\, a_n \,, \qquad \forall n > n_\varepsilon$$

is equivalent to

$$b_{m+1} < r\, b_m \,, \qquad \forall m \geq 0 \,.$$

Thus, the previous property yields $b_m < r^m b_0$, whence we get (5.16) by choosing $m = n - n_\varepsilon$.

A.2

Complements on limits and continuity

In this appendix, we first state and prove some results about limits, that are used in the subsequent proof of Theorem 4.10 concerning the algebra of limits. Next, we rigorously justify the limit behaviour of the most important elementary functions at the extrema of their domain, and we check the continuity of these functions at all points they are defined. At last, we provide the proofs of several properties of Napier's number stated in the text, and we show that this important number is irrational.

A.2.1 Limits

Now we discuss few results that will be useful later.

Theorem A.2.1 (local boundedness) *If a map f admits a finite limit for $x \to c$, there exist a neighbourhood $I(c)$ of c and a constant $M_f > 0$ such that*

$$\forall x \in \operatorname{dom} f \cap I(c) \setminus \{c\}\ ,\ \ |f(x)| \leq M_f\,.$$

Proof. Let $\ell = \lim\limits_{x \to c} f(x) \in \mathbb{R}$; the definition of limit with, say, $\varepsilon = 1$, implies the existence of a neighbourhood $I(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < 1\,.$$

By the triangle inequality (1.1), on such set

$$|f(x)| = |f(x) - \ell + \ell| \leq |f(x) - \ell| + |\ell| < 1 + |\ell|\,.$$

Therefore it is enough to choose $M_f = 1 + |\ell|$. □

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Theorem A.2.2 (**Theorem 4.2, strong form**) *If f admits non-zero limit (finite or infinite) for $x \to c$, then there are a neighbourhood $I(c)$ of c and a constant $K_f > 0$ for which*

$$\forall x \in \operatorname{dom} f \cap I(c) \setminus \{c\}, \qquad |f(x)| > K_f . \tag{A.2.1}$$

Proof. Let $\ell = \lim_{x \to c} f(x)$. If $\ell \in \mathbb{R} \setminus \{0\}$, and given for instance $\varepsilon = |\ell|/2$ in the definition of limit for f, there exists a neighbourhood $I(c)$ with $\forall x \in \operatorname{dom} f \cap I(c) \setminus \{c\}$, $|f(x) - \ell| < |\ell|/2$. Thus we have

$$|\ell| = |f(x) + \ell - f(x)| \leq |f(x)| + |f(x) - \ell| < |f(x)| + \frac{|\ell|}{2}$$

hence

$$|f(x)| > |\ell| - \frac{|\ell|}{2} = \frac{|\ell|}{2} .$$

The claim follows by taking $K_f = \frac{|\ell|}{2}$.

If $\ell \pm \infty$, then $\lim_{x \to c} |f(x)| = +\infty$ and it is sufficient to take $A = 1$ in the definition of limit to have $|f(x)| > 1$ in a neighbourhood $I(c)$ of c; in this case we may take in fact $K_f = 1$. □

Remark A.2.3 Notice that if $\ell > 0$, Theorem 4.2 ensures that on a suitable neighbourhood of c, possibly excluding c itself, the function is positive. Therefore the inequality in (A.2.1) becomes the more precise $f(x) > K_f$. Similarly for $\ell < 0$, in which case (A.2.1) reads $f(x) < -K_f$. In this sense Theorem A.2.2 is stronger than Theorem 4.2. □

The next property makes checking a limit an easier task.

Property A.2.4 *In order to prove that $\lim_{x \to c} f(x) = \ell \in \mathbb{R}$ it is enough to find a constant $C > 0$ such that for every $\varepsilon > 0$ there is a neighbourhood $I(c)$ with*

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < C\varepsilon . \tag{A.2.2}$$

Proof. Condition (3.8) follows indeed from (A.2.2) by choosing ε/C instead of ε. □

▶ Proof of Theorem 4.10, p. 96

Teorema 4.10 *Suppose f admits limit ℓ (finite or infinite) and g admits limit m (finite or infinite) for $x \to c$. Then*

$$\lim_{x\to c} [f(x) \pm g(x)] = \ell \pm m,$$
$$\lim_{x\to c} [f(x)\, g(x)] = \ell\, m,$$
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\ell}{m}$$

provided the right-hand-side expressions make sense. (In the last case one assumes $g(x) \neq 0$ on some $I(c)\setminus\{c\}$.)

Proof. The cases we shall prove are:

a) if $\ell \in \mathbb{R}$ and $m = -\infty$, then $\lim_{x\to c} \big(f(x) - g(x)\big) = +\infty$;
b) if $\ell, m \in \mathbb{R}$, then $\lim_{x\to c} f(x)g(x) \ = \ell\, m \in \mathbb{R}$;
c) if $\ell, m \in \mathbb{R}$ and $m \neq 0$, then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m} \in \mathbb{R}$;
d) if $\ell \in \mathbb{R} \setminus \{0\}$ or $\ell \pm \infty$, and $m = 0$, then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \infty$.

All remaining possibilities are left to the reader as exercise.

a) Let $A > 0$ be arbitrarily fixed.
By Theorem A.2.1 applied to f, there is a neighbourhood $I'(c)$ of c and there is a constant $M_f > 0$ such that for each $x \in \operatorname{dom} f \cap I'(c) \setminus \{c\}$, $|f(x)| \leq M_f$.
Moreover, $\lim_{x\to c} g(x) = -\infty$ is the same as saying that for any $B > 0$ there is an $I''(c)$ such that $g(x) < -B$ for every $x \in \operatorname{dom} g \cap I''(c)\setminus\{c\}$, i.e, $-g(x) > B$. Choose $B = A + M_f$ and set $I(c) = I'(c) \cap I''(c)$: then for all $x \in \operatorname{dom} f \cap \operatorname{dom} g \cap I(c) \setminus \{c\}$,

$$f(x) - g(x) > -M_f + B \geq A\,.$$

This proves that $\lim_{x\to c} \big(f(x) - g(x)\big) = +\infty$.

b) Fix $\varepsilon > 0$.
Assuming $\lim_{x\to c} f(x) = \ell \in \mathbb{R}$, as we have, tells

$$\exists I'(c) : \forall x \in \operatorname{dom} f,\ x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon\,,$$

while Theorem A.2.1 gives

$$\exists I''(c),\, \exists M_f > 0 : \forall x \in \operatorname{dom} f,\ x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad |f(x)| < M_f\,.$$

Analogously, $\lim_{x\to c} g(x) = m \in \mathbb{R}$ implies

$$\exists I'''(c) : \forall x \in \operatorname{dom} g,\ x \in I'''(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < \varepsilon\,.$$

Set now $I(c) = I'(c) \cap I''(c) \cap I'''(c)$; for all $x \in \operatorname{dom} f \cap \operatorname{dom} g \cap I(c) \setminus \{c\}$ we have

$$\begin{aligned} |f(x)g(x) - \ell m| &= |f(x)g(x) - f(x)m + f(x)m - \ell m| \\ &= |f(x)\big(g(x) - m\big) + \big(f(x) - \ell\big)m| \\ &\le |f(x)||g(x) - m| + |f(x) - \ell|\,|m| < (M_f + |m|)\varepsilon\,. \end{aligned}$$

This means that (A.2.2) holds with $C = M_f + |m|$.

c) Let $\varepsilon > 0$ be given.
From $\lim_{x\to c} f(x) = \ell \in \mathbb{R}$ and $\lim_{x\to c} g(x) = m \in \mathbb{R}$ it follows

$$\exists I'(c) : \forall x \in \operatorname{dom} f,\ x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon$$

and

$$\exists I''(c) : \forall x \in \operatorname{dom} g,\ x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < \varepsilon\,.$$

Since $m \neq 0$ moreover, Theorem A.2.2 guarantees there is a neighbourhood $I'''(c)$ of c together with a constant $K_g > 0$ such that $|g(x)| > K_g$, $\forall x \in \operatorname{dom} g$, $x \in I'''(c) \setminus \{c\}$.
Set $I(c) = I'(c) \cap I''(c) \cap I'''(c)$; then for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, $x \in I(c) \setminus \{c\}$

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{\ell}{m}\right| &= \left|\frac{f(x)m - \ell g(x)}{mg(x)}\right| = \frac{|f(x)m - \ell m + \ell m - \ell g(x)|}{|m||g(x)|} \\ &= \frac{\big|\big(f(x) - \ell\big)m + \ell\big(m - g(x)\big)\big|}{|m||g(x)|} \le \frac{|f(x) - \ell|\,|m| + |\ell|\,|g(x) - m|}{|m||g(x)|} \\ &< \frac{|m| + |\ell|}{|m|K_g}\,\varepsilon\,. \end{aligned}$$

Hence (A.2.2) holds for $C = \dfrac{|m| + |\ell|}{|m|K_g}$.

d) Fix a constant $A > 0$.
Using Theorem A.2.2 on f we know there are a neighbourhood $I'(c)$ of c and a $K_f > 0$ such that $\forall x \in \operatorname{dom} f \cap \in I'(c) \setminus \{c\}$, $|f(x)| > K_f$.
By hypothesis $\lim_{x\to c} g(x) = 0$, so choosing $\varepsilon = K_f/A$ ensures that there exists a neighbourhood $I''(c)$ of c with $|g(x)| < K_f/A$, for any $x \in \operatorname{dom} g \cap I''(c) \setminus \{c\}$.
Now let $I(c) = I'(c) \cap I''(c)$, so that for all $x \in \operatorname{dom} f \cap \operatorname{dom} g \cap I(c) \setminus \{c\}$

$$\left|\frac{f(x)}{g(x)}\right| > K_f \frac{A}{K_f} = A\,.$$

This shows that $\lim_{x\to c} \left|\dfrac{f(x)}{g(x)}\right| = +\infty$, which was our claim. □

A.2.2 Elementary functions

▶ Check of the limits in the table on p. 101

a)	$\lim_{x\to+\infty} x^\alpha = +\infty\,,$	$\lim_{x\to0^+} x^\alpha = 0$	$\alpha > 0$
b)	$\lim_{x\to+\infty} x^\alpha = 0\,,$	$\lim_{x\to0^+} x^\alpha = +\infty$	$\alpha < 0$
c)	$\lim_{x\to\pm\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$		
d)	$\lim_{x\to+\infty} a^x = +\infty\,,$	$\lim_{x\to-\infty} a^x = 0$	$a > 1$
e)	$\lim_{x\to+\infty} a^x = 0\,,$	$\lim_{x\to-\infty} a^x = +\infty$	$a < 1$
f)	$\lim_{x\to+\infty} \log_a x = +\infty\,,$	$\lim_{x\to0^+} \log_a x = -\infty$	$a > 1$
g)	$\lim_{x\to+\infty} \log_a x = -\infty\,,$	$\lim_{x\to0^+} \log_a x = +\infty$	$a < 1$
h)	$\lim_{x\to\pm\infty} \sin x\,, \quad \lim_{x\to\pm\infty} \cos x\,,$	$\lim_{x\to\pm\infty} \tan x$ do not exist	
i)	$\lim_{x\to\left(\frac{\pi}{2}+k\pi\right)^\pm} \tan x = \mp\infty\,, \quad \forall k \in \mathbb{Z}$		
l)	$\lim_{x\to\pm1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm1)$		
m)	$\lim_{x\to+1} \arccos x = 0 = \arccos 1\,,$	$\lim_{x\to-1} \arccos x = \pi = \arccos(-1)$	
n)	$\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$		

Proof.

a) Take the first limit. Fix $A > 0$ and set $B = A^{1/\alpha} > 0$. As power functions are monotone,

$$\forall x \in \mathbb{R}_+, \quad x > B \quad \Rightarrow \quad x^\alpha > B^\alpha = A\,,$$

so the requirement for the limit to hold (Definition 3.12) is satisfied.
As for the second limit, with a given $\varepsilon > 0$ we let $\delta = \varepsilon^{1/\alpha}$; again by monotonicity we have

$$\forall x \in \mathbb{R}_+, \quad x < \delta \quad \Rightarrow \quad x^\alpha > \delta^\alpha = \varepsilon\,.$$

The condition of Definition 3.15 holds.

b) These limits follow from a) by substituting $z = \frac{1}{x}$, which gives $x^\alpha = \frac{1}{z^{|\alpha|}}$. The algebra of limits and the Substitution theorem 4.15 allow to conclude.

c) The formula was proved in Example 4.14 iii).

d) Put $a = 1 + b$, with $b > 0$, in the first limit and use Bernoulli's inequality $a^n = (1+b)^n \geq 1 + nb$. Fix an arbitrary $A > 0$ and let $n \in \mathbb{N}$ be such that $1 + nb > A$. Since the exponential is monotone we obtain

$$\forall x \in \mathbb{R}, \quad x > n \quad \Rightarrow \quad a^x > a^n \geq 1 + nb > A\,,$$

hence the condition of Definition 3.12 holds for $B = n$.
The second limit is a consequence of the first, for

$$\lim_{x\to-\infty} a^x = \lim_{x\to-\infty} \frac{1}{a^{-x}} = \frac{1}{\lim\limits_{z\to+\infty} a^z} = 0\,.$$

e) These descend from d) using the identity $a^x = \dfrac{1}{(1/a)^x}$.

f) The limits of d) and Corollary 4.30 imply that the range of $y = a^x$ is the interval $(0, +\infty)$. Therefore the inverse $y = \log_a x$ is well defined on $(0, +\infty)$, and strictly increasing because inverse of a likewise map; its range is $(-\infty, +\infty)$. The claim then follows from Theorem 3.27.

g) A consequence of e), for the same reason as above.

h) That the first limit does not exist was already observed in Remark 4.19. In a similar way one can discuss the remaining cases.
More generally, notice that a non-constant periodic function does not admit limit for $x \to \pm\infty$.

i) Follows from the algebra of limits.

l)-m) The functions are continuous at the limit points (Theorem 4.33), making the results clear.

n) We can argue as in f) relatively to $y = \tan x$, restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$, and its inverse map $y = \arctan x$. □

▶ Proof of Proposition 3.20, p. 81

Proposition 3.20 *All elementary functions (polynomials, rational functions, powers, trigonometric functions, exponentials and their inverses) are continuous over their entire domains.*

Proof. The continuity of rational functions was established in Corollary 4.12. That, together with Theorems 4.17 and 4.33 on composites and inverses, implies in particular that power functions with positive rational exponent $y = x^{m/n} = \sqrt[m]{x^n}$ are continuous; the same holds for negative rational exponent $x^q = \frac{1}{x^{-q}}$ by using the algebra of limits. At last, powers with irrational exponent are continuous by definition $x^\alpha = e^{\alpha \log x}$ and because of Theorem 4.17, once the logarithm and the exponential function have been proven continuous.

As for sine and cosine, their continuity was ascertained in Example 3.17 iii), so the algebra of limits warrants continuity to the tangent and cotangent functions; by Theorem 4.33 we infer that the inverse trigonometric functions arcsine, arccosine, arctangent and arccotangent are continuous as well.

What remains to show is then the continuity of the exponential map only, because that of the logarithm will follow from Theorem 4.33. Let us consider the case $a > 1$, for if $0 < a < 1$, we can write $a^x = \frac{1}{(1/a)^x}$ and use the same argument. The identities

$$a^{x_1+x_2} = a^{x_1} a^{x_2}\,, \qquad a^{-x} = \frac{1}{a^x}$$

and the monotonicity

$$x_1 < x_2 \quad \Rightarrow \quad a^{x_1} < a^{x_2}$$

follow easily from the properties of integer powers and their inverses when the exponents are rational; for real exponents, we can apply the same argument using the definitions of exponential function and supremum.

First of all let us prove that $y = a^x$ is continuous at the right of the origin

$$\lim_{x\to 0^+} a^x = 1\,. \tag{A.2.3}$$

With $\varepsilon > 0$ fixed, we shall determine a $\delta > 0$ such that

$$0 \le x < \delta \quad \Rightarrow \quad 0 \le a^x - 1 < \varepsilon\,.$$

The exponential map being monotone, it suffices to find δ with $a^\delta - 1 < \varepsilon$, i.e., $a^\delta < 1 + \varepsilon$. Searching for δ of the form $\delta = \frac{1}{n}$, with n integer, the condition becomes $a < (1+\varepsilon)^n$. Bernoulli's inequality (5.15) implies $(1+\varepsilon)^n \ge 1 + n\varepsilon$. It is therefore enough to pick n so that $1 + n\varepsilon > a$, or $n > \frac{a-1}{\varepsilon}$. Thus (A.2.3) holds. Left-continuity at the origin is a consequence of

$$\lim_{x\to 0^-} a^x = \lim_{x\to 0^-} a^{-(-x)} = \lim_{x\to 0^-} \frac{1}{a^{-x}} = \frac{1}{\lim\limits_{z\to 0^+} a^z} = 1\,,$$

so the exponential map is indeed continuous at the origin. Eventually, from

$$\lim_{x\to x_0} a^x = \lim_{x\to x_0} a^{x_0+(x-x_0)} = a^{x_0} \lim_{x\to x_0} a^{x-x_0} = a^{x_0} \lim_{z\to 0} a^z = a^{x_0}\,,$$

we deduce that the function is continuous at every point $x_0 \in \mathbb{R}$. □

A.2.3 Napier's number

We shall prove some properties of the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$, $n > 0$, defining the Napier's number e (p. 72).

Property A.2.5 *The sequence $\{a_n\}$ is strictly increasing.*

Proof. Using Newton's formula (1.13) and (1.11), we may write

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \cdot 1 \cdot \left(1-\frac{1}{n}\right) \cdots \left(1-\frac{k-1}{n}\right); \end{aligned} \qquad \text{(A.2.4)}$$

similarly,

$$a_{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!} \cdot 1 \cdot \left(1-\frac{1}{n+1}\right) \cdots \left(1-\frac{k-1}{n+1}\right). \qquad \text{(A.2.5)}$$

We note that

$$\left(1-\frac{1}{n}\right) < \left(1-\frac{1}{n+1}\right), \quad \ldots \quad , \left(1-\frac{k-1}{n}\right) < \left(1-\frac{k-1}{n+1}\right)$$

so each summand of (A.2.4) is smaller than the corresponding term in (A.2.5). The latter sum, moreover, contains an additional positive summand labelled by $k = n+1$. Therefore $a_n < a_{n+1}$ for each n. □

Property A.2.6 *The sequence $\{a_n\}$ is bounded; precisely,*

$$2 < a_n < 3, \qquad \forall n > 1.$$

Proof. Since $a_1 = 2$, and the sequence is strictly monotone by the previous property, we have $a_n > 2$, $\forall n > 1$. Let us show that $a_n < 3$, $\forall n > 1$. By (A.2.4), and observing that $k! = 1 \cdot 2 \cdot 3 \cdots k \geq 1 \cdot 2 \cdot 2 \cdots 2 = 2^{k-1}$, it follows

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \frac{1}{k!} \cdot 1 \cdot \left(1-\frac{1}{n}\right) \cdots \left(1-\frac{k-1}{n}\right) < \sum_{k=0}^{n} \frac{1}{k!} \\ &= 1 + \sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \sum_{k=0}^{n-1} \frac{1}{2^k}. \end{aligned}$$

Example 5.27 will tell us that

$$\sum_{k=0}^{n-1} \frac{1}{2^k} = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right) < 2.$$

We conclude that $a_n < 3$. □

▶ Proof of Property 4.20, p. 105

Property 4.20 *The following limit holds*

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e}.$$

Proof. We start by considering the limit for $x \to +\infty$. Denoting by $n = [x]$ the integer part of x (see Examples 2.1), by definition $n \le x < n+1$; from that it follows $\frac{1}{n+1} < \frac{1}{x} \le \frac{1}{n}$, in other words $1+\frac{1}{n+1} < 1+\frac{1}{x} \le 1+\frac{1}{n}$. The familiar features of power functions yield

$$\left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{n+1}\right)^x < \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^x < \left(1+\frac{1}{n}\right)^{n+1}$$

hence

$$\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n+1}\right)^{-1} < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right). \quad \text{(A.2.6)}$$

When x tends to $+\infty$, n does the same. Using (3.3) we have

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \lim_{n\to+\infty}\left(1+\frac{1}{n}\right) = \mathrm{e};$$

the substitution $m = n+1$ similarly gives

$$\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n+1}\right)^{-1} = \mathrm{e}.$$

Applying the Second comparison theorem 4.5 to the three functions in (A.2.6) proves the claim for $x \to +\infty$. Now let us look at the case when x tends to $-\infty$. If $x < 0$ we can write $x = -|x|$, so

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{|x|}\right)^{-|x|} = \left(\frac{|x|-1}{|x|}\right)^{-|x|} = \left(\frac{|x|}{|x|-1}\right)^{|x|} = \left(1+\frac{1}{|x|-1}\right)^{|x|}.$$

Set $y = |x|-1$ and note y tends to $+\infty$ as x goes to $-\infty$. Therefore,

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y+1} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y \lim_{y\to+\infty}\left(1+\frac{1}{y}\right) = \mathrm{e}.$$

This concludes the proof. □

▶ Proof of the irrationality of Napier's number, p. 72

Property A.2.7 *Napier's number* e *is irrational, and lies between* 2 *and* 3*.*

Proof. Based on the First comparison theorem for sequences (p. 137, Theorem 4), from the previous property we quickly deduce

$$2 < \mathrm{e} \leq 3 . \tag{A.2.7}$$

Suppose, by contradiction, that e is a rational number, so that there exist two integers m_0 and $n_0 \neq 0$ such that $\mathrm{e} = \dfrac{m_0}{n_0}$. Recall that for any $n \geq 0$

$$\mathrm{e} = \sum_{k=0}^{n} \frac{1}{k!} + \frac{\mathrm{e}^{\bar{x}_n}}{(n+1)!}, \qquad 0 < \bar{x}_n < 1$$

(see Remark 7.4). From this,

$$n!\mathrm{e} = n!\frac{m_0}{n_0} = \sum_{k=0}^{n} \frac{n!}{k!} + \frac{\mathrm{e}^{\bar{x}_n}}{n+1} . \tag{A.2.8}$$

As the exponential map is monotone, and using (A.2.7), we deduce

$$1 = \mathrm{e}^0 < \mathrm{e}^{\bar{x}_n} < \mathrm{e} < 3 .$$

Choosing now $n \geq \max(3, n_0)$, the numbers $n!\dfrac{m_0}{n_0}$ and $\displaystyle\sum_{k=0}^{n} \frac{n!}{k!}$ are integers, whereas $\dfrac{\mathrm{e}^{\bar{x}_n}}{n+1}$ lies in the open interval between 0 and 1. The identity (A.2.8) then must be false, so e is irrational and equality in (A.2.7) never occurs. □

A.3

Complements on the global features of continuous maps

We first introduce the concept of subsequence of a given sequence, and establish a number of related properties. Among them, the Theorem of Bolzano-Weierstrass, which is a fundamental ingredient in the subsequent proof of the Theorem of Weierstrass concerning continuous functions on an interval of the real line; the proofs of other results for such functions are also provided. The appendix ends with the definition of uniform continuity and the discussion of some of its properties; these concepts will find application to the study of integral calculus and differential equations.

A.3.1 Subsequences

Theorem 2 on p. 137 states that every converging sequence is bounded. In general, though, the opposite implication is false. In fact, the sequence $a_n = (-1)^n$ does not converge despite being bounded ($|a_n| = 1, \ \forall n$). But if we take just the elements with even subscript, we obtain the constant sequence $\{b_k\}_{k\geq 0}$ where $b_k = a_{2k} = 1, \ k \geq 0$, which is patently convergent. Similarly if we take odd indexes only: the constant sequence $\{c_k\}_{k\geq 0}$ with $c_k = a_{2k+1} = -1, \ k \geq 0$, converges. Such sequences have been *extracted*, so to say, from the initial sequence $\{a_n\}_{n\geq 0}$, in the sense formalised below.

Definition A.3.1 *Let $\{a_n\}_{n\geq n^*}$ be a sequence and $\{n_k\}_{k\geq 0}$ a strictly increasing sequence of integers $\geq n^*$. The sequence $\{a_{n_k}\}_{k\geq 0}$ is said* **subsequence** *of $\{a_n\}_{n\geq n^*}$.*

Observe that the sequence $\{a_{n_k}\}_{k\geq 0}$ is a composite function, for it is obtained by composing the map $k \mapsto n_k$ with $n \mapsto a_n$.

Any subsequence of a converging or diverging sequence preserves the limit behaviour of the 'mother' sequence:

C. Canuto, A. Tabacco: *Mathematical Analysis I,* 2nd Ed.,
UNITEXT – La Matematica per il 3+2 84, DOI 10.1007/978-3-319-12772-9_A3,

Proposition A.3.2 *Let the sequence* $\{a_n\}_{n \geq n^*}$ *admit limit* $\lim_{n \to +\infty} a_n = \ell$*, finite or infinite. Then for any subsequence* $\{a_{n_k}\}_{k \geq 0}$

$$\lim_{k \to +\infty} a_{n_k} = \ell \, .$$

Proof. It is not that difficult to see, by induction, that

$$n_k \geq k \, , \qquad \forall k \geq 0 \, . \tag{A.3.1}$$

Clearly, $n_0 \geq 0$; supposing $n_k \geq k$ we have $n_{k+1} > n_k$ because the sequence is strictly increasing. That in turn implies $n_{k+1} \geq k + 1$, whence the claim follows.

Due to the First comparison theorem (p. 137, Theorem 4), the inequality (A.3.1) tells that the sequence $\{n_k\}$ diverges to $+\infty$. The result then follows from Theorem 4.15 adapted to sequences (whose proof is similar to the one given on p. 102). □

The fact that one can extract a converging subsequence from a bounded sequence, as we showed with the example $a_n = (-1)^n$, is a general and deep result. This is how it goes.

Theorem A.3.3 (**Bolzano-Weierstrass**) *A bounded sequence always admits a converging subsequence.*

Proof. Suppose $\{x_n\}_{n \geq n^*}$ is a bounded sequence by assuming

$$a \leq x_n \leq b \, , \qquad \forall n \geq n^* \, ,$$

for suitable $a, b \in \mathbb{R}$. We shall bisect the interval $[a, b]$ over and over, as in the proof of Theorem 4.23 of existence of zeroes. Set

$$a_0 = a \, , \qquad b_0 = b \, , \qquad \mathcal{N}_0 = \{n \geq n^*\} \, , \qquad n_0 = n^* .$$

Call c_0 the midpoint of $[a_0, b_0]$ and define the sets

$$\mathcal{N}_0^- = \{n \in \mathcal{N}_0 : x_n \in [a_0, c_0]\} \, , \qquad \mathcal{N}_0^+ = \{n \in \mathcal{N}_0 : x_n \in [c_0, b_0]\} \, .$$

Note $\mathcal{N}_0 = \mathcal{N}_0^- \cup \mathcal{N}_0^+$, where at least one of $\mathcal{N}_0^-, \mathcal{N}_0^+$ must be infinite because $\mathcal{N}_0$ is. If $\mathcal{N}_0^-$ is infinite, set

$$a_1 = a_0 \, , \qquad b_1 = c_0 \, , \qquad \mathcal{N}_1 = \mathcal{N}_0^- \, ;$$

otherwise,

$$a_1 = c_0 \, , \qquad b_1 = b_0 \, , \qquad \mathcal{N}_1 = \mathcal{N}_0^+ \, .$$

Now let n_1 be the first index $> n_0$ contained $\mathcal{N}_1$; we can make such a choice since $\mathcal{N}_1$ is infinite. Iterating the procedure (as always, in these situations, the Principle of Induction A.1.1 is required in order to make things formal), we can build a sequence of intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_k, b_k] \supset \ldots, \qquad \text{with} \quad b_k - a_k = \frac{b_0 - a_0}{2^k},$$

a sequence of infinite sets

$$\mathcal{N}_0 \supseteq \mathcal{N}_1 \supseteq \ldots \supseteq \mathcal{N}_k \supseteq \ldots$$

and a strictly increasing sequence of indices $\{n_k\}_{k \geq 0}$, $n_k \in \mathcal{N}_k$, such that

$$a_k \leq x_{n_k} \leq b_k, \qquad \forall k \geq 0.$$

Then just as in the proof of Theorem 4.23, there will be a unique $\ell \in [a, b]$ satisfying

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} b_k = \ell.$$

From the Second comparison theorem (Theorem 5 on p. 137) we deduce that the sequence $\{x_{n_k}\}_{k \geq 0}$, extracted from $\{x_n\}_{n \geq n^*}$, converges to ℓ. □

A.3.2 Continuous functions on an interval

▶ Proof of the Theorem of Weierstrass, p. 114

Theorem 4.31 (Weierstrass) *A continuous map f on a closed and bounded interval $[a, b]$ is bounded and admits minimum and maximum*

$$m = \min_{x \in [a,b]} f(x) \qquad \text{and} \qquad M = \max_{x \in [a,b]} f(x).$$

Consequently,

$$f([a, b]) = [m, M].$$

Proof. We will show first that f admits a maximum in $[a, b]$, in other words there exists $\xi \in [a, b]$ such that $f(x) \leq f(\xi)$, $\forall x \in [a, b]$. For this, let

$$M = \sup f([a, b]),$$

which is allowed to be both real or $+\infty$. In the former case the characterisation of the supremum, (1.7) *ii)*, tells that for any $n \geq 1$ there is $x_n \in [a, b]$ with

$$M - \frac{1}{n} < f(x_n) \leq d.$$

Letting n go to $+\infty$, from the Second comparison theorem (Theorem 5 on p. 137) we infer

$$\lim_{n\to\infty} f(x_n) = M.$$

In the other case, by definition of unbounded set (from above) we deduce that for any $n \geq 1$ there is $x_n \in [a,b]$ such that

$$f(x_n) \geq n.$$

The Second comparison theorem implies

$$\lim_{n\to\infty} f(x_n) = +\infty = M.$$

In either situation, the sequence $\{x_n\}_{n\geq 1}$ thus defined is bounded (it is contained in $[a,b]$). We are then entitled to use Theorem of Bolzano-Weierstrass and call $\{x_{n_k}\}_{k\geq 0}$ a convergent subsequence. Let ξ be its limit; since all x_{n_k} belong to $[a,b]$, necessarily $\xi \in [a,b]$. But $\{f(x_{n_k})\}_{k\geq 0}$ is a subsequence of $\{f(x_n)\}_{n\geq 0}$, so by Proposition A.3.2

$$\lim_{k\to\infty} f(x_{n_k}) = M.$$

The continuity of f at ξ implies

$$f(\xi) = f(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = M,$$

which tells us that M cannot be $+\infty$. Moreover, M belongs to the range of f, hence

$$M = \max f([a,b]).$$

Arguing in a similar fashion one proves that the number

$$m = \min f([a,b])$$

exists and is finite. The final claim is a consequence of Corollary 4.30. □

▶ Proof of Corollary 4.25, p. 111

Corollary 4.25 *Let f be continuous on the interval I and suppose it admits non-zero limits (finite or infinite) that are different in sign for x tending to the end-points of I. Then f has a zero in I, which is unique if f is strictly monotone on I.*

Proof. We indicate by α, β (finite or not) the end-points of I and call

$$\lim_{x\to\alpha^+} f(x) = \ell_\alpha \qquad \text{and} \qquad \lim_{x\to\beta^-} f(x) = \ell_\beta\,.$$

Should one end-point, or both, be infinite, these writings denote the usual limits at infinity.

We suppose $\ell_\alpha < 0 < \ell_\beta$, for otherwise we can swap the roles of ℓ_α and ℓ_β. By Theorem 4.2 there exist a right neighbourhood $I^+(\alpha)$ of α and a left neighbourhood $I^-(\beta)$ of β such that

$$\forall x \in I^+(\alpha)\,,\ f(x) < 0 \qquad \text{and} \qquad \forall x \in I^-(\beta)\,,\ f(x) > 0\,.$$

Let us fix points $a \in I^+(\alpha)$ and $b \in I^-(\beta)$ with $\alpha < a$, $b < \beta$. The interval $[a, b]$ is contained in I, hence f is continuous on it, and by construction $f(a) < 0 < f(b)$. Therefore f will vanish somewhere, in virtue of Theorem 4.23 (existence of zeroes) applied to $[a, b]$.

If f is strictly monotone, uniqueness follows from Proposition 2.8 on the interval I. □

▶ Proof of Theorems 4.32 and 4.33, p. 114

Let us prove a preliminary result before proceeding.

Lemma A.3.4 *Let f be continuous and invertible on an interval I. For any chosen points $x_1 < x_2 < x_3$ in I, then one, and only one, of*

(i) $$f(x_1) < f(x_2) < f(x_3)$$

or

(ii) $$f(x_1) > f(x_2) > f(x_3)$$

holds.

Proof. As f is invertible, hence one-to-one, the images $f(x_1)$ and $f(x_3)$ cannot coincide. Then either $f(x_1) < f(x_3)$ or $f(x_1) > f(x_3)$, and we claim that these cases imply (i) or (ii), respectively.

Suppose $f(x_1) < f(x_3)$, and assume by contradiction that (i) is false, so $f(x_2)$ does not lie strictly between $f(x_1)$ and $f(x_3)$. For instance,

$$f(x_1) < f(x_3) < f(x_2)$$

(if $f(x_2) < f(x_1) < f(x_3)$ the argument is the same). As f is continuous on the closed interval $[x_1, x_2] \subseteq I$, the Intermediate value theorem 4.29 prescribes that it will assume every value between $f(x_1)$ and $f(x_2)$ on $[x_1, x_2]$. In particular, there will be a point $\bar{x} \in (x_1, x_2)$ such that

$$f(\bar{x}) = f(x_3),$$

in contradiction to injectivity: $\bar{x}$ and x_3 are in fact distinct, because separated by x_2. □

Theorem 4.32 *A continuous function f on an interval I is one-to-one if and only if it is strictly monotone.*

Proof. Thanks to Proposition 2.8 we only need to prove the implication

$$f \text{ invertible on} I \quad \Rightarrow \quad f \text{ strictly monotone on } I\,.$$

Letting $x_1 < x_2$ be arbitrary points of I, we claim that if $f(x_1) < f(x_2)$ then f is strictly increasing on I ($f(x_1) > f(x_2)$ will similarly imply f strictly decreases on I).

Let $z_1 < z_2$ be points in I, and suppose both lie within (x_1, x_2); the other possibilities are dealt with in the same way. Hence we have

$$x_1 < z_1 < z_2 < x_2.$$

Let us use Lemma A.3.4 on the triple $x_1,\ z_1,\ x_2$: since we have assumed $f(x_1) < f(x_2)$, it follows

$$f(x_1) < f(z_1) < f(x_2).$$

Now we employ the triple $z_1,\ z_2,\ x_2$, to the effect that

$$f(z_1) < f(z_2) < f(x_2).$$

The first inequality in the above line tells f is strictly increasing, proving Theorem 4.32. □

Theorem 4.33 *Let f be continuous and invertible on an interval I. Then the inverse f^{-1} is continuous on the interval $J = f(I)$.*

Proof. The first remark is that J is indeed an interval, by Corollary 4.30. Using Theorem 4.32 we deduce f is strictly monotone on I: to fix ideas, suppose it is strictly increasing (having f strictly decreasing would not change the proof). By definition of a monotone map we have that f^{-1} is strictly increasing on J as well. But it is known that a monotone map admits at most discontinuities of the first kind (Corollary 3.28). We will show that f^{-1} cannot have this type either. By contradiction, suppose there is a jump point $y_0 = f(x_0) \in J = f(I)$ for f^{-1}. Equivalently, let

$$z_0^- = \sup_{y<y_0} f^{-1}(y) = \lim_{y\to y_0^-} f^{-1}(y)\,,$$

$$z_0^+ = \inf_{y>y_0} f^{-1}(y) = \lim_{y\to y_0^+} f^{-1}(y)\,,$$

and suppose $z_0^- < z_0^+$. Then inside (z_0^-, z_0^+) there will be at most one element $x_0 = f^{-1}(y_0)$ of the range $f^{-1}(J)$. Thus $f^{-1}(J)$ is not an interval. By definition of J, on the other hand, $f^{-1}(J) = I$ is an interval by hypothesis. In conclusion, f^{-1} must be continuous at each point of J. □

A.3.3 Uniform continuity

Let the map f be defined on the real interval I. Recall f is called continuous on I if it is continuous at each point $x_0 \in I$, i.e., for any $x_0 \in I$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \in I, \qquad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon\,.$$

In general $\delta = \delta(\varepsilon, x_0)$, meaning that δ depends on x_0, too. But if, for fixed $\varepsilon > 0$, we find $\delta = \delta(\varepsilon)$ independent of $x_0 \in I$, we say f is uniformly continuous on I. More precisely,

Definition A.3.5 *A function f is called* **uniformly continuous on** *I if, for any $\varepsilon > 0$, there is a $\delta > 0$ satisfying*

$$\forall x', x'' \in I, \qquad |x' - x''| < \delta \quad \Rightarrow \quad |f(x') - f(x'')| < \varepsilon\,. \tag{A.3.2}$$

Examples A.3.6

i) Let $f(x) = x^2$, defined on $I = [0,1]$. Then

$$|f(x') - f(x'')| = |(x')^2 - (x'')^2| = |x' + x''|\,|x' - x''| \leq 2|x' - x''|\,.$$

If $|x' - x''| < \frac{\varepsilon}{2}$ we see $|f(x') - f(x'')| < \varepsilon$, hence $\delta = \frac{\varepsilon}{2}$ fulfills (A.3.2) on I, rendering f uniformly continuous on I.

ii) Take $f(x) = x^2$ on $I = [0, +\infty)$. We want to prove by contradiction that f is not uniformly continuous on I. If it were, with $\varepsilon = 1$ for example, there would be a $\delta > 0$ satisfying (A.3.2). Choose $x' \in I$ and let $x'' = x' + \frac{\delta}{2}$, so that $|x' - x''| = \frac{\delta}{2} < \delta$; then

$$|f(x') - f(x'')| = |x' + x''|\,|x' - x''| < 1\,,$$

or

$$\Big(2x' + \frac{\delta}{2}\Big)\frac{\delta}{2} < 1\,.$$

Now letting x' tend to $+\infty$ we obtain a contradiction.

iii) Consider $f(x) = \sin x$ on $I = \mathbb{R}$. From

$$\sin x' - \sin x'' = 2 \sin \frac{x' - x''}{2} \cos \frac{x' + x''}{2}$$

we have

$$|\sin x' - \sin x''| \leq |x' - x''|\,, \qquad \forall x', x'' \in \mathbb{R}\,.$$

With a fixed $\varepsilon > 0$, $\delta = \varepsilon$ satisfies the requirement for uniform continuity.

iv) Let $f(x) = \frac{1}{x}$ on $I = (0, +\infty)$. Note that

$$|f(x') - f(x'')| = \left|\frac{1}{x'} - \frac{1}{x''}\right| = \frac{|x' - x''|}{x'x''}.$$

By letting x', x'' tend to 0, one easily verifies that f cannot be uniformly continuous on I.
But if we consider only $I_a = [a, +\infty)$, where $a > 0$ is fixed, then

$$|f(x') - f(x'')| \leq \frac{|x' - x''|}{a^2},$$

so $\delta = a^2\varepsilon$ satisfies the requirement on I_a, for any given $\varepsilon > 0$. □

Are there conditions guaranteeing uniform continuity? One answer is provided by the following result.

Theorem A.3.7 (**Heine-Cantor**) *Let f be a continuous map on the closed and bounded interval $I = [a, b]$. Then f is uniformly continuous on I.*

Proof. Let us suppose f is not uniformly continuous on I. This means that there exists an $\varepsilon > 0$ such that, for any $\delta > 0$, there are $x', x'' \in I$ with $|x' - x''| < \delta$ and $|f(x') - f(x'')| \geq \varepsilon$. Choosing $\delta = \frac{1}{n}$, $n \geq 1$, we find two sequences of points $\{x'_n\}_{n\geq 1}$, $\{x''_n\}_{n\geq 1}$ inside I such that

$$|x'_n - x''_n| < \frac{1}{n} \qquad \text{and} \qquad |f(x'_n) - f(x''_n)| \geq \varepsilon\,.$$

The condition on the left implies

$$\lim_{n\to\infty} (x'_n - x''_n) = 0\,,$$

while the one on the right implies that $f(x'_n) - f(x''_n)$ will not tend to 0 for $n \to \infty$. On the other hand the sequence $\{x'_n\}_{n\geq 1}$ is bounded, $a \leq x'_n \leq b$ for all n, so Theorem A.3.3, of Bolzano-Weierstrass, will give a subsequence $\{x'_{n_k}\}_{k\geq 0}$ converging to a certain $\bar{x} \in I$:

$$\lim_{k\to\infty} x'_{n_k} = \bar{x}\,.$$

Also the subsequence $\{x''_{n_k}\}_{k\geq 0}$ converges to $\bar{x}$, for

$$\lim_{k\to\infty} x''_{n_k} = \lim_{k\to\infty} [x'_{n_k} + (x''_{n_k} - x'_{n_k})] = \lim_{k\to\infty} x'_{n_k} + \lim_{k\to\infty} (x''_{n_k} - x'_{n_k}) = \bar{x} + 0 = \bar{x}\,.$$

Now, f being continuous at $\bar{x}$, we have

$$\lim_{k\to\infty} f(x'_{n_k}) = f\big(\lim_{k\to\infty} x'_{n_k}\big) = f(\bar{x}) \quad \text{and} \quad \lim_{k\to\infty} f(x''_{n_k}) = f\big(\lim_{k\to\infty} x''_{n_k}\big) = f(\bar{x})\,.$$

Then

$$\lim_{k\to\infty} \big(f(x'_{n_k}) - f(x''_{n_k})\big) = f(\bar{x}) - f(\bar{x}) = 0\,,$$

contradicting the fact that

$$|f(x'_{n_k}) - f(x''_{n_k})| \geq \varepsilon > 0\,, \qquad \forall k \geq 0\,.$$

□

A.4

Complements on differential calculus

This appendix is entirely devoted to the proof of important results of differential calculus. We first justify the main derivation formulas, then we prove the Theorem of de l'Hôpital. Our next argument is the study of differentiable and convex functions, for which we highlight logical links between convexity and certain properties of the first derivative. At last, we establish Taylor formulas with three forms of the remainder, i.e., Peano's, Lagrange's and the integral form.

A.4.1 Derivation formulas

▶ Proof of Theorem 6.4, p. 174

Theorem 6.4 (Algebraic operations) *Let $f(x), g(x)$ be differentiable maps at $x_0 \in \mathbb{R}$. Then the maps $f(x) \pm g(x)$, $f(x)g(x)$ and, if $g(x_0) \neq 0$, $\dfrac{f(x)}{g(x)}$ are differentiable at x_0. To be precise,*

$$\begin{aligned}
(f \pm g)'(x_0) &= f'(x_0) \pm g'(x_0), \\
(f\,g)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0), \\
\left(\frac{f}{g}\right)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}.
\end{aligned}$$

Proof. Let us start by (6.3). By Theorem 4.10 we have

$$\begin{aligned}
\lim_{x\to x_0} \frac{\big(f(x) \pm g(x)\big) - \big(f(x_0) \pm g(x_0)\big)}{x - x_0} &= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \pm \frac{g(x) - g(x_0)}{x - x_0}\right) \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \pm \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) \pm g'(x_0)\,.
\end{aligned}$$

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Next we prove (6.4). For this, recall that a differentiable map is continuous (Proposition 6.3), so $\lim_{x\to x_0} g(x) = g(x_0)$. Therefore

$$\begin{aligned}
&\lim_{x\to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x - x_0} \\
&\quad \lim_{x\to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x) + f(x_0)\,g(x) - f(x_0)\,g(x_0)}{x - x_0} \\
&\quad = \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\, g(x) + f(x_0)\, \frac{g(x) - g(x_0)}{x - x_0} \right) \\
&\quad = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0} g(x) + f(x_0) \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
&\quad = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)\,.
\end{aligned}$$

Eventually, we show (6.5). Since $\lim_{x\to x_0} g(x) = g(x_0) \neq 0$, Theorem 4.2 ensures there is a neighbourhood of x_0 where $g(x) \neq 0$. Then the function $\dfrac{f(x)}{g(x)}$ is well defined on such neighbourhood and we can consider its difference quotient

$$\begin{aligned}
&\lim_{x\to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0} = \lim_{x\to x_0} \frac{f(x)\,g(x_0) - f(x_0)\,g(x)}{g(x)\,g(x_0)\,(x - x_0)} \\
&\quad = \lim_{x\to x_0} \frac{f(x)\,g(x_0) - f(x_0)\,g(x_0) + f(x_0)\,g(x_0) - f(x_0)\,g(x)}{g(x)\,g(x_0)\,(x - x_0)} \\
&\quad = \lim_{x\to x_0} \frac{1}{g(x)\,g(x_0)} \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x_0) - f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right) \\
&\quad = \frac{1}{\big(g(x_0)\big)^2} \left(\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x_0) - \lim_{x\to x_0} f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right) \\
&\quad = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{\big(g(x_0)\big)^2}\,.
\end{aligned}$$

□

▶ Proof of Theorem 6.7, p. 175

Theorem 6.7 ("Chain rule") *Let $f(x)$ be differentiable at $x_0 \in \mathbb{R}$ and $g(y)$ a differentiable map at $y_0 = f(x_0)$. Then the composition $g \circ f(x) = g\big(f(x)\big)$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0) = g'\big(f(x_0)\big) f'(x_0).$$

Proof. Let us use the first formula of the finite increment (6.11) on g at the point $y_0 = f(x_0)$:

$$g(y) - g(y_0) = g'(y_0)(y - y_0) + o(y - y_0), \quad y \to y_0 .$$

By definition of 'little o', the above means that there exists a map φ such that $\lim_{y \to y_0} \varphi(y) = 0 = \varphi(y_0)$ satisfying

$$g(y) - g(y_0) = g'(y_0)(y - y_0) + \varphi(y)(y - y_0) , \quad \text{on a neighbourhood } I(y_0) \text{ of } y_0 .$$

As f is continuous at x_0 (Proposition 6.3), there is a neighbourhood $I(x_0)$ of x_0 such that $f(x) \in I(y_0)$ for all $x \in I(x_0)$. If we put $y = f(x)$ in the displayed relation, this becomes

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = g'(f(x_0)) \frac{f(x) - f(x_0)}{x - x_0} + \varphi(f(x)) \frac{f(x) - f(x_0)}{x - x_0} .$$

Observing that

$$\lim_{x \to x_0} \varphi(f(x)) = \lim_{y \to y_0} \varphi(y) = 0$$

by the Substitution theorem 4.15, we can pass to the limit and conclude

$$\begin{aligned} &\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &\quad = g'(f(x_0)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \varphi(f(x)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &\quad = g'(f(x_0)) f'(x_0) . \end{aligned}$$

□

▶ Proof of Theorem 6.9, p. 175

Theorem 6.9 (Derivative of the inverse function) *Suppose $f(x)$ is a continuous, invertible map on a neighbourhood of $x_0 \in \mathbb{R}$, and differentiable at x_0, with $f'(x_0) \neq 0$. Then the inverse map $f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))} .$$

Proof. The inverse map is continuous on a neighbourhood of y_0 in virtue of Theorem 4.33. Write $x = f^{-1}(y)$, so that on the same neighbourhood

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} .$$

By the Substitution theorem 4.15, with $x = f^{-1}(y)$, we have

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)} .$$

□

A.4.2 De l'Hôpital's Theorem

▶ Proof of de l'Hôpital's Theorem, p. 200

Theorem 6.41 (**de l'Hôpital**) *Let f, g be maps defined on a neighbourhood of c, except possibly at c, and such that*

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L,$$

where $L = 0, +\infty$ or $-\infty$. If f and g are differentiable around c, except possibly at c, with $g' \neq 0$, and if

$$\lim_{x\to c} \frac{f'(x)}{g'(x)}$$

exists (finite or not), then also

$$\lim_{x\to c} \frac{f(x)}{g(x)}$$

exists and equals the previous limit.

Proof. The theorem includes several statements corresponding to the values assumed by L and c, and the arguments used for the proofs vary accordingly. For this reason we have grouped the proofs together into cases.

a) The cases $L = 0$, $c = x_0^+, x_0^-, x_0$.
Let us suppose $c = x_0^+$. By assumption $\lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^+} g(x) = 0$, so we may extend both functions to x_0 (re-defining their values if necessary) by putting $f(x_0) = g(x_0) = 0$; thus f and g become right-continuous at x_0. Let $I^+(x_0)$ denote the right neighbourhood of x_0 where f, g satisfy the theorem, and take $x \in I^+(x_0)$. On the interval $[x_0, x]$ Theorem 6.25 is valid, so there is $t = t(x) \in (x_0, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(t)}{g'(t)}\,.$$

As $x_0 < t(x) < x$, the Second comparison theorem 4.5 guarantees that for x tending to x_0 also $t = t(x)$ approaches x_0. Now the Substitution theorem 4.15 yields

$$\lim_{x\to x_0^+} \frac{f(x)}{g(x)} = \lim_{x\to x_0^+} \frac{f'(t(x))}{g'(t(x))} = \lim_{t\to x_0^+} \frac{f'(t)}{g'(t)}\,,$$

and the proof ends.
We proceed similarly when $c = x_0^-$; the remaining case $c = x_0$ descends from the two one-sided limits.

b) The cases $L = 0$, $c = \pm\infty$.

Suppose $c = +\infty$. The substitution $z = \frac{1}{x}$ leads to consider the limit of the quotient $\dfrac{f(\frac{1}{z})}{g(\frac{1}{z})}$ for $z \to 0^+$. Because $\dfrac{\mathrm{d}}{\mathrm{d}z} f\left(\dfrac{1}{z}\right) = -\dfrac{1}{z^2} f'\left(\dfrac{1}{z}\right)$, and similarly for the map g, it follows

$$\lim_{z\to 0^+} \frac{\frac{\mathrm{d}}{\mathrm{d}z} f\left(\frac{1}{z}\right)}{\frac{\mathrm{d}}{\mathrm{d}z} g\left(\frac{1}{z}\right)} = \lim_{z\to 0^+} \frac{f'\left(\frac{1}{z}\right)}{g'\left(\frac{1}{z}\right)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} .$$

In this way we return to the previous case $c = 0^+$, and the result is proved. The same for $c = -\infty$.

c) The cases $L = \pm\infty$, $c = x_0^+, x_0^-, x_0$.

Assume $c = x_0^+$ and put $\displaystyle\lim_{x\to x_0^+} \frac{f'(x)}{g'(x)} = \ell$. When $\ell \in \mathbb{R}$, let $I^+(x_0)$ be the right neighbourhood of x_0 on which f and g satisfy the theorem. For every $\varepsilon > 0$ there exists $\delta_1 > 0$ with $x_0 + \delta_1 \in I^+(x_0)$ so that for all $x \in (x_0, x_0 + \delta_1)$ we have $\left|\dfrac{f'(x)}{g'(x)} - \ell\right| < \varepsilon$. On $[x, x_0 + \delta_1]$ Theorem 6.25 holds, hence there is $t = t(x) \in (x, x_0 + \delta_1)$ such that

$$\frac{f(x) - f(x_0 + \delta_1)}{g(x) - g(x_0 + \delta_1)} = \frac{f'(t)}{g'(t)} . \tag{A.4.1}$$

Write the ratio $\dfrac{f(x)}{g(x)}$ as

$$\frac{f(x)}{g(x)} = \psi(x) \frac{f'(t)}{g'(t)} ,$$

where, by (A.4.1),

$$\psi(x) = \frac{1 - \frac{g(x_0+\delta_1)}{g(x)}}{1 - \frac{f(x_0+\delta_1)}{f(x)}} , \qquad \text{with} \qquad \lim_{x\to x_0^+} \psi(x) = 1 ,$$

because $L = \pm\infty$. The last limit implies that there is a $\delta_2 > 0$, with $\delta_2 < \delta_1$, such that

$$|\psi(x)| \le 2 \qquad \text{and} \qquad |\psi(x) - 1| < \varepsilon$$

for every $x \in (x_0, x_0 + \delta_2)$. Therefore, for all $x \in (x_0, x_0 + \delta_2)$,

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \ell\right| &= \left|\psi(x)\frac{f'(t)}{g'(t)} - \psi(x)\ell + \psi(x)\ell - \ell\right| \\ &= |\psi(x)| \left|\frac{f'(t)}{g'(t)} - \ell\right| + |\psi(x) - 1||\ell| < \big(2 + |\ell|\big)\varepsilon . \end{aligned}$$

We conclude

$$\lim_{x\to x_0^+} \frac{f(x)}{g(x)} = \ell .$$

Let now $\ell = +\infty$; for all $A > 0$ there is $\delta_1 > 0$, with $x_0 + \delta_1 \in I^+(x_0)$, such that for all $x \in (x_0, x_0 + \delta_1)$ we have $\dfrac{f'(x)}{g'(x)} > A$. As before, using Theorem A.2.2 we observe that $\lim\limits_{x\to x_0^+} \psi(x) = 1$ implies the existence of a $\delta_2 > 0$, with $\delta_2 < \delta_1$, such that $\psi(x) \geq \frac{1}{2}$ for all $x \in (x_0, x_0 + \delta_2)$. Therefore, for every $x \in (x_0, x_0 + \delta_2)$,

$$\frac{f(x)}{g(x)} = \psi(x)\frac{f'(t)}{g'(t)} \geq \frac{1}{2}A\,,$$

proving the claim. The procedure is the same for $\ell = -\infty$.

An analogous proof holds for $c = x_0^-$, and $c = x_0$ is dealt with by putting the two arguments together.

d) The cases $L = \pm\infty$, $c = \pm\infty$.

As in b), we may substitute $z = \frac{1}{x}$ and use the previous argument. □

A.4.3 Convex functions

We begin with a lemma, which tells that local convexity (i.e., convexity on a neighbourhood of every point of I) is in fact a global condition (valid on all of I). intorno di ogni punto di I) è in realtà globale (cioè valida su tutto I).

Lemma A.4.1 *Let f be differentiable on the interval I. Then f is convex on I if and only if for every $x_0 \in I$*

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \qquad \forall x \in I\,. \tag{A.4.2}$$

Proof. Obviously, it is enough to show that if f is convex according to Definition 6.33 on I, then also (A.4.2) holds. To this end, one usefully notes that f is convex on I if and only if the map $g(x) = f(x)+ax+b$, $a, b \in \mathbb{R}$ is convex; in fact, requiring $f(x) \geq f(x_0)+f'(x_0)(x-x_0)$ is equivalent to $g(x) \geq g(x_0)+g'(x_0)(x-x_0)$.

Let then $x_0 \in I$ be fixed arbitrarily and consider the convex map $g(x) = f(x)-f(x_0)-f'(x_0)(x-x_0)$, which satisfies $g(x_0) = g'(x_0) = 0$. We have to prove that $g(x) \geq 0$, $\forall x \in I$. Suppose x_0 is not the right end-point of I and let us show $g(x) \geq 0$, $\forall x \in I$, $x > x_0$; a 'symmetry' argument will complete the proof.

Being g convex at x_0, we have $g(x) \geq 0$ on a (right) neighbourhood of x_0. It makes then sense to define

$$P = \{x > x_0 : g(s) \geq 0,\ \forall s \in [x_0, x]\}$$

and $x_1 = \sup P$.

If x_1 coincides with the right end-point of I, the assertion follows. Let us assume, by contradiction, x_1 lies inside I; By definition $g(x) \geq 0$, $\forall x \in [x_0, x_1)$, while in each (right) neighbourhood of x_1 there exist points $x \in I$ at which $g(x) < 0$. From this and the continuity of g at x_1 we deduce that necessarily $g(x_1) = 0$ (so, in particular, $x_1 = \max P$). We want to prove $g(x) = 0$, $\forall x \in [x_0, x_1]$. Once we have done that, then $g'(x_1) = 0$ (as g is differentiable at x_1 and constant on a left neighbourhood of the same point). Therefore the convexity of g at x_1 implies the existence of a neighbourhood of x_1 where $g(x) \geq 0$, against the definition of x_1.

It remains to prove $g(x) = 0$ in $[x_0, x_1]$. As $g(x) \geq 0$ on $[x_0, x_1]$ by definition, we assume, again by contradiction, that $M = \max\{g(x) : x \in [x_0, x_1]\} > 0$, and let $\bar{x} \in (x_0, x_1)$ be a pre-image of $g(\bar{x}) = M$. By Fermat's Theorem 6.21 $g'(\bar{x}) = 0$, so the convexity at $\bar{x}$ yields a neighbourhood of $\bar{x}$ on which $g(x) \geq g(\bar{x}) = M$; but M is the maximum of g on $[x_0, x_1]$, so $g(x) = M$ on said neighbourhood. Now define

$$Q = \{x > \bar{x} : g(s) = M,\ \forall s \in [\bar{x}, x]\}$$

and $x_2 = \sup Q$. The map g is continuous, hence $x_2 = \max Q$, and moreover $x_2 < x_1$ because $g(x_1) = 0$. As before, the hypothesis of convexity at x_2 leads to a contradiction. □

▶ Proof of Theorem 6.37, p. 193

> **Theorem 6.37** *Given a differentiable map f on the interval I,*
>
> *a) if f is convex on I, then f' is increasing on I.*
>
> *b1) If f' is increasing on I, then f is convex on I;*
>
> *b2) if f' is strictly increasing on I, then f is strictly convex on I.*

Proof.
a) Take $x_1 < x_2$ two points in I. From (A.4.2) with $x_0 = x_1$ and $x = x_2$ we obtain

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

while putting $x_0 = x_2$, $x = x_1$ gives

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2).$$

Combining the two inequalities yields the result.

b1) Let $x > x_0$ be chosen in I. The second formula of the finite increment of f on $[x_0, x]$ prescribes the existence of a point $\bar{x} \in (x_0, x)$ such that

$$f(x) = f(x_0) + f'(\bar{x})(x - x_0).$$

The map f' is monotone, so $f'(\bar{x}) \geq f'(x_0)$ hence (A.4.2). When $x < x_0$ the argument is analogous.

b2) In the proof for *b1)* we now have $f'(\bar{x}) > f'(x_0)$, whence (A.4.2) is strict (for $x \neq x_0$). □

A.4.4 Taylor formulas

We open this section by describing an interesting property of Taylor expansions. Observe to this end that if a map g is defined only at one point x_0, its Taylor polynomial of degree 0 can still be defined, by letting $Tg_{0,x_0}(x) = g(x_0)$.

Lemma A.4.2 *Let f be n times differentiable at x_0. The derivative of order h, $0 \le h \le n$, of the Taylor polynomial of f of degree n at x_0 coincides with the Taylor polynomial of $f^{(h)}$ of order $n-h$ at x_0:*

$$D^h Tf_{n,x_0}(x) = Tf^{(h)}_{n-h,x_0}(x)\,. \tag{A.4.3}$$

In particular,

$$D^h Tf_{n,x_0}(x_0) = f^{(h)}(x_0)\,, \qquad \forall h = 0, \ldots, n\,. \tag{A.4.4}$$

Proof. From Example 6.31 i) we know that

$$D^h(x-x_0)^k = \begin{cases} 0 & \text{if } h > k \\ \dfrac{k!}{(k-h)!}(x-x_0)^{k-h} & \text{if } h \le k\,. \end{cases}$$

Therefore

$$\begin{aligned} D^h Tf_{n,x_0}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} D^h(x-x_0)^k \\ &= \sum_{k=h}^{n} \frac{f^{(k)}(x_0)}{(k-h)!}(x-x_0)^{k-h}\,. \end{aligned}$$

Note also that

$$f^{(k)}(x_0) = f^{(h+k-h)}(x_0) = \big(f^{(h)}\big)^{(k-h)}(x_0)\,,$$

in other words differentiating $k-h$ times the derivative of order h produces the kth derivative. In this way, putting $\ell = k-h$ gives

$$\begin{aligned} D^h Tf_{n,x_0}(x) &= \sum_{k=h}^{n} \frac{\big(f^{(h)}\big)^{(k-h)}(x_0)}{(k-h)!}(x-x_0)^{k-h} \\ &= \sum_{\ell=0}^{n-h} \frac{\big(f^{(h)}\big)^{(\ell)}(x_0)}{\ell!}(x-x_0)^{\ell} = Tf^{(h)}_{n-h,x_0}(x)\,, \end{aligned}$$

which is (A.4.3). Formula (A.4.4) follows by recalling that the Taylor expansion at a point x_0 of a function coincides with the function itself at that point. □

▶ Proof of Theorem 7.1, p. 228

Theorem 7.1 (**Taylor formula with Peano's remainder**) *Let $n \geq 0$ and f be n times differentiable at x_0. Then the* **Taylor formula** *holds*

$$f(x) = Tf_{n,x_0}(x) + o\big((x-x_0)^n\big), \quad x \to x_0,$$

where

$$\begin{aligned} Tf_{n,x_0}(x) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k \\ &= f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n. \end{aligned}$$

Proof. We need to show that

$$L = \lim_{x \to x_0} \frac{f(x) - Tf_{n,x_0}(x)}{(x-x_0)^n} = 0\,.$$

The limit is an indeterminate form of type $\frac{0}{0}$; in order to apply de l'Hôpital's Theorem 6.41 we are lead to consider this

$$\lim_{x \to x_0} \frac{f'(x) - (Tf_{n,x_0})'(x)}{n(x-x_0)^{n-1}} = \lim_{x \to x_0} \frac{f'(x) - Tf'_{n-1,x_0}(x)}{n(x-x_0)^{n-1}}\,,$$

(in which Lemma A.4.2, with $h = 1$, was used); note that the other requirements of 6.41 are fulfilled.

For $n > 1$ we are still in presence of an indeterminate form $\frac{0}{0}$, so repeating $n-1$ times the argument above brings us to the limit

$$\begin{aligned} &\lim_{x \to x_0} \frac{f^{(n-1)}(x) - Tf^{(n-1)}_{1,x_0}(x)}{n!(x-x_0)} \\ &\qquad = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x-x_0)}{n!(x-x_0)} \\ &\qquad = \frac{1}{n!} \lim_{x \to x_0} \left(\frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x-x_0} - f^{(n)}(x_0) \right) = 0\,, \end{aligned}$$

by definition of nth derivative at x_0. This grants the green light to the use of de l'Hôpital's Theorem, and $L = 0$. □

▶ Proof of Theorem 7.2, p. 228

Theorem 7.2 (Taylor formula with Lagrange's remainder) *Let $n \geq 0$ and f differentiable n times at x_0, with continuous nth derivative, be given; suppose f is differentiable $n+1$ times around x_0, except possibly at x_0. Then the Taylor formula*

$$f(x) = Tf_{n,x_0}(x) + \frac{1}{(n+1)!} f^{(n+1)}(\bar{x})(x-x_0)^{n+1},$$

holds, for a suitable $\bar{x}$ between x_0 and x.

Proof. Let $\varphi(x) = f(x) - Tf_{n,x_0}(x)$ and $\psi(x) = (x-x_0)^{n+1}$. Using $(A.4.4)$, for $h = 0, \ldots, n$ we have

$$\varphi^{(h)}(x_0) = 0\,;$$

moreover, $\psi^{(h)}(x_0) = 0$ and $\psi^{(h)}(x) \neq 0$ for any $x \neq x_0$. Applying Theorem 6.25 to φ, ψ on the interval I_0 between x_0 and x, we know there is a point $x_1 \in I_0$ such that

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi(x) - \varphi(x_0)}{\psi(x) - \psi(x_0)} = \frac{\varphi'(x_1)}{\psi'(x_1)}\,.$$

The same recipe used on the maps $\varphi'(x)$, $\psi'(x)$ on the interval I_1 between x_0, x_1 produces a point $x_2 \in I_1 \subset I_0$ satisfying

$$\frac{\varphi'(x_1)}{\psi'(x_1)} = \frac{\varphi'(x_1) - \varphi'(x_0)}{\psi'(x_1) - \psi'(x_0)} = \frac{\varphi''(x_2)}{\psi''(x_2)}\,.$$

Iterating the argument eventually gives a $x_{n+1} \in I_0$ such that

$$\frac{\varphi(x)}{\psi(x)} = \cdots = \frac{\varphi^{(n+1)}(x_{n+1})}{\psi^{(n+1)}(x_{n+1})}\,.$$

But $\varphi^{(n+1)}(x) = f^{(n+1)}(x)$ and $\psi^{(n+1)}(x) = (n+1)!$, putting $\bar{x} = x_{n+1}$ in which yields the assertion. □

▶ Proof of Theorem 9.44, p. 338

Theorem 9.44 (Taylor formula with integral remainder) *Let $n \geq 0$ be an arbitrary integer, f differentiable $n+1$ times around a point x_0, with continuous derivative of order $n+1$. Then*

$$f(x) - Tf_{n,x_0}(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt\,.$$

Proof. We shall use the induction principle (see Appendix A.1). When $n = 0$, the formula reduces to the identity

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\, dt\,,$$

established in Corollary 9.42.

Supposing the statement true for a certain n, let us prove it for $n+1$. Integrating by parts and using the hypothesis,

$$\begin{aligned}
&\frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\, dt \\
&\quad = \frac{1}{(n+1)!}\left[f^{(n+1)}(t)(x-t)^{n+1}\Big|_{x_0}^{x} + (n+1)\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^{n}\, dt\right] \\
&\quad = -\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^{n}\, dt \\
&\quad = -\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + f(x) - Tf_{n,x_0}(x) \\
&\quad = f(x) - Tf_{n+1,x_0}(x)\,.
\end{aligned}$$

□

At last, we provide an example that illustrates how a more accurate piece of information may be extracted from the integral form of the remainder, as opposed to the Lagrange form.

Example A.4.3 Consider the MacLaurin expansion of the exponential function $f(x) = \mathrm{e}^x$ with remainder of order 1, both in Lagrange's form and in integral form. Assuming $x > 0$, if we use the former form we have for a suitable $\bar{x} \in (0, x)$

$$\mathrm{e}^x = 1 + x + \frac{1}{2}\mathrm{e}^{\bar{x}}x^2\,, \tag{A.4.5}$$

whereas with the latter form we obtain

$$\mathrm{e}^x = 1 + x + \int_0^x \mathrm{e}^t(x-t)\,\mathrm{d}t\,. \tag{A.4.6}$$

Since the exponential function is strictly increasing, it holds $\mathrm{e}^{\bar{x}} < \mathrm{e}^x$, hence, we deduce from (A.4.5) that the error due to approximating e^x by the polynomial $1 + x$ satisfies

$$0 < \mathrm{e}^x - (1+x) < \frac{1}{2}x^2\mathrm{e}^x\,. \tag{A.4.7}$$

On the other hand, if we look at the integral remainder, we easily check that the function $g(t) = \mathrm{e}^t(x-t)$ under the integral sign admits for $x \geq 1$ a strict maximum at $t = x - 1$, where it takes the value e^{x-1}. Hence,

$$0 < \int_0^x \mathrm{e}^t (x-t)\,\mathrm{d}t < \mathrm{e}^{x-1} \int_0^x \mathrm{d}t = \mathrm{e}^{x-1} x \,.$$

Therefore, we deduce from (A.4.6) that

$$0 < \mathrm{e}^x - (1+x) < \frac{1}{\mathrm{e}} x \mathrm{e}^x \,, \qquad x \geq 1 \,. \tag{A.4.8}$$

Since it is trivially seen that $\frac{1}{\mathrm{e}} x \mathrm{e}^x < \frac{1}{2} x^2 \mathrm{e}^x$ for $x \geq 1$, we conclude that (A.4.8) provides a more accurate estimate of the approximation error than (A.4.7) does. For instance, for $x = 1$ the error is

$$\mathrm{e}^1 - (1+1) = \mathrm{e} - 2 = 0.71828\ldots\,;$$

inequality (A.4.7) gives the upper bound $0.71828... < \frac{1}{2}\mathrm{e} = 1.35914...$, whereas (A.4.8) gives the bound $0.71828... < \frac{1}{\mathrm{e}}\mathrm{e} = 1$, which is sharper. □

A.5

Complements on integral calculus

We begin this appendix by checking the convergence of the two sequences that enter the definition of the Cauchy integral. We then consider the Riemann integral; we justify the integrability of relevant classes of functions, and we establish several properties of integrable functions and of the definite integral. We conclude by proving a few results concerning improper integrals.

A.5.1 The Cauchy integral

▶ Proof of Theorem 9.20, p. 320

Theorem 9.20 *The sequences $\{s_n\}$ and $\{S_n\}$ are convergent, and their limits coincide.*

Proof. We claim that for any $p \geq 1$

$$s_n \leq s_{pn}\,, \qquad S_{pn} \leq S_n\,.$$

In fact, subdividing the interval I_k in p subintervals I_{ki} $(1 \leq i \leq p)$ of equal width $\Delta x/p$, and letting

$$m_{ki} = \min_{x \in I_{ki}} f(x)\,,$$

it follows $m_k \leq m_{ki}$ for each i, hence

$$m_k \Delta x \leq \sum_{i=1}^{p} m_{ki} \frac{\Delta x}{p}\,.$$

Summing over k we obtain $s_n \leq s_{pn}$. The second inequality is similar.

Let now s_n, S_m be arbitrary sums. Since

$$s_n \leq s_{nm} \leq S_{nm} \leq S_m$$

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any lower sum is less or equal than any upper sum. Define

$$s = \sup_n s_n \qquad \text{and} \qquad S = \inf_n S_n .$$

We know $s \leq S_m$ holds for any m, so $s \leq S$. We wish to prove that $s = S$, and that such number is indeed the required limit. By the Heine-Cantor's Theorem A.3.7 the map f is uniformly continuous: given $\varepsilon > 0$, there is $\delta > 0$ such that if $x', x'' \in [a, b]$ then

$$|x' - x''| < \delta \qquad \text{implies} \qquad |f(x') - f(x'')| < \varepsilon .$$

Let n_ε be the integer such that $\frac{b-a}{n_\varepsilon} < \delta$. Take any $n \geq n_\varepsilon$; in each subinterval I_k of $[a, b]$ of width $\Delta x = \frac{b-a}{n}$ there exist points ξ_k and η_k such that

$$f(\xi_k) = m_k = \min_{x \in I_k} f(x) \qquad \text{and} \qquad f(\eta_k) = M_k = \max_{x \in I_k} f(x) .$$

As $|\eta_k - \xi_k| \leq \frac{b-a}{n} \leq \frac{b-a}{n_\varepsilon} < \delta$, it follows

$$M_k - m_k = f(\eta_k) - f(\xi_k) < \varepsilon .$$

Therefore

$$\begin{aligned} S_n - s_n &= \sum_{k=1}^{n} M_k \Delta x - \sum_{k=1}^{n} m_k \Delta x \\ &= \sum_{k=1}^{n} \big(M_k - m_k\big) \Delta x < \varepsilon \sum_{k=1}^{n} \Delta x = \varepsilon(b-a) . \end{aligned}$$

In other words, given $\varepsilon > 0$ there is an $n_\varepsilon > 0$ so that for all $n \geq n_\varepsilon$ we have $0 \leq S_n - s_n < \varepsilon(b-a)$. This implies

$$S - s \leq S_n - s_n < \varepsilon(b-a) .$$

Letting ε tend to 0, $S = s$ follows. In addition,

$$S - s_n \leq S_n - s_n < \varepsilon \qquad \text{if} \quad n \geq n_\varepsilon ,$$

that is,

$$\lim_{n \to \infty} s_n = S .$$

The same arguments may be adapted to show $\lim_{n \to \infty} S_n = S$. □

A.5.2 The Riemann integral

Throughout the section we shall repeatedly use the following result.

Lemma A.5.1 *Let f be a bounded function on $I = [a, b]$. Then f is integrable if and only if for any $\varepsilon > 0$ there exist two maps $h_\varepsilon \in \mathcal{S}_f^+$ and $g_\varepsilon \in \mathcal{S}_f^-$ such that*

$$\int_I h_\varepsilon - \int_I g_\varepsilon < \varepsilon \,.$$

Proof. According to the definition, f is integrable if and only if

$$\int_I f = \inf\left\{\int_I h : h \in \mathcal{S}_f^+\right\} = \sup\left\{\int_I g : g \in \mathcal{S}_f^-\right\} .$$

Let then f be integrable. Given $\varepsilon > 0$, by definition of lower and upper bound one can find a map $h_\varepsilon \in \mathcal{S}_f^+$ satisfying $\displaystyle\int_I h_\varepsilon - \int_I f < \varepsilon/2$ and, similarly, a function $g_\varepsilon \in \mathcal{S}_f^-$ such that $\displaystyle\int_I f - \int_I g_\varepsilon < \varepsilon/2$. Hence

$$\int_I h_\varepsilon - \int_I g_\varepsilon = \int_I h_\varepsilon - \int_I f + \int_I f - \int_I g_\varepsilon < \varepsilon \,.$$

Vice versa, using Definition 9.26 together with Property 9.27, one has

$$\int_I g_\varepsilon \le \underline{\int_I} f \le \overline{\int_I} f \le \int_I h_\varepsilon \,,$$

hence

$$\overline{\int_I} f - \underline{\int_I} f \le \int_I h_\varepsilon - \int_I g_\varepsilon < \varepsilon \,.$$

But ε is completely arbitrary, so $\displaystyle\underline{\int_I} f = \overline{\int_I} f$. In other words, f is integrable on $[a, b]$.

□

▶ Proof of Theorem 9.31, p. 327

Theorem 9.31 *Among the class of integrable maps on $[a, b]$ are*

a) continuous maps on $[a, b]$;
b) piecewise-continuous maps on $[a, b]$;
c) continuous maps on (a, b) which are bounded on $[a, b]$;
d) monotone functions on $[a, b]$.

Proof.
a) The Theorem of Weierstrass tells that f is bounded over $[a, b]$, and by Heine-Cantor's Theorem A.3.7 f is uniformly continuous on $[a, b]$. Thus for any given

$\varepsilon > 0$, there exists a $\delta > 0$ such that if $x', x'' \in [a,b]$ with $|x' - x''| < \delta$ then $|f(x') - f(x'')| < \varepsilon$. Let us consider a partition $\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that each interval $[x_{k-1}, x_k]$ has width $< \delta$ ($k = 1, \ldots, n$). We apply Weierstrass' Theorem 4.31 to each one of them: for every $k = 1, \ldots, n$, there are points $\xi_k, \eta_k \in [x_{k-1}, x_k]$ such that

$$f(\xi_k) = m_k = \min_{x \in [x_{k-1}, x_k]} f(x) \qquad \text{and} \qquad f(\eta_k) = M_k = \max_{x \in [x_{k-1}, x_k]} f(x)\,.$$

Since $|\eta_k - \xi_k| < \delta$,

$$M_k - m_k = f(\eta_k) - f(\xi_k) < \varepsilon\,.$$

Let $h_\varepsilon \in \mathcal{S}_f^+$ and $g_\varepsilon \in \mathcal{S}_f^-$ be defined by

$$h_\varepsilon(x) = \begin{cases} M_k & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a, \end{cases}$$

$$g_\varepsilon(x) = \begin{cases} m_k & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a. \end{cases}$$

For any $x \in [a,b]$ we have $h_\varepsilon(x) - g_\varepsilon(x) < \varepsilon$, hence

$$\int_I h_\varepsilon - \int_I g_\varepsilon = \int_I (h_\varepsilon - g_\varepsilon) < \int_I \varepsilon = (b-a)\varepsilon\,.$$

Given that ε is arbitrary, Lemma A.5.1 yields the result.

b) Call $\{x_1, x_2, \ldots, x_{n-1}\}$ the discontinuity points of f inside $[a,b]$, with $x_{k-1} < x_k$, and set $x_0 = a$ and $x_n = b$. For $k = 1, \ldots, n$, consider the continuous maps on $[x_{k-1}, x_k]$ defined as follows:

$$f_k(x) = \begin{cases} f(x) & \text{if } x \in (x_{k-1}, x_k), \\ \lim\limits_{x \to x_{k-1}^+} f(x) & \text{if } x = x_{k-1}, \\ \lim\limits_{x \to x_k^-} f(x) & \text{if } x = x_k\,. \end{cases}$$

Mimicking the proof of part *a)*, given $\varepsilon > 0$ there exist $h_{\varepsilon,k} \in \mathcal{S}_{f_k}^+$, $g_{\varepsilon,k} \in \mathcal{S}_{f_k}^-$ such that

$$h_{\varepsilon,k}(x) - g_{\varepsilon,k}(x) < \varepsilon\,, \qquad \forall x \in [x_{k-1}, x_k]\,.$$

Define $h_\varepsilon \in \mathcal{S}_f^+$ and $g_\varepsilon \in \mathcal{S}_f^-$ by

$$h_\varepsilon(x) = \begin{cases} h_{\varepsilon,k}(x) & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a, \end{cases}$$

$$g_\varepsilon(x) = \begin{cases} g_{\varepsilon,k}(x) & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a. \end{cases}$$

For any $x \in [a,b]$ then, $h_\varepsilon(x) - g_\varepsilon(x) < \varepsilon$; as before, Lemma A.5.1 ends the proof.

c) Fix $\varepsilon > 0$ so that $I_\varepsilon = [a+\varepsilon, b-\varepsilon] \subset [a,b]$. The map f is continuous on I_ε and we may find – as in part *a)* – two step functions defined on I_ε, say φ_ε and ψ_ε, such that

$$\varphi_\varepsilon(x) \le f(x) \le \psi_\varepsilon(x) \qquad \text{and} \qquad \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon\,, \quad \forall x \in I_\varepsilon\,.$$

Name $M = \sup\limits_{x\in I} f(x)$ and $m = \inf\limits_{x\in I} f(x)$ the supremum and infimum of f. Consider the step functions $h_\varepsilon \in \mathcal{S}_f^+$, $g_\varepsilon \in \mathcal{S}_f^-$ given by

$$h_\varepsilon(x) = \begin{cases} \psi_\varepsilon(x) & \text{if } x \in I_\varepsilon, \\ M & \text{if } x \notin I_\varepsilon, \end{cases} \qquad\qquad g_\varepsilon(x) = \begin{cases} \varphi_\varepsilon(x) & \text{if } x \in I_\varepsilon, \\ m & \text{if } x \notin I_\varepsilon. \end{cases}$$

Theorem 9.33 *i)* implies

$$\begin{aligned} \int_I h_\varepsilon - \int_I g_\varepsilon &= \int_{[a,a+\varepsilon]} (h_\varepsilon - g_\varepsilon) + \int_{I\varepsilon} (h_\varepsilon - g_\varepsilon) + \int_{[b-\varepsilon,b]} (h_\varepsilon - g_\varepsilon) \\ &= 2(M-m)\varepsilon + \int_{I\varepsilon} (h_\varepsilon - g_\varepsilon) \\ &< 2(M-m)\varepsilon + (b-a-2\varepsilon)\varepsilon < \big(2(M-m) + b - a\big)\varepsilon\,. \end{aligned}$$

Now Lemma A.5.1 allows to conclude.

d) Assume f is increasing. (In case f is decreasing, the proof is analogous.) Note first that f is bounded on $[a,b]$, for $f(a) \le f(x) \le f(b)$, $\forall x \in [a,b]$.

Given $\varepsilon > 0$, let n be a natural number such that $n > \dfrac{b-a}{n}$; split the interval into n parts, each $\dfrac{b-a}{n} < \varepsilon$ wide, and let $\{x_0, x_1, \ldots, x_n\}$ indicate the partition points. Introduce the step maps $h_n \in \mathcal{S}_f^+$, $g_n \in \mathcal{S}_f^-$ by

$$h_n(x) = \begin{cases} f(x_k) & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a, \end{cases}$$

$$g_n(x) = \begin{cases} f(x_{k-1}) & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a. \end{cases}$$

Then

$$\begin{aligned} \int_I h_n - \int_I g_n &= \sum_{k=1}^n f(x_k)(x_k - x_{k-1}) - \sum_{k=1}^n f(x_{k-1})(x_k - x_{k-1}) \\ &= \frac{b-a}{n} \sum_{k=1}^n \big(f(x_k) - f(x_{k-1})\big) = \frac{b-a}{n}\big(f(b) - f(a)\big) \\ &= \varepsilon\big(f(b) - f(a)\big)\,. \end{aligned}$$

Once again, the result follows from Lemma A.5.1. □

▶ Proof of Proposition 9.32, p. 328

Proposition 9.32 *If f is integrable on $[a,b]$, then*

i) *f is integrable on any subinterval $[c,d] \subset [a,b]$;*

ii) *$|f|$ is integrable on $[a,b]$.*

Proof.
i) If f is a step function the statement is immediate. More generally, let f be integrable over $[a,b]$; for $\varepsilon > 0$, Lemma A.5.1 yields maps $h_\varepsilon \in \mathcal{S}_f^+$, $g_\varepsilon \in \mathcal{S}_f^-$ such that

$$\int_a^b h_\varepsilon - \int_a^b g_\varepsilon = \int_a^b (h_\varepsilon - g_\varepsilon) < \varepsilon\,.$$

As

$$\int_c^d (h_\varepsilon - g_\varepsilon) \le \int_a^b (h_\varepsilon - g_\varepsilon) < \varepsilon\,,$$

the result is a consequence of Lemma A.5.1 applied to the function f restricted to $[c,d]$.

ii) Recall $|f| = f_+ + f_-$, where f_+ and f_- denote the positive and negative parts of f respectively. Thus it is enough to show that f_+ and f_- are integrable, for then we can use Theorem 9.33 *ii)*.

Let us prove f_+ is integrable. Given $\varepsilon > 0$, by Lemma A.5.1 there exist $h_\varepsilon \in \mathcal{S}_f^+$ and $g_\varepsilon \in \mathcal{S}_f^-$ such that $\int_a^b h_\varepsilon - \int_a^b g_\varepsilon < \varepsilon$. Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of $I = [a,b]$ adapted to both maps h_ε, g_ε. Consider the positive parts $h_{\varepsilon,+}$, $g_{\varepsilon,+}$ of the step functions. Having fixed an interval $I_k = [x_{k-1}, x_k]$, we may examine the three possible occurrencies $0 \le g_\varepsilon \le h_\varepsilon$, $g_\varepsilon \le 0 \le h_\varepsilon$ or $g_\varepsilon \le h_\varepsilon \le 0$. It is easy to check

$$g_{\varepsilon,+} \le f_+ \le h_{\varepsilon,+}$$

and

$$\int_{I_k} h_{\varepsilon,+} - \int_{I_k} g_{\varepsilon,+} \le \int_{I_k} h_\varepsilon - \int_{I_k} g_\varepsilon < \varepsilon\,.$$

Consequently, $h_{\varepsilon,+} \in \mathcal{S}_{f_+}^+$, $g_{\varepsilon,+} \in \mathcal{S}_{f_+}^-$, and

$$\int_I h_{\varepsilon,+} - \int_I g_{\varepsilon,+} = \sum_{k=1}^n \Big(\int_{I_k} h_{\varepsilon,+} - \int_{I_k} g_{\varepsilon,+} \Big) \le \sum_{k=1}^n \Big(\int_{I_k} h_\varepsilon - \int_{I_k} g_\varepsilon \Big) < \varepsilon\,.$$

Lemma A.5.1 yields then integrability for f_+.
A similar proof would tell that f_- is integrable as well. □

▶ Proof of Theorem 9.33, p. 329

Theorem 9.33 *Let f and g be integrable on a bounded interval I of the real line.*

i) (**Additivity with respect to the domain of integration**) *For any $a, b, c \in I$,*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

ii) (**Linearity**) *For any $a, b \in I$ and $\alpha, \beta \in \mathbb{R}$,*

$$\int_a^b \Big(\alpha f(x) + \beta g(x)\Big)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

iii) (**Positivity**) *Let $a, b \in I$, with $a < b$. If $f \geq 0$ on $[a, b]$ then*

$$\int_a^b f(x)\,\mathrm{d}x \geq 0.$$

If f is additionally continuous, equality holds if and only if f is the zero map.

iv) (**Monotonicity**) *Let $a, b \in I$, $a < b$. If $f \leq g$ in $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

v) (**Upper and lower bounds**) *Let $a, b \in I$, $a < b$. Then*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Proof. We shall directly prove statements *i)* -*v)* for generic integrable maps, for the case of step functions is fairly straightforward.

i) We shall suppose $a < c < b$, for the other instances descend from this and (9.18). By Proposition 9.32 *i)* f is integrable on the intervals $[a, b]$, $[a, c]$, $[c, b]$. Given $\varepsilon > 0$ moreover, let $g_\varepsilon \in \mathcal{S}_f^-$, $h_\varepsilon \in \mathcal{S}_f^+$ be such that

$$\int_a^b h_\varepsilon - \int_a^b g_\varepsilon < \varepsilon \qquad \text{and} \qquad \int_a^b g_\varepsilon \leq \int_a^b f \leq \int_a^b h_\varepsilon\,.$$

The property holds for step functions, so

$$\int_a^b g_\varepsilon = \int_a^c g_\varepsilon + \int_c^b g_\varepsilon \leq \int_a^c f + \int_c^b f \leq \int_a^c h_\varepsilon + \int_c^b h_\varepsilon = \int_a^b h_\varepsilon$$

and hence
$$\left|\int_a^b f - \int_a^c f - \int_c^b f\right| \le \int_a^b h_\varepsilon - \int_a^b g_\varepsilon < \varepsilon\,.$$
The claim follows because ε is arbitrary.

ii) We split the proof in two, and prove that

a) $\displaystyle\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$

b) $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\,.$

We start from a), and suppose $a<b$ for simplicity. When $\alpha = 0$ the result is clear, so let $\alpha > 0$. If $g \in \mathcal{S}_f^-$, $h \in \mathcal{S}_f^+$ then $\alpha g \in \mathcal{S}_{\alpha f}^-$ and $\alpha h \in \mathcal{S}_{\alpha f}^+$; thus
$$\begin{aligned}\alpha\int_a^b g(x)\,\mathrm{d}x = \int_a^b \alpha g(x)\,\mathrm{d}x &\le \underline{\int_a^b} \alpha f(x)\,\mathrm{d}x\\ &\le \overline{\int_a^b} \alpha f(x)\,\mathrm{d}x \le \int_a^b \alpha h(x)\,\mathrm{d}x = \alpha\int_a^b h(x)\,\mathrm{d}x\,.\end{aligned}$$

From $\displaystyle\alpha\int_a^b g(x)\,\mathrm{d}x \le \underline{\int_a^b} \alpha f(x)\,\mathrm{d}x$, taking the upper bound of the integrals $\int_a^b g$ as g varies in $\mathcal{S}_f^-$, and using the integrability of f on $[a,b]$, we obtain
$$\alpha\underline{\int_a^b} f(x)\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x \le \underline{\int_a^b}\alpha f(x)\,\mathrm{d}x\,;$$

similarly from $\displaystyle\overline{\int_a^b} \alpha f(x)\,\mathrm{d}x \le \alpha\int_a^b h(x)\,\mathrm{d}x$ we get
$$\overline{\int_a^b}\alpha f(x)\,\mathrm{d}x \le \alpha\overline{\int_a^b} f(x)\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x\,.$$

In conclusion,
$$\alpha\int_a^b f(x)\,\mathrm{d}x \le \underline{\int_a^b}\alpha f(x)\,\mathrm{d}x \le \overline{\int_a^b}\alpha f(x)\,\mathrm{d}x \le \alpha\int_a^b f(x)\,\mathrm{d}x$$

hence $\displaystyle\alpha\int_a^b f(x)\,\mathrm{d}x = \int_a^b \alpha f(x)\,\mathrm{d}x$.

When $\alpha < 0$, the proof is the same because $g \in \mathcal{S}_f^-$, $h \in \mathcal{S}_f^+$ satisfy $\alpha g \in \mathcal{S}_{\alpha f}^+$ and $\alpha h \in \mathcal{S}_{\alpha f}^-$.

Now part b). Take $f_1 \in \mathcal{S}_f^-$, $f_2 \in \mathcal{S}_f^+$, $g_1 \in \mathcal{S}_g^-$, $g_2 \in \mathcal{S}_g^+$; then $f_1 + g_1 \in \mathcal{S}_{f+g}^-$, $f_2 + g_2 \in \mathcal{S}_{f+g}^+$, and

$$\begin{aligned}\int_a^b f_1(x)\,\mathrm{d}x + \int_a^b g_1(x)\,\mathrm{d}x &= \int_a^b \big(f_1(x)+g_1(x)\big)\,\mathrm{d}x \le \underline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \\ &\le \overline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \le \int_a^b \big(f_2(x)+g_2(x)\big)\,\mathrm{d}x \\ &= \int_a^b f_2(x)\,\mathrm{d}x + \int_a^b g_2(x)\,\mathrm{d}x\,.\end{aligned}$$

Fix g_1, f_2 and g_2, and take the upper bound of the integrals $\displaystyle\int_a^b f_1(x)\,\mathrm{d}x$ as $f_1 \in \mathcal{S}_f^-$ varies:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x + \int_a^b g_1(x)\,\mathrm{d}x &\le \underline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \\ &\le \overline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \le \int_a^b f_2(x)\,\mathrm{d}x + \int_a^b g_2(x)\,\mathrm{d}x\,;\end{aligned}$$

varying g_1 in $\mathcal{S}_g^-$ and taking the upper bound of the integrals $\displaystyle\int_a^b g_1(x)\,\mathrm{d}x$ we find

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x &\le \underline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \\ &\le \overline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \le \int_a^b f_2(x)\,\mathrm{d}x + \int_a^b g_2(x)\,\mathrm{d}x\,.\end{aligned}$$

Now we may repeat the argument fixing g_2 and varying $f_2 \in \mathcal{S}_f^+$ first, then varying $g_2 \in \mathcal{S}_g^+$, to obtain

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x &\le \underline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \\ &\le \overline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\,.\end{aligned}$$

iii) The zero map g belongs in $\mathcal{S}_f^-$ (it is constant), hence

$$0 = \int_a^b g(x)\,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x\,.$$

Suppose f continuous; clearly $f(x) = 0$ forces $\displaystyle\int_a^b f(x)\,\mathrm{d}x = 0$. We shall prove the opposite implication: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = 0$ implies $f(x) = 0$. If, by contradiction, $f(\bar{x}) \ne 0$ for a certain $\bar{x} \in (a, b)$, Theorem A.2.2 would give a neighbourhood

$I_\delta(\bar{x}) = (\bar{x}-\delta, \bar{x}+\delta) \subset [a,b]$ and a constant $K_f > 0$, for any $x \in I_\delta(\bar{x})$. The step function

$$g(x) = \begin{cases} K_f & \text{if } x \in I_\delta(\bar{x}) \\ 0 & \text{if } x \notin I_\delta(\bar{x}) \end{cases}$$

would belong to $\mathcal{S}_f^-$, and

$$\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x = \delta K_f > 0\,,$$

a contradiction. Therefore $f(x) = 0$ for all $x \in (a,b)$, and by continuity f must vanish also at the end-points a, b.

iv) This follows directly from *iii)*, noting $h(x) = g(x) - f(x) \geq 0$.

v) Proposition 9.32 *ii)* says that $|f|$ is integrable over $[a,b]$. But $f = f_+ - f_-$ (f_+ and f_- are the positive and negative parts of f respectively), so the linearity proven in part *ii)* yields

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f_+(x)\,\mathrm{d}x - \int_a^b f_-(x)\,\mathrm{d}x\,.$$

Using the triangle inequality, property *iii)* ($f_+, f_- \geq 0$) and the relation $|f| = f_+ + f_-$, we eventually have

$$\begin{aligned} \left|\int_a^b f(x)\,\mathrm{d}x\right| &\leq \left|\int_a^b f_+(x)\,\mathrm{d}x\right| + \left|\int_a^b f_-(x)\,\mathrm{d}x\right| = \int_a^b f_+(x)\,\mathrm{d}x + \int_a^b f_-(x)\,\mathrm{d}x \\ &= \int_a^b \big(f_+(x) + f_-(x)\big)\,\mathrm{d}x = \int_a^b |f(x)|\,\mathrm{d}x\,. \end{aligned}$$

□

A.5.3 Improper integrals

▶ Check of property (10.3), p. 362

$$\int_1^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x \quad \text{converges, but} \quad \int_1^{+\infty} \left|\frac{\sin x}{x}\right|\,\mathrm{d}x \quad \text{diverges.}$$

Proof. We explain first why $\displaystyle\int_1^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$ converges. Let us integrate by parts over each interval $[1,a]$ with $a > 1$, by putting $f(x) = \frac{1}{x}$ and $g'(x) = \sin x$; since $f'(x) = -\frac{1}{x^2}$, $g(x) = -\cos x$, it follows

$$\int_1^a \frac{\sin x}{x}\,\mathrm{d}x = -\left.\frac{\cos x}{x}\right|_1^a - \int_1^a \frac{\cos x}{x^2}\,\mathrm{d}x\,;$$

the last integral is known to converge from Example 10.8. Thus the map $\dfrac{\sin x}{x}$ has a well-defined improper integral over $[1, +\infty)$.

Now let us convince ourselves that $\dfrac{\sin x}{x}$ is not absolutely integrable on $[1, +\infty)$. Since $|\sin x| \le 1$ for any x, we have

$$\left|\frac{\sin x}{x}\right| \ge \frac{\sin^2 x}{x} = \frac{1}{2}\frac{1-\cos 2x}{x}\,.$$

We claim the integral $\displaystyle\int_1^{+\infty} \frac{1-\cos 2x}{x}\,\mathrm{d}x$ diverges, hence the Comparison test (Theorem 10.5) forces $\displaystyle\int_1^{+\infty} \left|\frac{\sin x}{x}\right|\,\mathrm{d}x$ to diverge as well. In fact,

$$\int_1^{+\infty} \frac{1-\cos 2x}{x}\,\mathrm{d}x = \int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x - \int_1^{+\infty} \frac{\cos 2x}{x}\,\mathrm{d}x\,.$$

While the first integral on the right-hand side diverges, the second one converges, as can be proved by the same procedure as above. Therefore $\displaystyle\int_1^{+\infty} \frac{1-\cos 2x}{x}\,\mathrm{d}t$ diverges, and the function $\dfrac{\sin x}{x}$ cannot be absolutely integrable. □

▶ Proof of Theorem 10.10, p. 363

Theorem 10.10 (Asymptotic comparison test) *Suppose the function $f \in \mathcal{R}_{\text{loc}}([a, +\infty))$ is infinitesimal of order α, for $x \to +\infty$, with respect to $\varphi(x) = \frac{1}{x}$. Then*

i) if $\alpha > 1$, $f \in \mathcal{R}([a, +\infty))$;

ii) if $\alpha \le 1$, $\displaystyle\int_a^{+\infty} f(x)\,\mathrm{d}x$ diverges.

Proof. Since $f(x) \sim \frac{1}{x^\alpha}$ for $x \to +\infty$, we may assume the map f has constant sign for x sufficiently large, for instance when $x > A > 0$. Without loss of generality we may also take f strictly positive, for otherwise we could just change sign. Moreover, for $x \to +\infty$,

$$f(x) \sim \frac{1}{x^\alpha} \qquad \Rightarrow \qquad f(x) = O\Big(\frac{1}{x^\alpha}\Big) \quad \text{and} \quad \frac{1}{x^\alpha} = O\big(f(x)\big)\,;$$

otherwise said, there exist positive constants c_1, c_2 such that

$$\frac{c_1}{x^\alpha} \le f(x) \le \frac{c_2}{x^\alpha}\,, \qquad \forall x > A\,.$$

In order to conclude, it suffices to use the Comparison test (Theorem 10.5) jointly with Example 10.4. □

▶ Proof of Theorem 10.13, p. 364

Theorem 10.13 (Integral test) *Let f be continuous, positive and decreasing on $[k_0, +\infty)$, for $k_0 \in \mathbb{N}$. Then*

$$\sum_{k=k_0+1}^{\infty} f(k) \le \int_{k_0}^{+\infty} f(x)\,dx \le \sum_{k=k_0}^{\infty} f(k)\,.$$

Therefore the integral and the series share the same behaviour:

a) $\displaystyle\int_{k_0}^{+\infty} f(x)\,dx$ *converges* $\iff$ $\displaystyle\sum_{k=k_0}^{\infty} f(k)$ *converges;*

b) $\displaystyle\int_{k_0}^{+\infty} f(x)\,dx$ *diverges* $\iff$ $\displaystyle\sum_{k=k_0}^{\infty} f(k)$ *diverges.*

Proof. Since f decreases, for any $k \ge k_0$ we have

$$f(k+1) \le f(x) \le f(k)\,, \qquad \forall x \in [k, k+1]\,,$$

and as the integral is monotone,

$$f(k+1) \le \int_{k}^{k+1} f(x)\,dx \le f(k)\,.$$

Then for all $n \in \mathbb{N}$ with $n > k_0$ we obtain

$$\sum_{k=k_0+1}^{n+1} f(k) \le \int_{k_0}^{n} f(x)\,dx \le \sum_{k=k_0}^{n} f(k)$$

(after re-indexing the first series). Passing to the limit for $n \to +\infty$ and recalling f is positive and continuous, we conclude. □

Tables and Formulas

Recurrent formulas

$$\cos^2 x + \sin^2 x = 1, \qquad \forall x \in \mathbb{R}$$

$$\sin x = 0 \quad \text{se } x = k\pi, \quad \forall k \in \mathbb{Z}, \qquad \cos x = 0 \quad \text{se} \quad x = \frac{\pi}{2} + k\pi$$

$$\sin x = 1 \quad \text{se} \quad x = \frac{\pi}{2} + 2k\pi, \qquad \cos x = 1 \quad \text{se} \quad x = 2k\pi$$

$$\sin x = -1 \quad \text{se} \quad x = -\frac{\pi}{2} + 2k\pi, \qquad \cos x = -1 \quad \text{se} \quad x = \pi + 2k\pi$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin 2x = 2\sin x\cos x\,, \qquad \cos 2x = 2\cos^2 x - 1$$

$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x-y}{2}\sin\frac{x+y}{2}$$

$$\sin(x+\pi) = -\sin x\,, \qquad \cos(x+\pi) = -\cos x$$

$$\sin(x+\frac{\pi}{2}) = \cos x\,, \qquad \cos(x+\tfrac{\pi}{2}) = -\sin x$$

$$a^{x+y} = a^x a^y\,, \qquad a^{x-y} = \frac{a^x}{a^y}\,, \qquad (a^x)^y = a^{xy}$$

$$\log_a(xy) = \log_a x + \log_a y\,, \quad \forall x, y > 0$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y, \quad \forall x, y > 0$$

$$\log_a(x^y) = y\log_a x, \quad \forall x > 0\,,\ \forall y \in \mathbb{R}$$

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Fundamental limits

$$\lim_{x\to+\infty} x^\alpha = +\infty\,, \qquad \lim_{x\to 0^+} x^\alpha = 0\,, \qquad \alpha > 0$$

$$\lim_{x\to+\infty} x^\alpha = 0\,, \qquad \lim_{x\to 0^+} x^\alpha = +\infty\,, \qquad \alpha < 0$$

$$\lim_{x\to\pm\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$$

$$\lim_{x\to+\infty} a^x = +\infty\,, \qquad \lim_{x\to-\infty} a^x = 0\,, \qquad a > 1$$

$$\lim_{x\to+\infty} a^x = 0\,, \qquad \lim_{x\to-\infty} a^x = +\infty\,, \qquad a < 1$$

$$\lim_{x\to+\infty} \log_a x = +\infty\,, \qquad \lim_{x\to 0^+} \log_a x = -\infty\,, \qquad a > 1$$

$$\lim_{x\to+\infty} \log_a x = -\infty\,, \qquad \lim_{x\to 0^+} \log_a x = +\infty\,, \qquad a < 1$$

$$\lim_{x\to\pm\infty} \sin x\,, \qquad \lim_{x\to\pm\infty} \cos x\,, \qquad \lim_{x\to\pm\infty} \tan x \qquad \text{do not exist}$$

$$\lim_{x\to\left(\frac{\pi}{2}+k\pi\right)^\pm} \tan x = \mp\infty\,,\ \forall k \in \mathbb{Z}\,, \quad \lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$$

$$\lim_{x\to\pm1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm1)$$

$$\lim_{x\to+1} \arccos x = 0 = \arccos 1\,, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,, \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to\pm\infty} \left(1+\frac{a}{x}\right)^x = e^a\,, \quad a\in\mathbb{R}\,, \quad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log a}\,,\ a>0; \quad \text{in particular,} \quad \lim_{x\to 0}\frac{\log(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{a^x-1}{x} = \log a\,, \quad a>0; \qquad \text{in particular,} \quad \lim_{x\to 0}\frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha\,, \quad \alpha\in\mathbb{R}$$

Derivatives of elementary functions

$f(x)$	$f'(x)$
x^α	$\alpha x^{\alpha-1}\,, \qquad \forall \alpha \in \mathbb{R}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1+\tan^2 x = \dfrac{1}{\cos^2 x}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
a^x	$(\log a)\, a^x$
$\log_a \lvert x \rvert$	$\dfrac{1}{(\log a)\, x}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

Differentiation rules

$$\big(\alpha f(x) + \beta g(x)\big)' = \alpha f'(x) + \beta g'(x)$$

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

$$\big(g(f(x))\big)' = g'(f(x))f'(x)$$

Maclaurin's expansions

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^k}{k!} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots + \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots + \left|\binom{-\frac{1}{2}}{m}\right| \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

Integrals of elementary functions

$f(x)$	$\int f(x)\,\mathrm{d}x$
x^α	$\dfrac{x^{\alpha+1}}{\alpha+1}+c\,, \qquad \alpha \neq -1$
$\dfrac{1}{x}$	$\log\|x\|+c$
$\sin x$	$-\cos x+c$
$\cos x$	$\sin x+c$
e^x	e^x+c
$\sinh x$	$\cosh x+c$
$\cosh x$	$\sinh x+c$
$\dfrac{1}{1+x^2}$	$\arctan x+c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x+c$
$\dfrac{1}{\sqrt{1+x^2}}$	$\log(x+\sqrt{x^2+1})+c=\operatorname{sett}\sinh x+c$
$\dfrac{1}{\sqrt{x^2-1}}$	$\log(x+\sqrt{x^2-1})+c=\operatorname{sett}\cosh x+c$

Integration rules

$$\int \big(\alpha f(x)+\beta g(x)\big)\,\mathrm{d}x=\alpha\int f(x)\,\mathrm{d}x+\beta\int g(x)\,\mathrm{d}x$$

$$\int f(x)g'(x)\,\mathrm{d}x=f(x)g(x)-\int f'(x)g(x)\,\mathrm{d}x$$

$$\int \frac{\varphi'(x)}{\varphi(x)}\,\mathrm{d}x=\log|\varphi(x)|+c$$

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x=\int f(y)\,\mathrm{d}y \quad \text{where} \quad y=\varphi(x)$$

Index

C. Canuto, A. Tabacco: *Mathematical Analysis I,* 2nd Ed.,
UNITEXT – La Matematica per il 3+2 84, DOI 10.1007/978-3-319-12772-9,

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