A.1 Orthogonal Coordinate Systems

A.1.1 Cartesian (Rectangular) Coordinate System

The unit vectors are denoted by \( \mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{\hat{z}} \) in the Cartesian system. By convention, \((\mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{\hat{z}})\) triplet form a right-handed system and obey the following cyclic relations

\[
\mathbf{\hat{x}} \times \mathbf{\hat{y}} = \mathbf{\hat{z}}, \quad \mathbf{\hat{z}} \times \mathbf{\hat{x}} = \mathbf{\hat{y}}, \quad \mathbf{\hat{y}} \times \mathbf{\hat{z}} = \mathbf{\hat{x}}
\] (A.1)

Differential length is defined as

\[
d\ell = dx \mathbf{\hat{x}} + dy \mathbf{\hat{y}} + dz \mathbf{\hat{z}}
\] (A.2)

Differential surface areas with normal vectors \( \mathbf{\hat{x}}, \mathbf{\hat{y}}, \) and \( \mathbf{\hat{z}} \) are respectively defined as

\[
\begin{align*}
ds_x &= dy \, dz \, \mathbf{\hat{x}} \\
ds_y &= dx \, dz \, \mathbf{\hat{y}} \\
ds_z &= dx \, dy \, \mathbf{\hat{z}}
\end{align*}
\] (A.3)

The differential volume element is

\[
dv = dx \, dy \, dz
\] (A.4)

A.1.2 Cylindrical Coordinate System

The unit vectors are denoted by \( \mathbf{\hat{\rho}}, \mathbf{\hat{\phi}}, \mathbf{\hat{z}} \) in the cylindrical system. By convention, \((\mathbf{\hat{\rho}}, \mathbf{\hat{\phi}}, \mathbf{\hat{z}})\) triplet form a right-handed system and obey the following cyclic relations
\[ \hat{\rho} \times \hat{\phi} = \hat{z}, \quad \hat{z} \times \hat{\rho} = \hat{\phi}, \quad \hat{\phi} \times \hat{z} = \hat{\rho} \quad (A.5) \]

Differential length is defined as
\[ d\ell = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{z} \quad (A.6) \]

Differential surface areas with normal vectors \( \hat{\rho}, \hat{\phi}, \) and \( \hat{z} \) are respectively defined as
\[
\begin{align*}
    ds_\rho &= \rho d\phi dz \hat{\rho} \\
    ds_\phi &= d\rho dz \hat{\phi} \\
    ds_z &= \rho d\rho d\phi \hat{z}
\end{align*} \quad (A.7)
\]

The differential volume element is
\[ dv = \rho d\rho d\phi dz \quad (A.8) \]

### A.1.3 Spherical Coordinate System

The unit vectors are denoted by \( \hat{r}, \hat{\theta}, \hat{\phi} \) in the spherical system. By convention, \((\hat{r}, \hat{\theta}, \hat{\phi})\) triplet form a right-handed system and obey the following cyclic relations
\[ \hat{r} \times \hat{\theta} = \hat{\phi}, \quad \hat{\phi} \times \hat{r} = \hat{\theta}, \quad \hat{\theta} \times \hat{\phi} = \hat{r} \quad (A.9) \]

Differential length is defined as
\[ d\ell = dr \hat{r} + r \sin \theta d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi} \quad (A.10) \]

Differential surface areas with normal vectors \( \hat{r}, \hat{\theta}, \) and \( \hat{\phi} \) are respectively defined as
\[
\begin{align*}
    ds_r &= r^2 \sin \theta d\theta d\phi \hat{r} \\
    ds_\theta &= r \sin \theta dr d\phi \hat{\theta} \\
    ds_\phi &= r dr d\theta \hat{\phi}
\end{align*} \quad (A.11)
\]

The differential volume element is
\[ dv = r^2 \sin \theta dr d\theta d\phi \quad (A.12) \]
A.2 Coordinate Transformations

Consider a point \( P \) whose coordinates in Cartesian, cylindrical, and spherical coordinate systems, respectively are \((x, y, z)\), \((\rho, \phi, z)\), and \((r, \theta, \phi)\). The Cartesian and cylindrical coordinates are related by

\[
\begin{align*}
x &= \rho \cos \phi \\
y &= \rho \sin \phi
\end{align*}
\]  
(A.13)

The Cartesian and spherical coordinates are related by

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta 
\end{align*}
\]  
(A.14)

The spherical and cylindrical coordinates are related by

\[
\begin{align*}
\rho &= r \sin \theta \\
z &= r \cos \theta
\end{align*}
\]  
(A.15)

A.3 Vector Transformations

Consider a three-dimensional vector \( \mathbf{F} \). We can express \( \mathbf{F} \) in terms of the unit vectors in Cartesian, cylindrical, and spherical systems respectively as:

\[
\begin{align*}
\mathbf{F} &= F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \\
\mathbf{F} &= F_\rho \hat{\rho} + F_\phi \hat{\phi} + F_z \hat{z} \\
\mathbf{F} &= F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi}
\end{align*}
\]  
(A.16)

A.3.1 Cartesian-Cylindrical Vector Transformations

Cartesian to cylindrical components are related by

\[
\begin{align*}
F_\rho &= F_x \cos \phi + F_y \sin \phi \\
F_\phi &= -F_x \sin \phi + F_y \cos \phi \\
F_z &= F_z
\end{align*}
\]  
(A.17)
Cylindrical to Cartesian components are related by
\[
\begin{align*}
F_x &= F_\rho \cos \phi - F_\phi \sin \phi \\
F_y &= F_\rho \sin \phi + F_\phi \cos \phi \\
F_z &= F_z
\end{align*}
\] (A.18)

\section*{A.3.2 Cartesian-Spherical Vector Transformations}

Cartesian to spherical components are related by
\[
\begin{align*}
F_r &= F_x \sin \theta \cos \phi + F_y \sin \theta \sin \phi + F_z \cos \theta \\
F_\theta &= F_x \cos \theta \cos \phi + F_y \cos \theta \sin \phi - F_z \sin \theta \\
F_\phi &= -F_x \sin \phi + F_y \cos \phi
\end{align*}
\] (A.19)

Spherical to Cartesian components are related by
\[
\begin{align*}
F_x &= F_r \sin \theta \cos \phi + F_\theta \cos \theta \cos \phi - F_\phi \sin \phi \\
F_y &= F_r \sin \theta \sin \phi + F_\theta \cos \theta \sin \phi - F_\phi \cos \phi \\
F_z &= F_r \cos \theta - F_\theta \sin \theta
\end{align*}
\] (A.20)

\section*{A.3.3 Cylindrical-Spherical Vector Transformations}

Cylindrical to spherical components are related by
\[
\begin{align*}
F_r &= F_\rho \sin \theta + F_z \cos \theta \\
F_\theta &= F_\rho \cos \theta - F_z \sin \theta \\
F_\phi &= F_\phi
\end{align*}
\] (A.21)

Spherical to cylindrical components are related by
\[
\begin{align*}
F_\rho &= F_r \sin \theta + F_\theta \cos \theta \\
F_\phi &= F_\phi \\
F_z &= F_r \cos \theta - F_\theta \sin \theta
\end{align*}
\] (A.22)
Appendix B
Vector Calculus

In the following, scalar functions are denoted by lower-case letters and vector functions are denoted by upper-case bold letters. The notations for vector components are same as shown in (A.17).

B.1 Differential Operators

B.1.1 Cartesian System

$$\nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \quad (B.1)$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (B.2)$$

$$\nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} \quad (B.3)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (B.4)$$
B.1.2 Cylindrical System

\[
\nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}
\]  \hspace{0.5cm} (B.5)

\[
\nabla \cdot \mathbf{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}
\]  \hspace{0.5cm} (B.6)

\[
\nabla \times \mathbf{F} = \left( \frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) \hat{\rho} + \left( \frac{\partial F_\rho}{\partial \phi} - \frac{\partial F_z}{\partial \rho} \right) \hat{\phi} + \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho F_\phi) - \frac{\partial F_\rho}{\partial \phi} \right) \hat{z}
\]  \hspace{0.5cm} (B.7)

\[
\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}
\]  \hspace{0.5cm} (B.8)

B.1.3 Spherical System

\[
\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}
\]  \hspace{0.5cm} (B.9)

\[
\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 F_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( F_\theta \sin \theta \right) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi}
\]  \hspace{0.5cm} (B.10)

\[
\nabla \times \mathbf{F} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( F_\phi \sin \theta \right) - \frac{\partial F_\theta}{\partial \phi} \right] \hat{r}
\]  \hspace{0.5cm} \hspace{1cm} + \left[ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} \left( r F_\phi \right) \right] \hat{\theta}
\]  \hspace{1cm} + \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r F_\theta \right) - \frac{\partial r F_r}{\partial \theta} \right] \hat{\phi}
\]  \hspace{0.5cm} (B.11)
Appendix B: Vector Calculus

\[ \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \] (B.12)

B.2 Differentiation

\[ \nabla \cdot (f \mathbf{F}) = f \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f \] (B.13)

\[ \nabla \cdot (\mathbf{E} \times \mathbf{F}) = \mathbf{F} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{F} \] (B.14)

\[ \nabla \times (f \mathbf{F}) = \nabla f \times \mathbf{F} + f \nabla \times \mathbf{F} \] (B.15)

\[ \nabla \times (\mathbf{E} \times \mathbf{F}) = \mathbf{E} \nabla \cdot \mathbf{F} - \mathbf{F} \nabla \cdot \mathbf{E} + (\mathbf{F} \cdot \nabla) \mathbf{E} - (\mathbf{E} \cdot \nabla) \mathbf{F} \] (B.16)

\[ \nabla (\mathbf{E} \cdot \mathbf{F}) = (\mathbf{E} \cdot \nabla) \mathbf{F} + (\mathbf{F} \cdot \nabla) \mathbf{E} + \mathbf{E} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{E}) \] (B.17)

\[ \nabla \times \nabla \times \mathbf{F} = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \] (B.18)

\[ \nabla \cdot (\nabla \times \mathbf{F}) = 0 \] (B.19)

\[ \nabla \times \nabla f = 0 \] (B.20)

B.3 Integration Theorems

In the following \( V \) is a three-dimensional volume with differential volume element \( dv \), \( S \) is a closed two-dimensional surface enclosing \( V \) with differential surface
element $ds$ and outward unit normal vector $\mathbf{n}$. An open surface is denoted by $\Omega$ and its bounding contour by $C$.

$$\int_V \nabla \cdot \mathbf{F} \, dv = \oint_S \mathbf{F} \cdot \mathbf{n} \, ds \quad \text{(Divergence Theorem)} \quad (B.21)$$

$$\int_V \nabla f \, dv = \oint_S f \mathbf{n} \, ds \quad (B.22)$$

$$\int_V \nabla \times \mathbf{F} \, dv = \oint_S \mathbf{n} \times \mathbf{F} \, ds \quad (B.23)$$

$$\int_V \left( f \nabla^2 g - g \nabla^2 f \right) \, dv = \oint_S \left( f \nabla g - g \nabla f \right) \cdot \mathbf{n} \, ds \quad (B.24)$$

$$\int_{\Omega} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, ds = \oint_C \mathbf{F} \cdot d\ell \quad \text{(Stoke’s Theorem)} \quad (B.25)$$

$$\int_{\Omega} \mathbf{n} \times \nabla f \, ds = \oint_C f \, d\ell \quad (B.26)$$
Appendix C
Bessel Functions

C.1 Gamma Function

In the 18th century, Swiss mathematician Leonhard Euler concerned himself with the problem of interpolating the factorial function \( n! = n(n - 1)(n - 2) \ldots 3 \cdot 2 \cdot 1 \) between non-integer values. This problem eventually led him to gamma function defined as

\[
\Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} \, dt \quad (C.1)
\]

The integral in (C.1) is convergent for all complex numbers except negative integers and zero. It can be shown that for positive integers

\[
\Gamma(n) = (n - 1)!
\]

Two useful properties of the gamma function are

\[
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \quad \text{(Reflection Formula)} \quad (C.3)
\]

\[
2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2}) = \sqrt{\pi}\Gamma(2z) \quad \text{(Duplication Formula)} \quad (C.4)
\]

It is important to note that the reflection formula in (C.3) is only valid for non-integer values of \( z \).
C.2 Bessel Functions

Bessel functions are the solutions of Bessel’s equation

\[
\frac{d^2 y}{dz^2} + \frac{1}{z} \frac{dy}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) y = 0 \tag{C.5}
\]

where \(\nu\) which may be complex is called the order of the equation and the solution.

C.2.1 Bessel Functions of the First Kind \(J_\nu(z)\)

The series representation for the Bessel function of the first kind of order \(\nu\) is

\[
J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k + \nu + 1)} \left(\frac{z}{2}\right)^{2k+\nu} \tag{C.6}
\]

It can be easily verified that \(J_\nu(z)\) satisfies the differential equation in (C.5). Bessel functions of integer order are of particular interest in electromagnetic problems. For integer values of \(\nu\) the expression in (C.6) is simplified to

\[
J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + n)!} \left(\frac{z}{2}\right)^{2k+n} , \quad n = 0, 1, 2, \ldots \tag{C.7}
\]

Furthermore it can be shown that for \(n = 0, 1, 2, \ldots\)

\[
J_{-n}(z) = (-1)^n J_n(z) \tag{C.8}
\]

Figure C.1 shows the plot for Bessel functions of the first kind of orders 0, 1, 2, and 3. As it can be seen from (C.6) the analytical expression for Bessel functions is rather complicated. In many electromagnetic boundary value problems, depending on the application, only Bessel functions of small arguments (for near-field analysis) and large arguments (for far-field analysis) are of interest. In these cases it is much more convenient to use concise asymptotic expressions which have been derived for Bessel functions. It can be shown that

\[
\lim_{z \to 0^+} J_n(z) \simeq \begin{cases} 
1 & \text{if } n = 0 \\
\left(\frac{z}{n!}\right)^2 & \text{if } n = 1, 2, 3, \ldots
\end{cases} \tag{C.9}
\]
and

\[
\lim_{z \to \infty} J_n(z) \simeq \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{\pi}{4} - \frac{n\pi}{2} \right)
\]  \hspace{1cm} (C.10)

Figure C.2 compares the values for \( J_1(z) \) and \( J_2(z) \) with their large argument asymptotic expression from (C.10) which have been denoted by \( \tilde{J}_1(z) \) and \( \tilde{J}_2(z) \) respectively.
Generating Function

The expression

\[ w(z, t) = e^{\frac{z}{2}(t^{-1}/t)}, \quad 0 < |t| < \infty \]  \hspace{1cm} (C.11)\]

is called the generating function for \( J_n(z) \). It can be shown that

\[ w(z, t) = e^{\frac{z}{2}(t^{-1}/t)} = \sum_{n=-\infty}^{\infty} J_n(z)t^n \]  \hspace{1cm} (C.12)\]

The Bessel generating function has many useful applications in electromagnetic boundary value problems. For example setting \( t = je^{j\phi} \) in (C.12) gives us the expansion of a plane wave in terms of an infinite sum of cylindrical waves

\[ e^{jz\cos \phi} = \sum_{n=-\infty}^{\infty} j^n J_n(z)e^{jn\phi} \]  \hspace{1cm} (C.13)\]

\section*{C.2.2 Bessel Functions of the Second Kind (Neumann Functions) \( N_\nu(z) \)}

Bessel functions of the second kind, also known as the Neumann functions are defined by the formula

\[ N_\nu(z) = \frac{\cos(\nu\pi) J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)} \]  \hspace{1cm} (C.14)\]

By inspection it can be verified that \( N_\nu(z) \) satisfies the differential equation in (C.5). For integer values of \( \nu \) the expression in (C.14) is an indeterminate form and has to be evaluated at the limit

\[ N_n(z) = \lim_{\nu \to n} N_\nu(z), \quad n = 1, 2, 3, \ldots \]  \hspace{1cm} (C.15)\]

The limit in (C.15) can be evaluated using l’Hospital’s rule, leading to

\[ N_n(z) = \frac{2}{\pi} J_n(z) \ln \left( \frac{z}{2} \right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)! \left( \frac{2k-n}{z^2} \right)}{k!} \]

\[ - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{2k+n}{z^2} \right) [\psi(k+n+1) + \psi(k+1)]}{k!(n+k)!} \]  \hspace{1cm} (C.16)\]
where $\psi(z)$ is the digamma function defined as

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$ (C.17)

Similar to Bessel functions of the first kind, it can be shown that for $n = 0, 1, 2, \ldots$

$$N_n(z) = (-1)^n N_n(z)$$ (C.18)

Figure C.3 shows the plot for Neumann functions of orders 0, 1, 2, and 3. As it can be seen from the plot, $N_n$ contains a singularity at zero due to the logarithm term in (C.16). The asymptotic expressions for $N_n(z)$ are

$$\lim_{z \to 0^+} N_n(z) \simeq \begin{cases} \frac{2}{\pi} \ln \left( \frac{\gamma z}{2} \right) & \text{if } n = 0 \\ -\frac{(n-1)!}{\pi} \left( \frac{2}{z} \right)^n & \text{if } n = 1, 2, 3, \ldots \end{cases}$$ (C.19)

where $\gamma = 1.78107 \ldots$ is Euler’s constant and

$$\lim_{z \to \infty} N_n(z) \simeq \sqrt{\frac{2}{\pi z}} \sin \left( z - \frac{\pi}{4} - \frac{n\pi}{2} \right)$$ (C.20)

Figure C.4 compares the values for $N_1(z)$ and $N_2(z)$ with their large argument asymptotic expression from (C.20) which have been denoted by $\tilde{N}_1(z)$ and $\tilde{N}_2(z)$ respectively.
Fig. C.4 Large argument asymptotic expressions for $N_1(z)$ and $N_2(z)$

### C.2.3 Bessel Functions of the Third Kind (Hankel Functions)

Bessel functions of the first kind and Neumann functions are the orthogonal eigenfunctions of the Bessel equation, hence any linear combination of them is also a solution to (C.5). Two particularly important such combinations are the Hankel function of the first kind of order $n$ defined as

$$H_n^{(1)}(z) = J_n(z) + j N_n(z)$$  \hspace{1cm} (C.21)

and the Hankel function of the second kind of order $n$ defined as

$$H_n^{(2)}(z) = J_n(z) - j N_n(z)$$  \hspace{1cm} (C.22)

Hankel functions are often encountered in electromagnetic radiation problems. For large arguments, the asymptotic expressions for Hankel functions are

$$\lim_{z \to \infty} H_n^{(1)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp \left[ j \left( z - \frac{\pi}{4} - \frac{n\pi}{2} \right) \right]$$  \hspace{1cm} (C.23)

$$\lim_{z \to \infty} H_n^{(2)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp \left[ -j \left( z - \frac{\pi}{4} - \frac{n\pi}{2} \right) \right]$$  \hspace{1cm} (C.24)
C.2.4 Formulas for Bessel Functions

In the following $Z_n(z)$ represents $J_n(z)$, $N_n(z)$, $H_n^{(1)}(z)$, $H_n^{(2)}(z)$ or any linear combinations of these functions.

**Recurrence and Differentiation Formulas**

\[ 2Z'_n(z) = Z_{n-1}(z) - Z_{n+1}(z) \]  
(C.25)

\[ \frac{2n}{z} Z_n(z) = Z_{n-1}(z) + Z_{n+1}(z) \]  
(C.26)

\[ J_n(z)N'_n(z) - N_n(z)J'_n(z) = \frac{2}{z\pi} \]  
(C.27)

\[ J_n(z)J'_{-n}(z) - J_{-n}(z)J'_n(z) = -\frac{2}{z\pi} \sin(n\pi) \]  
(C.28)

**Integrals**

\[ \int z^{n+1} Z_n(z) \, dz = z^{n+1} Z_{n+1}(z) \]  
(C.29)

\[ \int z^{1-n} Z_n(z) \, dz = -z^{1-n} Z_{n-1}(z) \]  
(C.30)

\[ \int z Z_n^2(kz) \, dz = \frac{z^2}{2} \left[ Z_n^2(kz) - Z_{n-1}(kz)Z_{n+1}(kz) \right] \]  
(C.31)

\[ J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(n\phi - z \sin \phi) \, d\phi \]  
(C.32)