Solutions to Exercises

Exercises of Chapter 2

1. (a) Conjunctive interpretation.
   (b) Disjunctive interpretation.
   (c) Conjunctive interpretation.
   (d) Disjunctive interpretation.

2. (a) The proportion of students that speak 3 or more languages is between 0.1 (at least, student no. 1 satisfies this condition) and 0.9 (student no. 3 only speaks two different languages.) Those proportions coincide with the lower and upper probabilities of the set $[3, \infty)$. In fact:

$$P_*([3,\infty)) = P(\{s_i \in \Omega : \Gamma(s_i) \subseteq [3,\infty), \Gamma(s_i) \neq \emptyset\}) = P(\{s_1\}) = 0.1,$$
$$P^*([3,\infty)) = P(\{s_i \in \Omega : \Gamma(s_i) \cap [3,\infty) \neq \emptyset\}) = P(\Omega \setminus \{s_3\}) = 0.9.$$

   (b) The minimum value of the expectation is $\min E(\Gamma) = 0.1 \cdot 4 + 0.9 \cdot 2 = 2.2$. The maximum is $\max E(\Gamma) = 0.1 \cdot 2 + 0.1 \cdot 3 + 0.8 \cdot 6 = 5.3$. The expected number of languages spoken is a number bounded by those numbers.
   (c) Let us consider the random variables $X = \min \Gamma$ and $Y = \max \Gamma$. Their respective expectations 2.2 and 5.3 (they do coincide with the minimum and the maximum of the expectation of Γ). Their respective variances are, therefore:

$$\mathrm{Var}(X) = 0.9 \cdot (2 - 2.2)^2 + 0.1 \cdot (4 - 2.2)^2 = 0.36, \text{ and}$$
$$\mathrm{Var}(Y) = 0.1 \cdot (2 - 5.3)^2 + 0.1 \cdot (3 - 5.2)^2 + 0.8(6 - 5.2)^2 = 2.01.$$

   Their half sum is $\frac{\mathrm{Var}(X)+\mathrm{Var}(Y)}{2} = \frac{0.36+2.01}{2} = 1.185$ and it measures the dispersion of the images of Γ. It belongs to the parametrized family of scalar variances defined by Lubiano Reference 34 of Chap. 2.
   (d) All we know about the actual variance of the number of languages spoken is that it belongs to the set of variances of the measurable selections of Γ. The minimum of this set coincides with the variance of the selection Z_1 defined as follows:

I. Couso et al., *Random Sets and Random Fuzzy Sets as Ill-Perceived Random Variables*,
SpringerBriefs in Computational Intelligence, DOI: 10.1007/978-3-319-08611-8

$$Z_1(s_1) = 4,\ Z_1(s_2) = 3,\ Z_1(s_3) = 2,\ Z_1(s_i) = 3,\ i = 4, \ldots, 10,$$

whose variance is $\text{Var}(Z_1) = 0.2$. It is strictly lower than the variances of X and Y.
On the other hand, the maximum of this set is the variance of one of the "most dispersed" measurable selection of Γ. There are several selections with the maximum variance. One of them is the random variable Z defined as follows:

$$Z_2(s_1) = 6,\ Z_2(s_2) = Z_2(s_3) = Z_2(s_4) = Z_2(s_5) = Z(6) = 2,\ Z_2(s_i) = 2,\\ i = 7, \ldots, 10,$$

whose variance is $\text{Var}(Z_2) = 4$. It is strictly greater than the variances of X and Y and, therefore, also greater than their half sum. According to this example, the minimum value of the variances of the selections of a random interval can be strictly lower than the variances of both extremes, as well as the maximum variance can be strictly higher than the maximum of the variances of both extremes. In general, the pair of variances of the extremes do not determine any of the bounds for the set of possible values of the actual variance.

3. Let us consider an arbitrary Borel measurable set $A \in \beta_{\mathbb{R}}$. Let us consider the dense subset of A, $D = \mathbb{Q} \cap A$. Let us consider the multi-valued mapping $\text{Int}(\Gamma) : \Omega \to \wp(\mathbb{R})$ that assigns, to each $\omega \in \Omega$ the interior of $\Gamma(\omega)$, i.e., the open interval $\text{Int}(\Gamma)(\omega) = (T_n(\omega), T_x(\omega))$. The upper inverse of A can be expressed as follows:

$$\Gamma^*(A) = \{\omega \in \Omega : \text{Int}(\Gamma)(\omega) \cap A \neq \emptyset\} \cup T_n^{-1}(A) \cup T_x^{-1}(A).$$

Furthermore, we can easily check that $\text{Int}(\Gamma)(\omega) \cap A \neq \emptyset$ if and only if $\text{Int}(\Gamma)(\omega) \cap D \neq \emptyset$, for every $\omega \in \Omega$. Therefore, the upper inverse of the open random interval can be expressed as follows:

$$\begin{aligned}\text{Int}(\Gamma)^*(A) &= \text{Int}(\Gamma)^*(D) = \cup_{d\in D}\text{Int}(\Gamma)^*(\{d\})\\ &= \cup_{d\in D}\left[T_n^{-1}(-\infty, d) \cap T_x^{-1}((d, \infty)\right].\end{aligned}$$

Therefore, $\Gamma^*(A) = (\cup_{d\in D}[T_n^{-1}(-\infty, d) \cap T_x^{-1}((d, \infty)]) \cup T_n^{-1}(A) \cup T_x^{-1}(A)$. Thus, we can express the upper probability of A in terms of the joint probability, as the probability that the 2-dimensional random vector takes values in the set:

$$(\cup_{d\in D}[(-\infty, d) \times (d, \infty)]) \cup (A \times \mathbb{R}) \cup (\mathbb{R} \times A).$$

4. Let us first assume that Γ is strongly measurable. The anti-image of any singleton $\{B\}$, for an arbitrary $B \in \wp(U)$, can be expressed in terms of the upper images as follows:

$$\Gamma^{-1}(\{B\}) = \cap_{C : C\subseteq B, C\neq\emptyset}\Gamma^*(C) \cap [\Gamma^*(B^c)]^c.$$

Conversely, let us now suppose that the set $\Gamma^{-1}(\{B\})$ is measurable for every $B \in \wp(U)$. Then, the upper inverse of any subset $A \in \wp(U)$ can be written as a finite union of sets of that form, $\Gamma^*(A) = \cup_{B\cap A\neq\emptyset}\Gamma^{-1}(\{B\})$, and therefore, it is measurable.

5. If X is measurable selection of Γ, then, $X(\omega) \in \Gamma(\omega), \forall\, \omega \in \Omega$ and therefore:

$$\Gamma(\omega) \subseteq A \Rightarrow X(\omega) \in A \Rightarrow \Gamma(\omega) \cap A \neq \emptyset,\ \forall\, A \in \wp(U).$$

Thus,

$$\begin{aligned}\Gamma_*(A) = \{\omega \in \Omega : \Gamma(\omega) \subseteq A\} &\subseteq \{\omega \in \Omega : X(\omega) \in A\} \\ &\subseteq \{\omega \in \Omega : \Gamma(\omega) \cap A \neq \emptyset\},\ \forall\, A \in \wp(U),\end{aligned}$$

and hence,

$$P_*(A) = P(\Gamma_*(A)) \leq P_X(A) \leq P(\Gamma^*(A)) = P^*(A),\ \forall\, A \in \wp(U).$$

In order to prove the second part, let us consider an arbitrary set $A \in \wp(U)$ and let us consider the following three disjoint families of sets:

$$\begin{aligned}\mathcal{F}_1 &= \{B \in \wp(U) : B \cap A \neq \emptyset, B \cap A^c = \emptyset\} = \{B_1, \ldots, B_k\} \\ \mathcal{F}_2 &= \{B \in \wp(U) : B \cap A \neq \emptyset, B \cap A^c \neq \emptyset\} = \{B_{k+1}, \ldots, B_l\} \\ \mathcal{F}_3 &= \{B \in \wp(U) : B \cap A = \emptyset\} = \{B_{l+1}, \ldots, B_{2^{\#U}}\}.\end{aligned}$$

Now, let us select an arbitrary element $b_i \in B_i$ for each $i \in \{1, \ldots, k\}$, two arbitrary elements $b_i^X \in B \cap A^c$ and $b_i^Y \in B \cap A$, for each $i \in \{k+1, \ldots, l\}$, and an arbitrary $b_i \in B$ for each $i \in \{l+1, \ldots, 2^{\#U}\}$.
Let us now define the mappings $X_A, Y_A : \Omega \to U$ as follows:

$$X_A(\omega) = \begin{cases} b_i & \text{if } \Gamma(\omega) = B_i, i \in \{1, \ldots, k\} \cup \{l+1, \ldots, 2^{\#U}\}, \\ b_i^X & \text{if } \Gamma(\omega) = B_i, i \in \{k+1, \ldots, l\}. \end{cases}$$

$$Y_A(\omega) = \begin{cases} b_i & \text{if } \Gamma(\omega) = B_i, i \in \{1, \ldots, k\} \cup \{l+1, \ldots, 2^{\#U}\}, \\ b_i^Y & \text{if } \Gamma(\omega) = B_i, i \in \{k+1, \ldots, l\}. \end{cases}$$

Since Γ is assumed to be strongly measurable, and according to the last exercise, we can easily check that the sets $X^{-1}(C)$ and $Y^{-1}(C)$ are measurable, for every $C \in \wp(U)$. Furthermore, $X_A^{-1}(A) = \Gamma_*(A)$ and $Y_A^{-1}(A) = \Gamma^*(A)$ and therefore, we have that $P_{X_A}(A) = P_*(A)$ and $P_{Y_A}(A) = P^*(A)$. Moreover, by construction, X_A and Y_A are selections of Γ.
(Let the reader notice that, when the images of Γ are singletons, we have that $\Gamma^{-1}(\mathcal{F}_2) = \emptyset$ and therefore X_A and Y_A do coincide. Otherwise, they do not, and, when, in addition, $P(\Gamma \in \mathcal{F}_2) \neq 0$, they induce different probability measures on U.)

6. (a) Let us consider the set function $m : \wp(U) \to [0, 1]$ defined as follows:

$$m(B) = P(\{\omega \in \Omega : \Gamma(\omega) = B\}) = P_\Gamma(\{B\}),\ \forall\, B \in \wp(U).$$

We observe that $m(\emptyset) = 0$ and $\sum_{B \in \wp(U)} m(B) = 1$, and therefore m is, formally speaking, a basic mass assignment. Furthermore, the upper and lower probabilities induced by Γ can be defined as functions of m as follows:

$$\begin{aligned} P^*(A) &= P(\{\omega \in \Omega : \Gamma(\omega) \cap A \neq \emptyset\}) \\ &= \sum_{B\,:\,B \cap A \neq \emptyset} P(\{\omega \in \Omega : \mathcal{F}(\omega) = B\}) \\ &= \sum_{B\,:\,B \cap A \neq \emptyset} m(B), \\ P_*(A) &= P(\{\omega \in \Omega : \Gamma(\omega) \cap A \neq \emptyset\}) \\ &= \sum_{B\,:\,B \subseteq A} P(\{\omega \in \Omega : \mathcal{F}(\omega) = B\}) \\ &= \sum_{B\,:\,B \subseteq A} m(B). \end{aligned}$$

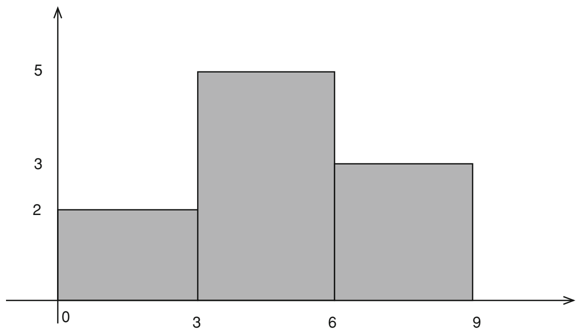
Therefore, P^* and P_* do respectively satisfy the properties of plausibility and belief functions.

(b) Given a belief function, $\text{Bel} : \wp(U) \to [0, 1]$, there exists a unique mass assignment $m : \wp(U) \to [0, 1]$ such that $\text{Bel}(A) = \sum_{B\,:\,B \subseteq A} m(B),\ \forall\, A \in \wp(U)$. Such a mass function is called the Möbius transform of Bel, and it can be expressed as follows [45] in terms on Bel. Let $F_1, \ldots, F_k \subseteq U$ denote the focal sets of m. Let us define the multi-valued mapping Γ: $[0, 1] \to \wp(U)$ as follows:

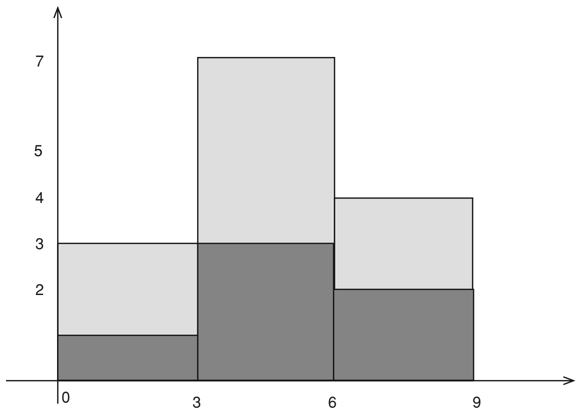
$$\Gamma(x) = \begin{cases} F_1 & \text{if } 0 \leq x < m(F_1) \\ \ldots & \\ F_i & \text{if } \sum_{j=1}^{i-1} m(B_j) \leq x < \sum_{j=1}^{i} m(B_j) \\ \ldots & \\ F_n & \text{if } \sum_{j=1}^{n-1} m(B_j) \leq x \leq 1. \end{cases}$$

We can easily to check that $P(\Gamma = F_i) = m(F_i), \forall\, i = 1, \ldots, n$.

7. The histogram associated to the original (precise) data:



The "imprecise" histogram is the following one. It represents the collection of histograms where the respective heights are between the minimum and the maximum heights, and the sum of the three heights is equal to 10.



8. The maximum value for the variance (maximum of Kruse's variance) in both cases is reached, for instance, by the random variable X that respectively takes the values -10 and 10 for the outcomes h and t, respectively. This random variable is a measurable selection of both random sets. On the other hand, the minimum value is taken for any constant random variable, for instance, for the random variable $Y(h) = Y(t) = 0$, which is a measurable selection of both random sets. Moreover, for any value between 0 and 100 it can be defined a measurable selection of Γ_2 (and therefore, also a selection of Γ_1) with such a variance. Thus, Kruse's variance is equal to the interval [0, 100] in both cases.
On the other hand, the probability envelope of Γ_2 is included in the probability envelope associated to Γ_1, since $\Gamma_2(\omega) \subseteq \Gamma_1(\omega), \omega = h, t$. Furthermore, the inclusion is strict. For instance, the probability measure that assigns probability 1 to the value 10 belongs to the envelope of Γ_1 but not to the probability envelope of Γ_2.

9. The probability envelopes of Γ and Γ' are, respectively,

$$\mathcal{P}(\Gamma) = \left\{ \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right) \right\}$$

and

$$\mathcal{P}(\Gamma') = \left\{ \left(\frac{i}{150,} \frac{150-i}{150}\right) : i = 50, \ldots, 150 \right\}$$
$$= \left\{ (p, 1-p) : \frac{1}{3} \leq p \leq \frac{2}{3}, \text{ and } 150p \in \mathbb{N} \right\}.$$

But both credal sets do coincide and they are the convex hull of both envelopes:

$$M(P^*_{\Gamma}) = M(P^*_{\Gamma'}) \left\{ (p, 1-p) : \frac{1}{3} \leq p \leq \frac{2}{3} \right\}.$$

The minimum and maximum values of the variance, if we range the whole credal set are $\frac{2}{9}$ and $\frac{1}{4}$, respectively. They coincide with the bounds of $Var(\Gamma')$, but they do not with the bounds of $\mathrm{Var}(\Gamma)$. The maximum of $\mathrm{Var}(\Gamma)$ is equal to $\frac{2}{9}$ that is strictly lower than $\frac{1}{4}$.

10. Let us use denote by A_i the set $A_i = \Gamma^{-1}([a_i, b_i]) = \{\omega \in \Omega : \Gamma(\omega) = [a_i, b_i]\} \subseteq \Omega, \forall i = 1, \ldots, n$.
On the one hand, we can easily check that the variance of any measurable selection, X of Γ belongs to the interval of values determined in Eq. 2.19, because $X(\omega) \in [a_i, b_i], \forall \omega \in A_i, \forall i = 1, \ldots, n$ and furthermore $E(X) \in E(\Gamma)$.
Let us now check that the interval of values determined in Eq. 2.19 strictly includes the set of possible variances (Kruse's variance):

$$\mathrm{Var}(\Gamma) = \{\mathrm{Var}(X) : X \in S(\Gamma)\}.$$

We will prove that the value $\sum_{i=1}^n p_i \cdot \max\{(a_i - \max E(\Gamma))^2 p_i, (b_i - \min E(\Gamma))^2\}$ belongs to $\oplus_{i=1}^n [[a_i, b_i] \ominus E(\Gamma)]^2 \odot p_i$, but not to Kruse's variance. Let us consider the random variable $Z(\omega) = \max\{[a_i - \max E(\Gamma)]^2, [b_i - \min E(\Gamma)]^2, \forall \omega \in A_i, i = 1, \ldots, n$. We observe that $0 \leq (X(\omega) - E(X))^2 \leq Z(\omega), \ \forall \omega \in \Omega, \forall X \in S(\Gamma)$. Therefore $\mathrm{Var}(X) \leq E(Z)$ and $\mathrm{Var}(X) = E(Z)$ if and only if $[X(\omega) - E(X)]^2 = Z(\omega), c.s.(P)$. Let us suppose that there exists some measurable selection, $X \in S(\Gamma)$ such that $\mathrm{Var}(X) = E(Z)$. On the other hand, since X is a selection of Γ, we have that $E(X) = \sum_{i=1}^n (p_{i1} a_i + p_{i2} b_i)$, where $p_{i1} + p_{i2} = p_i$. This expectation is strictly higher than $\min E(\mathcal{F})$ and strictly lower than $\max E(\Gamma)$, unless $p_{i1} = p_i, \forall i$ or $p_{i2} = p_i, \forall i$. According to this, $\mathrm{Var}(X) < E(Z)$, unless one of those two alternatives happens. Let us suppose that $p_{i1} = p_i, \ \forall i$ (the proof under the second hypothesis is similar). In that case, $X = \max \mathcal{F}$ c.s.(P) and therefore there exists a collection of events

$C_1, \ldots, C_1$ such that $C_i \subseteq A_i$, $P(C_i) = P(A_i)$ and $X(\omega) = b_i$, $\forall\, \omega \in C_i$. Then we get a contradiction, if we suppose that $a_i < b_i$, for some $i \in \{1, \ldots, n\}$, because in that case, $\mathrm{Var}(X) = \sum_{i=1}^{n} p_i(b_i - \max E(\Gamma))^2$ is strictly lower than $\max\{(a_i - \max E(\Gamma))^2, (b_i - \min E(\Gamma))^2\} = (a_i - \max E(\Gamma))^2$ and $E(X) = \max E(\Gamma)$.

11. (a) We have not been provided with the joint distribution of the random vector (X_0, Y_0), but we would expect them not to be independent (indeed, a positive correlation would be expected).
    (b) The random interval Γ_1 is a constant, and therefore it is stochastically independent from any other random set defined on Ω. In particular, it is stochastically independent from Γ_2.

12. Let us define the random sets Γ_1 and Γ_2 as follows:

$$\Gamma_1(i,j) = \begin{cases} \{\text{red}\} & i = 1, \ldots, 5, \\ \{\text{white}\} & i = 6, 7, \qquad \forall\, j = 1, \ldots, 10. \\ \{\text{red, white}\} & i = 8, 9, 10 \end{cases}$$

$$\Gamma_2(i,j) = \begin{cases} \{\text{red}\} & j = 1, 2, 3, \\ \{\text{white}\} & j = 4, 5, 6, \qquad \forall\, i = 1, \ldots, 10. \\ \{\text{red, white}\} & j = 7, 8, 9, 10 \end{cases}$$

We consider the Laplace probability measure on $\{1, \ldots, 10\} \times \{1, \ldots, 10\}$ that assigns probability 0.01 to each singleton $\{(i, j)\}$. The mass functions associated to the respective marginal probabilities are determined as follows:

$$P(\Gamma_1 = \{\text{red}\}) = 0.5,\ P(\Gamma_1 = \{\text{white}\}) = 0.2,\ P(\Gamma_1 = \{\text{red, white}\}) = 0.3$$
$$P(\Gamma_2 = \{\text{red}\}) = 0.3,\ P(\Gamma_2 = \{\text{white}\}) = 0.3,\ P(\Gamma_2 = \{\text{red, white}\}) = 0.4.$$

Furthermore, their joint probability is the product of both probability measures, and therefore they are stochastically independent.
The upper probability induced by the (Cartesian product) random set $\Gamma = \Gamma_1 \times \Gamma_2 : \{1, \ldots, 10\} \times \{1, \ldots, 10\} \to \wp(\{\text{red, white}\} \times \{\text{red, white}\})$ is determined by the upper probabilities of Cartesian products, that can be calculated as follows:

$$\begin{aligned} P^*_\Gamma(A \times B) &= P(\{(i,j) : \mathcal{F}(i,j) \cap A \times B \neq \emptyset\}) \\ &= P(\{(i,j) : \mathcal{F}_1(i,j) \cap A \neq \emptyset, \mathcal{F}_2(i,j) \cap B \neq \emptyset\}) \\ &= P(\{(i,j) : \mathcal{F}_1(i,j) \cap A \neq \emptyset\} \cdot P(\{(i,j) : \mathcal{F}_2(i,j) \cap B \neq \emptyset\}) \\ &= P^*_{\Gamma_1}(A) \cdot P^*_{\Gamma_2}(B),\ \forall, A, B \subseteq \{\text{red, white}\}. \end{aligned}$$

According to the above information, the “marginal” upper probabilities are determined as follows:

A	{Red}	{White}	{Red,White}
$P^*_{\Gamma_1}(A)$	0.8	0.5	1

A	{Red}	{White}	{Red,White}
$P^*_{\Gamma_2}(A)$	0.7	0.7	1

The credal set associated to Γ is the set of probability measures dominated by P^*_{Γ},

$$\{P : P(C) \leq P^*(C),\ \forall C \in \wp(\{\text{red, white}\} \times \{\text{red, white}\}\}.$$

13. The probability measure, $P_{(X_1,X_2)} : \wp(\{r, w\} \times \{r, w\}) \to [0, 1]$, associated to the joint experiment cannot be expressed as a product. In fact, there exists a stochastic dependence between the random variables X_1 and X_2, that represent the colours of both balls. Let us notice, for instance, that
    - $P_{(X_1,X_2)}(\{(r, r)\}) = 0.15 + 0.2 \cdot \frac{1}{4} + 0.09 \cdot \frac{1}{6} + 0.12 \cdot \frac{1}{4}$
    - $P_{X_1}(\{r\}) = 0.5 + 0.09 \cdot \frac{1}{6} + 0.09 \cdot \frac{5}{6} + 0.12 \cdot \frac{1}{2}$, and
    - $P_{X_2}(\{r\}) = 0.3 + 0.2 \cdot \frac{1}{6} + 0.06 \cdot \frac{5}{6} + 0.12 \cdot \frac{1}{2}$

    Thus, $P_{(X_1,X_2)}(\{(r, r)\}) = 0.245$ does not coincide with $P_{X_1}(\{r\}) \cdot P_{X_2}(\{r\}) = 0.65 \cdot 0.46$.
    In the above expressions, r stands for "red" and w stands for "white".
14. The multi-valued mappings Γ_1 and Γ_2 defined as follows:

    $$\Gamma_1(i) = \Gamma_2(i) = \begin{cases} \{\text{r}\} & i = 1, \ldots, 5, \\ \{\text{r, w}\} & i = 6, \ldots, 10. \end{cases}$$

    represent our incomplete knowledge about the final colour of each ball. They are not independent. In fact:

    $$P(\Gamma_1 = \{r\}) = P(\Gamma_2 = \{r\}) = P(\Gamma_1 = \{r\}, \Gamma_2 = \{r\}) = 0.5, \quad \text{and}$$
    $$P(\Gamma_1 = \{r, w\}) = P(\Gamma_2 = \{r, w\}) = P(\Gamma_1 = \{r, w\}, \Gamma_2 = \{r, w\}) = 0.5,$$

    and therefore, $P(\Gamma_1 = A, \Gamma_2 = B) \neq P(\Gamma_1 = A) \times P(\Gamma_2 = B)$, in general.
15. The joint probability induced by the random vector (X_1, X_2) that represents the color of both balls can be factorized as the product of both marginals. In fact, the joint probability assigns the respective probabilities $\frac{9}{16}$, $\frac{3}{16}$, $\frac{3}{16}$ and $\frac{1}{16}$ to the respective possible outcomes (r, r), (r, w), (w, r) and (w, w).
16. In these conditions we have that:

    $$P(\Gamma_i = [0, \infty)|\Gamma_{i+1} = [k, \infty)) = 1, \text{ but } P(\Gamma_i = [0, \infty)|\Gamma_{i+1} \subseteq (-\infty, k)) = 0.$$

    Therefore, Γ_{i+1} and Γ_i are not stochastically independent, $i = 1, \ldots, n-1$.

Exercises of Chapter 3

1. (a) Random fuzzy object. (Based on the ontic semantic of fuzzy sets.)
   (b) Ill-known random variable. (Based on the epistemic semantic of fuzzy sets.)
2. (a) The induced possibility distribution is defined as follows:

$$\pi(x)\begin{cases} 0 & \text{if } x \in (-\infty, -5] \cup [5, \infty) \\ 0.05 & \text{if } x \in [-5, -3) \cup (3, 5] \\ 0.35 & \text{if } x \in [-3, -2) \cup (2, 3] \\ 1 & \text{if } x \in [-2, 2]. \end{cases}$$

   (b) The degree of possibility of that interval is:

$$\Pi([d_0 - 5, d_0 - 2.5]) = \sup_{x \in [d_0 - 5, d_0 - 2.5]} \pi(x) = \pi(d_0 - 2.5) = 0.35.$$

3. It is easy to check, if we have into account that any trapezoidal fuzzy number is determined by four points in the plane, $(x_1, 0)$, $(x_2, 1)$, $(x_3, 1)$ and $(x_4, 0)$, and therefore it can be represented by the 4-dimensional vector (x_1, x_2, x_3, x_4). Thus, we can define a bijective function between the family of trapezoidal fuzzy sets and $\mathbb{R}^4$, $b : \mathcal{F}_T(\mathbb{R}) \to \mathbb{R}^4$. Let us consider the random vector **:** $\Omega \to \mathbb{R}^4$, defined as the composition of $\tilde{X}(\omega)$ with b. Since both random objects, $\tilde{X}$ and $\mathbf{X}$ are connected by means of a bijective function, the probability distribution of each of them is determined by the distribution of the other one.
4. The scalar variance can be expressed as a function of the variances of the mid points and spreads of the 10 outcomes of the fuzzy random variable. Taking into account that the quality of our information about each weight is the same for all the 10 objects, the variance of the spreads is null. Therefore, in this example, the scalar variance would just quantify the dispersion of the mid-points, i.e., the dispersion of the 10 displayed quantities.
   On the other hand, the fuzzy variance would be a fuzzy number representing the incomplete information about the variance of the actual weights of the 10 objects. Thus, the membership value $\text{Var}(\tilde{X})(x)$ would represent the degree of possibility that the actual variance coincides with x. We can equivalently say that, according to the available information, the α-cut $[\text{Var}(\tilde{X})]_\alpha$ contains the variance of the true weights with probability greater than or equal to $1 - \alpha$.
5. According to Exercise 3 of this chapter any triangular fuzzy random variable $\tilde{X}$ can be expressed as a bijective function of a 3-dimensional random vector $\mathbf{X} = b^{-1} \circ \tilde{X}$. Analogously, any pair of triangular fuzzy random variables $(\tilde{X}_1, \tilde{X}_2)$ can be expressed as a function of a 2-dimensional random vector $(\mathbf{X}_1, \mathbf{X}_2)$, whose components are, in turn, 3-dimensional random vectors. The joint probability of $(\tilde{X}_1, \tilde{X}_2)$ is univocally determined by the joint probability of $(\mathbf{X}_1, \mathbf{X}_2)$. Something similar can be said about the marginal distributions. Therefore, we can easily check that the joint probability of $(\tilde{X}_1, \tilde{X}_2)$ can be decomposed as the product of its marginals if and only if the joint probability induced by $(\mathbf{X}_1, \mathbf{X}_2)$ can be also decomposed as a product.