Appendices

A.1 Appendix 1: Basic Concepts of Topology

This section is a quick reminder of basic ideas in topology. Readers can find more details and proofs in any standard text such as Kelley [116] or Simmons [187].

Let S be a set. A *topology* on S is a collection τ of subsets of S which has the following properties:

- $S, \emptyset \in \tau$,
- τ is closed under arbitrary unions and finite intersections.

The pair (S, τ) is then called a *topological space*, and the sets in τ are called the *open sets* for the topology. A *basis* for a topology τ is a subcollection $\sigma \subset \tau$ so that every set in τ is a union of sets from σ. The space (S, τ) is said to be *second countable* if it has a countable basis.

Let $A \subset S$. The *induced, relative* or *subspace* topology for A is the collection $\tau_A = \{A \cap U; U \in \tau\}$. Then (A, τ_A) is itself a topological space which is called a (topological) *subspace* of (S, τ).

Suppose that S is equipped with topologies τ_1 and τ_2 wherein $\tau_1 \subset \tau_2$. In this case we say that τ_1 is *weaker* (or *coarser*) that τ_2. Equivalently τ_2 is *stronger* (or *finer*) than τ_1. The *discrete topology* τ_D on an arbitrary set S is that topology for which every subset of S is open. It is the strongest topology on S in that if τ is any other topology, then $\tau \subset \tau_D$. If S is equipped with the discrete topology we call it a *discrete space*.

Let (S_1, τ_1) and (S_2, τ_2) be topological spaces. A mapping $f : S_1 \to S_2$ is said to be *continuous* if $f^{-1}(U) \in \tau_1$ whenever $U \in \tau_2$. A mapping f is said to be a *homeomorphism* between (S_1, τ_1) and (S_2, τ_2) if it is bijective and continuous and if f^{-1} is also continuous.

If $x \in S$, a *neighbourhood* of x is a set $N \subseteq S$ such that there exists some $U \in \tau$ for which $x \in U \subseteq N$. If N is a neighbourhood of x and $N \in \tau$, we call it an *open neighbourhood*. The space (S, τ) is said to be *Hausdorff* if for all distinct $x, y \in S$ there exist open neighbourhoods U_x of x and V_y of y so that $U_x \cap V_y = \emptyset$.

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A sequence $(x_n, n \in \mathbb{N})$ of points in S is said to *converge* to a *limit* $x \in S$ if given any neighbourhood U of x there exists $N \in \mathbb{N}$ so that $x_n \in U$ for all $n > N$. If (S, τ) is Hausdorff, then convergent sequences in S have unique limits. If (S_i, τ_i) for $i = 1, 2$ are topological spaces wherein (S_1, τ_1) is second countable, then $f : S_1 \to S_2$ is continuous if and only if given any sequence $(x_n, n \in \mathbb{N})$ in S_1 that converges to $x \in S_1$, then the sequence $(f(x_n), n \in \mathbb{N})$ converges to $f(x)$ in S_2.

A collection $\mathcal{U} := \{U_i, i \in I\}$ of open sets in τ is an *open cover* for S if $S = \bigcup_{i \in I} U_i$. A *subcover* of a cover $\mathcal{U}$ is a subcollection of $\mathcal{U}$ which is itself an open cover for S. The space (S, τ) is said to be *compact* if every open cover of S has a *finite* subcover. The continuous image of a compact set is compact. A discrete space is compact if and only if it has a finite number of elements. If (S, τ) is locally compact and Hausdorff, its *one-point compactification* is the compact Hausdorff space (S_∞, τ_∞) where $S_\infty = S \cup \{\infty\}$ (with $\infty \notin S$), and τ_∞ comprises the open sets of S (now regarded as subsets of S_∞), S_∞ itself, and the complements in S_∞ of compact subsets of S.

A subset of S is said to be *closed* if its complement is open. A closed subset of a compact space is compact. Every compact subset of a Hausdorff space is closed. In particular, if (S, τ) is Hausdorff, then $\{x\}$ is closed for all $x \in S$.

If A is a subset of S, then its *closure* $\overline{A}$ is the intersection of all closed sets containing A. We always have $A \subseteq \overline{A}$ but $A = \overline{A}$ if and only if A is closed. A subset A of S is said to be *relatively compact* if $\overline{A}$ is compact. The space (S, τ) is said to be *locally compact* if every point in S has a neighbourhood which has compact closure. The set A is said to be *dense* in (S, τ) if $\overline{A} = S$. The space (S, τ) is said to be *separable* if S contains a countable dense subset. If (S, τ) is second countable, then it is separable.

A space (S, τ) is *connected* if we cannot write $S = A \cup B$ for two non-empty, disjoint open sets A and B. The continuous image of a connected set is connected. If (S, τ) is an arbitrary topological space and $x \in S$, the *component* of x is the largest connected subset of S that contains x. Components are always closed sets. The set of all components $\{A_i, i \in I\}$ in S forms a *partition* of S in that $A_i \cap A_j = \emptyset$ whenever $i \neq j$ and $S = \bigcup_{i \in I} A_i$. The corresponding equivalence relation $\sim$ on S is given by $x \sim y$ if there exists a connected set containing both x and y. A space (S, τ) is said to be *locally connected* if every neighbourhood of every point in S contains a connected neighbourhood of that point.

A property of a space (S, τ) is said to be a *topological property* if it is preserved under homeomorphisms. The Hausdorff property, compactness, local compactness, connectedness and local connectedness are all topological properties.

If $\{(S_i, \tau_i), i = 1, 2, \ldots, n\}$ are a (finite) collection of topological spaces, the *product topology* τ on the Cartesian product $S := S_1 \times S_2 \times \cdots \times S_n$ is defined via the basis $U_1 \times U_2 \times \cdots \times U_n$ where $U_i \in \tau_i (1 \leq i \leq n)$. The space (S, τ) is compact (respectively, connected, Hausdorff, locally compact, locally connected) if (S_i, τ_i) is for all $i = 1, 2, \ldots, n$. The product topology can be extended to arbitrary collections $\{(S_\alpha, \tau_\alpha), \alpha \in I\}$, where I is some index set, but the construction is different from that given for finite index sets. We won't give the details here (they can be found in the standard texts) but we'll quote the famous *Tychonoff theorem* which states that the product of an arbitrary collection of compact spaces is itself compact. Kelley (p. 143) writes that this "is unquestionably the most useful theorem on compactness. It is probably the most important single theorem of general topology." In relation to the examples discussed in Sect. 1.1, we see that the infinite torus group Π^{∞} (which is the product of countably many tori) is compact.

Finally we state a corollary of the celebrated *Urysohn lemma* which says that if X is a locally compact Hausdorff space, K is a compact subset of X and U is an open set with $K \subset U$, then there exists a function $h \in C_c(X, \mathbb{R})$ for which $1_K \leq h \leq 1_U$. From this we can deduce the following:

Proposition A.0.1 *If X is a locally compact Hausdorff space then*

1. $C_c(X, \mathbb{R})$ *is dense in* $C_0(X, \mathbb{R})$.
2. $C_c(X, \mathbb{C})$ *is dense in* $C_0(X, \mathbb{C})$.

Proof

1. Let $f \in C_0(X, \mathbb{R})$. Then given $\epsilon > 0$ there exists a compact set K with $||f 1_{K^c}||_\infty < \frac{\epsilon}{2}$. Now apply the corollary to Urysohn's lemma, taking U to be the union of open neighbourhoods of each point in K. Thus we can assert the existence of $h \in C_c(X, \mathbb{R})$ for which $1_K \leq h \leq 1_U$. In particular, it follows that $hf \in C_c(X, \mathbb{R})$ and

$$\begin{aligned} ||f - hf||_\infty &\leq ||f - 1_K f||_\infty + ||1_K f - hf||_\infty \\ &\leq ||f - 1_K f||_\infty + ||1_K f - 1_U f||_\infty \\ &= ||f - 1_K f||_\infty + ||1_{U \setminus K} f||_\infty \\ &\leq 2||f 1_{K^c}||_\infty < \epsilon. \end{aligned}$$

2. Follows easily from (1) by approximating real and imaginary parts. □

A.2 Appendix 2: Concerning Left/Right Uniform Continuity

We begin by discussing some useful facts about topological groups G. First some notation. If $A, B \subset G$ we write $AB := \{gh; g \in A, h \in B\}$. In particular if $g \in G$, then $gA := \{g\}A$ and $Ag := A\{g\}$. We also write $A^{-1} := \{g \in G; g^{-1} \in A\}$.

Fact 1. It is pointed out in Sect. 1.1 that for each $g \in G$, left translation l_g and right translation r_g are homeomorphisms of G. It follows that if U is a neighbourhood of e, then both gU and Ug are neighbourhoods of g.

We say that $V \subset G$ is *symmetric* if $V^{-1} = V$.

Fact 2. Given any neighbourhood U of e there exists a symmetric neighbourhood v of e so that $VV \subseteq U$. To see this it's sufficient to use continuity of the group operation from $G \times G$ to G to deduce that there exist neighbourhoods W_1 and W_2 of e so that $W_1 W_2 \subseteq U$. Now take $V = W_1 \cap W_1^{-1} \cap W_2 \cap W_2^{-1}$.

Fact 3. Let $f : G \to \mathbb{R}$ be continuous and fix $g \in G$. Then given any $\epsilon > 0$ there exists a neighbourhood U_g of g so that so that $g' \in U_g \Rightarrow |f(g) - f(g')| < \epsilon$. By Fact 1 we can find a neighbourhood W_g of e so that $gW_g \subseteq U_g$ and for all $h \in W_g, |f(g) - f(gh)| < \epsilon$.

Now we are ready to prove the main result.

Theorem A.0.1 *Every continuous function of compact support on G is both left and right uniformly continuous.*

Proof We follow Folland [68], Proposition 2.6, p. 34 very closely (see also Higgins [98] Proposition 21, p. 67–68). We only deal with right uniform continuity here as the other case works in the same way. Let $f \in C_c(G)$ and $\epsilon > 0$. Write $K := \text{supp}(f)$ and let $g \in K$ be arbitrary. Then by Fact 3 we can find a neighbourhood W_g of e so that for all $h \in W_g, |f(g) - f(gh)| < \frac{\epsilon}{2}$. By Fact 2, there exists a symmetric neighbourhood V_g of e so that $V_g V_g \subseteq W_g$. By Fact 1, the sets $\{gV_g, g \in G\}$ cover K and so there exists $g_1, \ldots, g_N \in G$ so that $K \subseteq \bigcup_{j=1}^N g_j V_{g_j}$. Then there exists some j so that $g_j^{-1} g \in V_{g_j}$. Now for all $h \in V := \bigcap_{j=1}^N V_{g_j}$ we have $gh = g_j(g_j^{-1}g)h \in g_j W_{g_j}$ while $g = g_j(g_j^{-1}g) \in g_j W_{g_j}$. We then have

$$|f(gh) - f(g)| \leq |f(gh) - f(g_j)| + |f(g_j) - f(g)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

A similar argument works when $gh \in K$ and the result follows. □

Corollary A.0.1 *Every continuous function on G that vanishes at infinity is both left and right uniformly continuous.*

Proof We only prove left uniform continuity as the same argument works in the other case. Let $f \in C_0(G)$. Since $C_c(G)$ is dense in $C_0(G)$, given any $\epsilon > 0$ there exists $h \in C_c(G)$ so that $||f - h||_\infty < \frac{\epsilon}{3}$. By Theorem A.2.1 there exists an open neighbourhood U of e so that $||L_g h - h||_\infty < \frac{\epsilon}{3}$ for all $g \in U$. It follows that for all $g \in U$,

$$\begin{aligned} ||L_g f - f||_\infty &\leq ||L_g f - L_g h||_\infty + ||L_g h - h||_\infty + ||h - f||_\infty \\ &= ||L_g h - h||_\infty + 2||h - f||_\infty \\ &< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon. \end{aligned}$$

□

A.3 Appendix 3: Manifolds

This appendix gathers together some useful facts about manifolds. For proofs and more detailed discussion we recommend a dedicated text on differential geometry such as Helgason [88] or Warner [214]. Spivak [189] gives an excellent quick introduction and contains a very nice account of integration. For Riemannian manifolds, we particularly recommend Chavel [43].

A.3.1 Manifolds, Tangent Spaces, Vector Fields

A Hausdorff, second countable topological space M is a *(real, topological) manifold* of dimension d if each point in M has an open neighbourhood that is homeomorphic to an open set in $\mathbb{R}^d$. It follows immediately that M is locally compact. If U is an open neighbourhood of $p \in M$ and ϕ is a homeomorphism from U to an open subset of $\mathbb{R}^d$, we call (U, ϕ) a *chart* or *co-ordinate system* at p. For each $q \in U$, we write $\phi(q) = (x_1(q), \ldots, x_d(q))$, and call $x_1, \ldots, x_d$ *local co-ordinates* at U. We will also use the notation $q_i = x_i(q)(1 \leq i \leq d)$.

An *atlas* is a collection of charts $\{(U_\alpha, \phi_\alpha); \alpha \in I\}$ so that $(U_\alpha, \alpha \in I)$ covers M. If M has an atlas for which the mappings $\phi_\alpha \circ \phi_\beta^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ are C^∞ for all $\alpha, \beta \in I$, then M is said to be a C^∞-manifold. Henceforth whenever we use the word "manifold" we will always assume that it is C^∞.

A function $f : M \to \mathbb{R}$ is said to be *smooth* or C^∞ if $f \circ \phi_\alpha^{-1}$ is C^∞ from $\phi_\alpha(U_\alpha)$ to $\mathbb{R}$ for all $\alpha \in I$. Let $C^\infty(M, \mathbb{R})$ denote the set of all smooth real-valued functions on M. It is a (real) algebra under the usual pointwise operations.

A *tangent vector* $X(p)$ at $p \in M$ is a linear functional on $C^\infty(M, \mathbb{R})$ which satisfies the local derivation property

$$X(p)(fg) = f(p)X(p)(g) + (X(p)f)g(p), \tag{A.1}$$

for each $f, g \in C^\infty(M, \mathbb{R})$. In local co-ordinates we can write $X(p) = \sum_{i=1}^d X_i(p)$ $\frac{\partial}{\partial x_j}$ where $X_1(p), \ldots, X_d(p) \in \mathbb{R}$. The set of all tangent vectors at p forms a d-dimensional real linear space called the *tangent space* to p at M. We denote it by $T_p(M)$. We can give the set $T(M) := \bigcup_{p \in M} T_p(M)$ the structure of a $2d$-dimensional manifold, and $T(M)$ is called the *tangent bundle* to M.

Let M_1 and M_2 be manifolds of dimension d_1 and d_2 (respectively) and let $\{(U_\alpha, \phi_\alpha); \alpha \in I\}$ be an atlas for M_1 and $\{(V_\beta, \psi_\beta); \beta \in J\}$ be an atlas for M_2. A mapping $f : M_1 \to M_2$ is said to be C^∞ if $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is C^∞ from $\mathbb{R}^{d_1}$ to $\mathbb{R}^{d_2}$ for all $\alpha \in I, \beta \in J$. A C^∞-mapping $f : M_1 \to M_2$ is said to be a *diffeomorphism* if it is a bijection and f^{-1} is also a C^∞-mapping.

Suppose that M_1 and M_2 are manifolds, and that $\Phi : M_1 \to M_2$ is a C^∞ mapping. Let $p \in M_1$ and $X_p \in T_p(M_1)$. Then if $q = \Phi(p)$, there exists $Y_q \in T_q(M_2)$ so that for all $f \in C^\infty(M_2, \mathbb{R})$,

$$Y_q f = X_p(f \circ \Phi).$$

We write $Y_q = d\Phi_p(X_p)$ and call the linear map $d\Phi_p : T_p(M_1) \to T_q(M_2)$ the *differential* of Φ (at p). It is common to write $d\Phi$ instead of $d\Phi_p$.

The following composition property is very important. If M_1, M_2 and M_3 are manifolds and $\Phi_1 : M_1 \to M_2$ and $\Phi_2 : M_2 \to M_3$ are C^∞ mappings, then for all $p \in M_1$:

$$d(\Phi_2 \circ \Phi_1)_p = (d\Phi_2)_q \circ (d\Phi_1)_p, \tag{A.2}$$

where $q = \Phi_1(p)$.

A C^∞ mapping X from M to $T(M)$ is said to be a *vector field* if $X(p) \in T_p(M)$ for all $p \in M$. In each local co-ordinate system (U, ϕ) we can write $X(p) = \sum_{i=1}^{d} X_i(p)\frac{\partial}{\partial x_j}$, as we did for individual tangent vectors but now X_i is a C^∞ function from $\phi(U)$ to $\mathbb{R}$. From an analytic point of view, a vector field X can be regarded as a first order differential operator mapping $C^\infty(M, \mathbb{R})$ to itself by the prescription $(Xf)(p) = X(p)f$ for $p \in M$. Indeed, the local derivation property (A.1) now extends to a global derivation property on the algebra $C^\infty(M, \mathbb{R})$:

$$X(fg) = X(f)g + fX(g).$$

The collection of all vector fields on M forms a real linear space $\mathcal{L}(M)$. It is also an (infinite-dimensional) real Lie algebra, where the Lie bracket is defined by

$$[X, Y]f = X(Yf) - Y(Xf),$$

for each $X, Y \in \mathcal{L}(M)$, $f \in C^\infty(M)$.

If M_1 and M_2 are manifolds, and $\Phi : M_1 \to M_2$ is a C^∞ mapping then the vector fields $X \in \mathcal{L}(M_1)$ and $Y \in \mathcal{L}(M_2)$ are said to be Φ*-related* if $d\Phi(X_p) = Y_{\Phi(p)}$ for all $p \in M$. If X_i is Φ-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is Φ-related to $[Y_1, Y_2]$.

If M and N are manifolds and $\phi : M \to N$ is C^∞ we say that ϕ is an *immersion* if $d\phi_p$ is non-singular for each $p \in M$, and we say that (M, ϕ) is a *submanifold* of N if ϕ is an injective immersion.

If K and U are compact and open sets (respectively) with $K \subset U \subset M$, then there exists a mapping $\phi \in C_c^\infty(M)$ which is called a *cut-off function* for which $\text{supp}(\phi) \subset U$, $0 \leq \phi \leq 1$ and $\phi(p) = 1$ for all $p \in K$. We can imitate the proof of Proposition A.1.1, using cut-off functions instead of Urysohn's lemma, to show that $C_c^\infty(M, \mathbb{R})$ is dense in $C_0(M, \mathbb{R})$, and hence that $C_c^\infty(M, \mathbb{C})$ is dense in $C_0(M, \mathbb{C})$.

A.3.2 Differential Forms

If $f \in C^\infty(M, \mathbb{R})$, then since $T_x(\mathbb{R}) \cong \mathbb{R}$ for all $x \in \mathbb{R}$, we have $df(X) = Xf$ for all $f \in C^\infty(M, \mathbb{R})$. The mapping $df : \mathcal{L}(M) \to C^\infty(M, \mathbb{R})$ is called a *one-form*.[1] If we localise at $p \in M$, then df_p is a linear mapping from $T_p(M)$ to $\mathbb{R}$, i.e. an element of the algebraic dual $T_p^*(M)$ to $T_p(M)$ which is called the *cotangent space* to M at p.

We can form tensor products of r tangent and s co-tangent spaces at a point $p \in M$ to obtain the linear space

[1] The most general one-forms are finite linear combinations of the df's.

$$\bigotimes_p^{(r,s)}(M) = \underbrace{T_p(M) \otimes \cdots \otimes T_p(M)}_{r \text{ copies}} \otimes \underbrace{T_p^*(M) \otimes \cdots \otimes T_p^*(M)}_{s \text{ copies}}.$$

The set $\bigotimes^{(r,s)}(M) := \bigcup_{p \in M} \bigotimes_p^{(r,s)}(M)$ can be given the structure of a real d^{r+s}-dimensional manifold, and is called the (r, s)-tensor bundle over M. An (r, s) *tensor field* is a C^∞ mapping F from M to $\bigotimes^{(r,s)}(M)$ such that $F(p) \in \bigotimes_p^{(r,s)}(M)$ for each $p \in M$. A tensor field is said to be *covariant* if $s = 0$ and *contravariant* if $r = 0$. So for example a vector field is covariant (with $r = 1$) and a one-form is contravariant (with $s = 1$).

Recall that if V is a real d-dimensional vector space and $V^{\otimes^r}$ is its r-fold tensor product, then the subspace $\Lambda^r(V) \subseteq V^{\otimes^r}$ of *alternating r-tensors* is spanned by

$$\mathrm{Alt}(v_1, \ldots, v_r) := \frac{1}{r!} \sum_{\sigma \in \Sigma(r)} \mathrm{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)},$$

where $v_1, \ldots, v_r \in V$ and $\Sigma(r)$ is the symmetric group on r letters.[2] We have $\Lambda^1(V) = V$ and $\Lambda^r(V) = \{0\}$ for $r > d$. Note also that $\dim(\Lambda^r(V)) = \binom{d}{r}$, and so in particular $\dim(\Lambda^d(V)) = 1$. It follows that $\Lambda^d(V) - \{0\}$ has two connected components, and an *orientation* on V is a specific choice of one of these.

If $v = \mathrm{Alt}(v_1, \ldots, v_r) \in \Lambda^r(V)$ and $w = \mathrm{Alt}(w_1, \ldots, w_s) \in \Lambda^s(V)$, their *wedge product* $v \wedge w \in \Lambda^{r+s}(V)$ is

$$v \wedge w := \frac{(r+s)!}{r!s!} \mathrm{Alt}(v_1, \ldots, v_r, w_1, \ldots w_s).$$

Furthermore,

$$v \wedge w = (-1)^{rs} w \wedge v.$$

Consider the linear space $\Lambda^r(T_p^*(M))$ of alternating r-tensors over the cotangent space at $p \in M$. We can again give a manifold structure to $\Lambda^r(M) := \bigcup_{p \in M} \Lambda^r(T_p^*(M))$ (indeed, $\Lambda^r(M)$ is a *sub-bundle* of $\bigotimes^{(0,r)}(M)$). A smooth *r-form* ω is a C^∞ mapping from M to $\Lambda^r(M)$ such that for each $p \in M$, $\omega_p \in \Lambda^r(T_p^*(M))$. In particular, if $X_1, \ldots, X_r \in \mathcal{L}(M)$,

$$\omega(X_1, \ldots, X_r)(p) = \omega_p(X_1(p), \ldots, X_r(p)).$$

In local co-ordinates (U, ϕ) at $p \in M$ an r-form ω can be written

$$\omega(p) = \sum_{\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, d\}} \omega_{i_1, \ldots, i_r}(p) dx_{i_1} \wedge \cdots dx_{i_r},$$

[2] The *sign* $\mathrm{sign}(\sigma)$ of a permutation σ is defined to be 1 if it is even, and -1 if it is odd.

where $\omega_{i_1,\ldots,i_r}$ are smooth mappings from $\phi(U)$ to $\mathbb{R}$ and $dx_j\left(\frac{\partial}{\partial x_k}\right) = \delta_{jk}$.

We denote the linear space of all C^∞ r-forms on M by $\Omega^r(M)$. In fact $\Omega^r(M)$ is a left $C^\infty(M)$-module where for $f \in C^\infty(M), \omega \in \Omega^r(M)$, $f\omega$ is the r-form whose value at $p \in M$ is $f(p)\omega_p$.

If M_1 and M_2 are manifolds and $\Phi : M_1 \to M_2$ is a C^∞ mapping, then its *pullback* Φ^* is the linear mapping from $\Omega^r(M_2)$ to $\Omega^r(M_1)$ defined by

$$\Phi^*(\omega)(X_1, \ldots, X_r)(p) = \omega_{\Phi(p)}(d\Phi_p(X_1(p)), \ldots, d\Phi_p(X_r(p))),$$

for all $X_1, \ldots, X_r \in \mathcal{L}(M_1)$, $p \in M_1$.

A.3.3 Integration on Manifolds

Suppose that M is a connected d-dimensional manifold. Define $O \in \Lambda^d(M)$ by $O = \bigcup_{p\in M}\{0_p\}$ where 0_p is the zero vector in $\Lambda^d(T_p^*(M))$. Since $\Lambda^d(T_p^*(M)) - \{0_p\}$ has exactly two connected components for each $p \in M$, it follows that $\Lambda^d(M) - \{O\}$ has at most two of these. We say that M is *orientable* if $\Lambda^d(M) - \{O\}$ has exactly two connected components, and a choice of component is called an *orientation* on M. If M is not connected then it is orientable if each connected component is orientable in the above sense. Equivalently, the manifold M is orientable if and only if there is a nowhere-vanishing d-form on M. From now on we assume that M is *oriented*, i.e. it is equipped with an orientation. We say that a d-form ω is *positive* if (in each connected component of M) it belongs to the component corresponding to the chosen orientation.

Now let $f \in C_c^\infty(M, \mathbb{R})$ and assume that $\text{supp}(f) \subseteq U$ where (U, ϕ) is a chart. Let $\omega \in \Lambda^d(M)$ and suppose that $\omega(q) = h_\omega(x_1(q), \ldots, x_d(q))dx^1 \wedge \cdots \wedge dx^d$ for all $q \in U$. Then we may define the *integral* of the form $f\omega$ by

$$\int_M f\omega = \int_{\mathbb{R}^d} f \circ \phi^{-1}(x_1, \ldots, x_d)h_\omega(x_1, \ldots, x_d)dx_1 \ldots dx_d.$$

If ω is positive and (U, ϕ) is chosen (as it may be) so that $dx^1 \wedge \cdots \wedge dx^d$ is positive in $\phi(U)$, then $h_\omega > 0$ in $\phi(U)$. To define more general integrals we need a *partition of unity*, which is a collection of functions $(\psi_i, i \in I)$ in $C^\infty(M)$ for which

(i) At each $p \in M$ only finitely many of the ψ_is are non-zero,
(ii) For each $i \in I$ $\text{supp}(\psi_i)$ is compact,
(iii) For each $p \in M, i \in I, \psi_i(p) \geq 0$ and $\sum_{i\in I} \psi_i(p) = 1$.

Now suppose that $(U_\alpha, \alpha \in J)$ is an open cover of M. We say that a partition of unity is *subordinate* to this cover if for each $i \in I$ there exists $\alpha \in J$ such that $\text{supp}(\psi_i) \subseteq U_\alpha$. The key fact we need is that given a manifold M with an atlas

$\{(U_\alpha, \phi_\alpha), \alpha \in J\}$ there exists a partition of unity that is subordinate to the cover $(U_\alpha, \alpha \in J)$.

For arbitrary $\omega \in \Lambda^d(M)$ we may now define

$$\int_M \omega = \sum_{i \in I} \int_M \phi_i \omega,$$

whenever the right hand side converges absolutely, where the integral $\int_M \phi_i \omega$ is defined by the prescription we gave earlier. In particular, if M is compact, then we can cover M with finitely many of the U_αs, and so the series is a finite one.

In the discussion so far we have insisted that ω be a smooth form, but this is not essential. In fact it is sufficient for it to be continuous. Now let $\omega \in \Gamma^d(M)$ be positive and let $f \in C_c(M, \mathbb{R})$. Then we may define $\int_M f\omega$ as above. But $f \to \int_M f\omega$ is then a linear functional defined on $C_c(M, \mathbb{R})$, which is positive in that $f \geq 0 \Rightarrow \int_M f\omega \geq 0$. Hence by the Riesz representation theorem (see Appendix A.5), there is a unique regular Borel measure μ_ω on M so that

$$\int_M f(x)\mu_\omega(dx) = \int_M f\omega,$$

for all $f \in C_c(M, \mathbb{R})$, where the integral on the left hand side is in the Lebesgue sense. We call μ_ω the measure induced by the form ω. We use this process to construct Haar measure on a Lie group at the end of Sect. 1.3.2.

Let $g \in \bigotimes^{0,2}(M)$. Then for each $p \in M$, g_p is a real bilinear form on $T_p(M)$ which varies smoothly with p. We say that g is a *Riemannian metric* on M if g_p is an inner product on $T_p(M)$ for all $p \in M$. The pair (M, g) is then called a *Riemannian manifold*. The metric g induces a metric (in the usual sense of "metric spaces") on M which we denote as ρ_g by the prescription:

$$\rho_g(p, q) = \inf_\gamma \int_0^1 g_{\gamma(t)}(d\gamma(t), d\gamma(t))^{\frac{1}{2}} dt,$$

where the infimum is taken over all paths γ from [0, 1] to M such that $\gamma(0) = p$ and $\gamma(1) = q$. Note that paths are required to be continuous and at least piecewise differentiable. Every Riemannian manifold is orientable, and there exists a d-form called the *volume element* whose value at p (in local co-ordinates) is $\sqrt{g_p} dx_1 \wedge \cdots \wedge dx_d$, where g_p is the determinant of the matrix whose (i, j)th component is $g_p\left(\frac{\partial}{\partial i}, \frac{\partial}{\partial j}\right)$.

A.4 Appendix 4: The Symplectic and Spin Groups

The material in this appendix can be found in standard monographs on Lie groups such as Chevellay [45] or Bröcker and tom Dieck [36].

A.4.1 Quaternions and the Symplectic Group

The *quaternions* are the elements of the four dimensional real division algebra Q generated by $\{1, i, j, k\}$ satisfying the relations

$$i^2 = j^2 = k^2 = -1,$$

and

$$ij = -ji = k,\ jk = -kj = i, ki = -ik = j.$$

Any $q \in Q$ may be written $q = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, and its *conjugate* is $\tilde{q} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$. We obtain a norm $|\cdot|$ on Q by the prescription

$$|q| = (\tilde{q}q)^{\frac{1}{2}} = (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{\frac{1}{2}}.$$

We may also regard Q as a two dimensional complex vector space with basis $\{1, j\}$ by writing

$$q = \alpha_0 + \alpha_1\sqrt{-1} + j(\alpha_2 - \alpha_3\sqrt{-1}). \tag{A.3}$$

We can then regard the Cartesian product Q^n as a $4n$-dimensional real vector space or as a $2n$-dimensional complex vector space. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in Q^n$ we define their *symplectic product* to be

$$x \cdot y = \sum_{i=1}^{n} x_i \tilde{y}_i.$$

The symplectic product is a (real) inner product on Q^n and we define the *sympletic group* $Sp(n)$ to be the set of all matrices A in $M_n(Q)$ for which

$$Ax \cdot Ay = x \cdot y,$$

for all $x, y \in Q^n$.

By using (A.3) we can equivalently characterise $Sp(n)$ as the group of all $A \in U(2n)$ for which $A^T JA = J$, where $J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and this latter approach leads naturally to the description of the Lie algebra $\mathbf{sp(n)}$ given in Sect. 1.3.1.

Quaternions were invented by William Rowan Hamilton (1805–1865) to generalise complex numbers to higher dimensions. Hamilton was also responsible for the reformulation of classical mechanics that is now known as *Hamiltonian mechanics*. The symplectic group arises as the natural symmetry group in this context (see e.g. Dragt [58]).

A.4.2 Clifford Algebras and the Spin Group

Let V be a real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. The *Clifford algebra* $\mathrm{Cl}(V)$ over V is the real algebra generated by $\{q(v), v \in V\}$ and the identity I such that $q(\cdot)$ is linear and for each $u, v \in V$:

$$q(u)q(v) + q(v)q(u) = -2\langle u, v\rangle I. \tag{A.4}$$

Note that in particular $q(v)^2 = -||v||^2 I$ for each $v \in V$. If $\dim(V) = n$, then $\dim(\mathrm{Cl}(V)) = 2^n$.

From now on we take V to be $\mathbb{R}^n$ with its usual inner product and write $\mathrm{Cl}(n) := \mathrm{Cl}(\mathbb{R}^n)$. Then $\mathrm{Cl}(n)$ is generated by I and $\{q_1, \ldots, q_n\}$, where $q_i := q(e_i)$ and $\{e_1, \ldots, e_n\}$ is the natural basis in $\mathbb{R}^n$, i.e. for $1 \leq i \leq n, e_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0)$. Indeed, if $v = \sum_{i=1}^n v_i e_i \in \mathbb{R}^n$, then $q(v) = \sum_{i=1}^n v_i q_i$. The generating relations (A.4) are then equivalent to

$$q_i^2 = -I \text{ and } q_j q_k + q_k q_j = 0 \tag{A.5}$$

for each $1 \leq i \leq n, 1 \leq j < k \leq n$.

There is a natural vector space $\mathbb{Z}_2$ grading on $\mathrm{Cl}(n)$, so that we can write

$$\mathrm{Cl}(n) = \mathrm{Cl}(n)_+ \oplus \mathrm{Cl}(n)_-,$$

where $\mathrm{Cl}(n)_+$ is the vector space spanned by I and $\{q_{i_1} \ldots q_{i_{2m}}; 1 \leq i_1 < \cdots < i_{2m} \leq n, m \in \mathbb{N}\}$ and $\mathrm{Cl}(n)_-$ is the vector space spanned by $\{q_{j_1} \ldots q_{j_{2p-1}}; 1 \leq j_1 < \cdots < j_{2p-1} \leq n, p \in \mathbb{N}\}$. Note that $\mathrm{Cl}(n)_+$ is a subalgebra of $\mathrm{Cl}(n)$, but $\mathrm{Cl}(n)_-$ is not closed under products, indeed

$$\mathrm{Cl}(n)_+\mathrm{Cl}(n)_+ \subseteq \mathrm{Cl}(n)_+,\ \mathrm{Cl}(n)_+\mathrm{Cl}(n)_- \subseteq \mathrm{Cl}(n)_-,\ \mathrm{Cl}(n)_-\mathrm{Cl}(n)_+ \subseteq \mathrm{Cl}(n)_- \\ \text{and } \mathrm{Cl}(n)_-\mathrm{Cl}(n)_- \subseteq \mathrm{Cl}(n)_+.$$

We now define three groups, the third of which is the main object of interest:

$\mathrm{Cl}^\times(n)$ is the group of all invertible elements in $\mathrm{Cl}(n)$.

$$\text{Pin}(n) \text{ is the group generated by } \{q(v) \in \text{Cl}_n^{\times}; ||v|| = 1\}.$$

$$\text{Spin}(n) = \text{Pin}(n) \cap \text{Cl}(n)_+ \text{ is the } \textit{spin group}.$$

We will not give a full proof here that $\text{Spin}(n)$ is the covering group of $SO(n)$ for $n \geq 3$, but we will demonstrate that there is indeed a natural relationship between these two groups. Let W be the real vector space spanned by $\{q_1, \ldots, q_n\}$ and consider the automorphism $T(g)(v) = gvg^{-1}$ for $g \in \text{Spin}(n)$, $v \in \text{Cl}(n)$. It can be shown that $T(g) : W \subseteq W$, and so there is an $n \times n$ matrix $(T_{ij}(g))$ such that $T(g)q_i = \sum_{j=1}^{n} T_{ij}(g)q_j$ for each $1 \leq i \leq n$. Hence for arbitrary $w \in W$ with $w = \sum_{i=1}^{n} w_i q_i$ we have

$$T(g)w = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} w_i T_{ij}(g) \right).$$

Now since $T(g)$ is an automorphism we have $T(g)v^2 = T(g)v \cdot T(g)v$. Computing both sides of this identity and using (A.5), we obtain

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{n} w_i T_{ij}(g) \right)^2 = \sum_{i=1}^{n} \alpha_i^2,$$

i.e. the matrix $T(g)$ is orthogonal. Now for each $g, h \in \text{Spin}(n)$, $T(gh) = T(g)T(h)$ hence $\det(T(gh)) = \det(T(g))\det(T(h))$, and so we must have $\det(T(g)) = 1$ for all $g \in \text{Spin}(n)$. Hence we deduce that $T : \text{Spin}(n) \to SO(n)$ is a homomorphism.

Spin groups and associated spin structures play a key role in differential geometry and its applications to quantum physics, particularly through the study of Dirac operators, and applications to index theory. For a nice introduction to this rich area, see the article by Boi [33] (it has a very large bibliography for those who want to explore further), or the book by Roe [168].

A.5 Appendix 5: Measures on Locally Compact Spaces

In this section we gather together some useful results about measures on locally compact spaces, most of which can be found in either Cohn [50] or Bauer [21].

Throughout this section X is always assumed to be locally compact topological space and $\mathcal{B}(X)$ is its Borel σ-algebra, i.e. the smallest σ-algebra of subsets of X containing all open sets of X. Let μ be a Borel measure on X, i.e. a measure defined on $(X, \mathcal{B}(X))$. We say that μ is *outer regular* if for all $A \in \mathcal{B}(X)$,

$$\mu(A) = \inf\{\mu(O); A \subset O, O \text{ is open in X}\}.$$

The measure μ is said to be *inner regular* if

$$\mu(A) = \sup\{\mu(C); C \subset A, C \text{ is compact in X}\}.$$

We say that μ is *regular* if it is both inner and outer regular, and also satisfies $\mu(C) < \infty$ for all compact sets C in X.

Here are some useful facts about regular measures:

- If X is second countable and Hausdorff, then every Borel measure defined on X that is finite on compact sets is regular.
- If X is second countable and Hausdorff, then every regular measure defined on X is σ-finite.

Proposition A.0.2 *If μ is a non-trivial regular Borel measure on a locally compact Hausdorf space X then*

1. $C_c(X, \mathbb{R})$ *is dense in* $L^p(X, \mathcal{B}(X), \mu; \mathbb{R})$ *for* $1 \le p < \infty$,
2. $C_c(X, \mathbb{C})$ *is dense in* $L^p(X, \mathcal{B}(X), \mu; \mathbb{C})$ *for* $1 \le p < \infty$.

Proof

1. If $E \in \mathcal{B}(X)$ with $\mu(E) < \infty$ then by regularity of μ, given any $\epsilon > 0$, we can find a compact K and an open U with $K \subset E \subset U$ and $\mu(U) < \mu(K) + \epsilon$. By a corollary to Urysohn's lemma (see Appendix A.1) there exists $h \in C_c(X, \mathbb{R})$ so that $1_K \le h \le 1_U$. Moreover

$$||1_E - h||_p^p \le ||1_U - 1_K||_p^p = \mu(U \setminus K) < \epsilon,$$

and the result follows.
2. follows from (1) by approximating real and imaginary parts.

□

Note If M is a manifold, then the result of Proposition A.5.1 may be refined to prove that $C_c^\infty(M, \mathbb{R})$ is dense in $L^p(M, \mathcal{B}(M), \mu; \mathbb{R})$ (with an analogous result for the complex case) by using cut-off functions instead of Urysohn's lemma (see e.g. the argument in Grigor'yan [74] pp. 20–21).

A Borel measure μ on X is said to be *locally finite* if every point of X has an open neighbourhood U such that $\mu(U) < \infty$, and μ is said to be a *Radon measure* if it is both inner regular and locally finite.

Proposition A.0.3 *1. A Borel measure μ on a locally compact topological space X is locally finite if and only if it is finite on compacta.*
2. Every regular Borel measure on X is a Radon measure.

Proof

1. Suppose that μ is locally finite and let K be a compact subset of X. Then every point in K has an open neighbourhood that has finite measure, and the collection of all such neighbourhoods is an open cover for K. It follows that there are a finite number of these, say $V_1, \ldots, V_N$, which also cover K. But then

$$\mu(K) \leq \mu\left(\bigcup_{i=1}^{N} V_i\right) \leq \sum_{i=1}^{N} \mu(V_i) < \infty.$$

   The converse is a consequence of the definition of local compactness.
2. This follows immediately from (1), and the definitions.

□

It can be shown that if X is second countable and Hausdorff (and not necessarily locally compact), then every inner regular Borel measure on X which is finite on compact sets is a Radon measure and so if X is also locally compact, then every Borel measure that is finite on compact sets is regular and Radon. In particular:

If M is a smooth manifold , then every Borel measure on M that is finite on compact sets is regular and Radon. In particular, every finite Borel measure on M is regular and Radon.

We now describe the Riesz representation theorem, which is used extensively in this book. A (real) linear functional I on $C_c(X, \mathbb{R})$ is said to be *positive* if $f \geq 0 \Rightarrow I(f) \geq 0$.

Theorem A.0.2 (Riesz representation theorem) *Let X be locally compact and Hausdorff. If I is a positive linear functional on* $C_c(X, \mathbb{R})$*, then there is a unique regular Borel measure* μ *on X such that for all* $f \in C_c(X, \mathbb{R})$*,*

$$I(f) = \int_X f(x)\mu(dx).$$

A *signed Borel measure* on X is a σ-additive set function ν defined on $\mathcal{B}(X)$ and taking values in $[-\infty, \infty]$ for which $\nu(\emptyset) = 0$. It is said to be *finite* if $|\nu(X)| \leq \infty$. The *Jordan decomposition* of a signed measure ν states that we have a unique decomposition $\nu = \nu_+ - \nu_-$, where ν_+ and ν_- are positive measures at least one of which is finite. The *total variation* of ν is defined to be $||\nu|| := \nu_+(X) + \nu_-(X)$. The concepts of (inner, outer) regularity as defined above extend to signed measures in an obvious way. Let $\mathcal{M}(X)$ denote the set of all regular signed measures on X which have finite total variation. It becomes a real vector space with respect to the operations

$$(\nu_1 + \nu_2)(A) = \nu_1(A) + \nu_2(A),$$

$$\text{and} \ \ (\lambda\nu)(A) = \lambda\nu(A),$$

for $\nu_1, \nu_2, \nu \in \mathcal{M}(X)$ and $\lambda \in \mathbb{R}$, and is a real Banach space with respect to the total variation norm.

If f is a measurable function defined on X and ν is a signed measure on X, we can define

$$\int_X f(x)\nu(dx) := \int_X f(x)\nu_+(dx) - \int_X f(x)\nu_-(dx),$$

provided at least one of the integrals appearing on the right hand side is finite.

The next result is a variant of the Riesz representation theorem. Let $C_0(X, \mathbb{R})$ be the real Banach space (under the supremum norm $||\cdot||_\infty$) of all real-valued continuous functions on X which vanish at infinity. Note that if ν is a finite signed measure on X, then I_ν is a bounded linear functional on $C_0(X, \mathbb{R})$, where for each $f \in C_0(X, \mathbb{R})$

$$I_\nu(f) := \int_X f(x)\nu(dx).$$

Indeed we easily see that $|I_\nu(f)| \leq ||f||_\infty ||\nu||$ and so $||I_\nu|| \leq ||\nu||$.

Theorem A.0.3 *If X is locally compact and Hausdorff, then the mapping $\nu \to I_\nu$ is an isometric isomorphism of $\mathcal{M}(X)$ onto the continuous dual of $C_0(X, \mathbb{R})$.*

An immediate consequence of Theorem A.5.2 is the useful identity:

$$||\nu|| = \sup\left\{\left|\int_X f(x)\nu(dx)\right|; ||f||_\infty = 1\right\}.$$

A.6 Appendix 6: Compact, Hilbert-Schmidt and Trace-Class Operators

The material is this appendix is often founds in texts on functional analysis such as Sects. 4.2, 5.5 and 5.6 of Davies [54] and Sects. VI.5 and 6 of Reed and Simon [166]. We also found the on-line notes of Walter [213] to be particularly helpful.

A.6.1 Compact Operators

Let E and F be real or complex Banach spaces. A bounded linear operator $T : E \to F$ is said to be *compact* if it maps the unit ball in E to a compact set in F. Equivalently, whenever $(x_n, n \in \mathbb{N})$ is a bounded sequence in E, then $(Tx_n, n \in \mathbb{N})$

has a convergent subsequence in F. From now on we will take $E = F = H$ to be a complex separable Hilbert space.

Theorem A.0.4 *If* $T : H \to H$ *is compact, then there exist complete orthonormal bases* $(e_n, n \in \mathbb{N})$ *and* $(f_n, n \in \mathbb{N})$ *and a sequence of complex numbers* $(c_n, n \in \mathbb{N})$ *with* $\lim_{n\to\infty} c_n = 0$ *such that*

$$T = \sum_{n=1}^{\infty} c_n \langle \cdot, e_n \rangle f_n. \tag{A.6}$$

A bounded linear operator on H is of *finite rank* if it has finite range. Let $\mathcal{F}(H)$ be the linear space of all finite rank operators on H. Then Theorem A.6.1 expresses the fact that every compact operator in H is a uniform limit of finite rank operators.

Theorem A.0.5 (Riesz-Schauder) *If* $T : H \to H$ *is compact, then its spectrum* $\sigma(T)$ *is discrete and has no limit points except possibly* $\{0\}$. *If* $\lambda \in \sigma(T)$, *then* λ *is an eigenvalue of* T *having at most finite multiplicity.*

Theorem A.0.6 (Hilbert-Schmidt) *If* $T : H \to H$ *is compact and self-adjoint, then it has a complete orthonormal basis of eigenvectors, and the corresponding eigenvalues* $\{\lambda_n, n \in \mathbb{N}\}$ *satisfy* $\lim_{n\to\infty} \lambda_n = 0$.

In fact if T is both compact and self adjoint, then we can write the representation (A.6) as

$$T = \sum_{n=1}^{\infty} \lambda_n \langle \cdot, e_n \rangle e_n, \tag{A.7}$$

where e_n is the normalised eigenvector corresponding to $\lambda_n (n \in \mathbb{N})$.

Let $\mathcal{C}(H)$ be the linear space of all compact operators on H.

1. $\mathcal{C}(H)$ is a closed subspace of $\mathcal{L}(H)$.
2. $\mathcal{C}(H)$ is a (two-sided) *-ideal of $\mathcal{L}(H)$, i.e. $T \in \mathcal{C}(H)$ if and only if $T^* \in \mathcal{C}(H)$ and if $T \in \mathcal{C}(H)$, $X, Y \in \mathcal{L}(H)$, then $XTY \in \mathcal{C}(H)$.

A.6.2 Hilbert-Schmidt Operators

We say that a bounded linear operator T acting on H is Hilbert-Schmidt if $\sum_{n=1}^{\infty} ||Te_n||^2 < \infty$ for some (and hence all) complete orthonormal basis $(e_n, n \in \mathbb{N})$ in H. Let $\mathcal{I}_2(H)$ denote the linear space of all Hilbert-Schmidt operators on H. It is itself a complex Hilbert space with respect to the inner product

$$\langle S, T \rangle_2 = \sum_{n=1}^{\infty} \langle Se_n, Te_n \rangle,$$

where $S, T \in \mathcal{I}_2(H)$. The associated norm will be denoted $|| \cdot ||_2$.

1. $\mathcal{I}_2(H)$ is a (two-sided) *-ideal of $\mathcal{L}(H)$.
2. $\mathcal{I}_2(H) \subseteq \mathcal{C}(H)$ and if $T \in \mathcal{C}(H)$, then $T \in \mathcal{I}_2(H)$ if and only if $\sum_{n=1}^{\infty} |c_n|^2 < \infty$ in (A.6) and we then have

$$||T||_2 = \left(\sum_{n=1}^{\infty} |c_n|^2 \right)^{\frac{1}{2}}. \tag{A.8}$$

3. $\mathcal{F}(H)$ is $|| \cdot ||_2$-dense in $\mathcal{I}_2(H)$.

We have a very precise description of Hilbert-Schmidt operators acting in L^2 spaces.

Theorem A.0.7 *Let (S, Σ, μ) be a measure space wherein the σ-algebra Σ is countably generated. Let $H = L^2(S, \Sigma, \mu)$. Then $T \in \mathcal{I}_2(H)$ if and only if for all $f \in H, x \in S$*

$$(Tf)(x) = \int_S f(y)k(x, y)\mu(dy),$$

where the kernel $k \in L^2(S \times S, \Sigma \otimes \Sigma, \mu \times \mu)$. Moreover

$$||T||_2^2 = \int_S \int_S |k(x, y)|^2 \mu(dx)\mu(dy).$$

Proof (Sketch). First we consider sufficiency. Since Σ is countably generated, H is separable. Let $(e_n, n \in \mathbb{N})$ be a complete orthonormal basis for H. Then $(e_m \otimes \overline{e_n}, m, n \in \mathbb{N})$ is a complete orthonormal basis for $H \otimes H = L^2(S \times S, \Sigma \otimes \Sigma, \mu \times \mu)$. Given $k \in H \otimes H$, we may assert that there exists $(c_{m,n}, m, n \in \mathbb{N}) \in l^2(\mathbb{N}^2)$ so that

$$k = \sum_{m,n=1}^{\infty} c_{m,n} e_m \otimes \overline{e_n}.$$

Then for all $k \in \mathbb{N}, x \in S$,

$$Te_k(x) = \int_S \left(\sum_{m,n=1}^{\infty} c_{m,n} e_m(x) \overline{e_n(y)} \right) e_k(y)\mu(dy) = \sum_{m=1}^{\infty} c_{m,k} e_m(x).$$

Hence

$$\sum_{m=1}^{\infty} ||Te_k||^2 = \sum_{m,k=1}^{\infty} |c_{m,k}|^2 = ||k||_{H \otimes H}^2 < \infty,$$

and so $T \in \mathcal{I}_2(H)$. For necessity it can be shown that $T \in \mathcal{I}_2(H)$ has a kernel of the form $k = \sum_{m,n=1}^{\infty} c_{m,n} e_m \otimes \overline{e_n}$ by making an approximation with finite-rank operators. □

A.6.3 Trace-Class Operators

If $T \in \mathcal{L}(H)$ the operator T^*T is positive and self-adjoint. Hence by spectral theory, it has a positive self-adjoint square root which we denote as $|T| = (T^*T)^{\frac{1}{2}}$. We say that T is *trace class* if

$$\mathrm{Tr}(|T|) := \sum_{n=1}^{\infty} \langle |T| e_n, e_n \rangle < \infty,$$

for some (and hence all) orthonormal basis $(e_n, n \in \mathbb{N})$ of H. The linear space of all trace class operators on H is denoted $\mathcal{I}_1(H)$. It is a Banach space with respect to the norm $|| \cdot ||_1$, where $||T||_1 := \mathrm{Tr}(|T|)$ for $T \in \mathcal{I}_1(H)$.

1. $\mathcal{I}_1(H)$ is a (two-sided) *-ideal of $\mathcal{L}(H)$.
2. $\mathcal{I}_1(H) \subseteq \mathcal{C}(H)$ and if $T \in \mathcal{C}(H)$, then $T \in \mathcal{I}_1(H)$ if and only if $\sum_{n=1}^{\infty} |c_n| < \infty$ in (A.6).
3. $\mathcal{F}(H)$ is $|| \cdot ||_1$-dense in $\mathcal{I}_1(H)$.

Note also that if T is of the form (A.6), then $|T| = \sum_{n=1}^{\infty} |c_n| \langle \cdot, e_n \rangle e_n$ and

$$||T||_1 = \sum_{n=1}^{\infty} |c_n|. \tag{A.9}$$

Theorem A.0.8 *$\mathcal{I}_1(H)$ is continuously embedded into $\mathcal{I}_2(H)$. Indeed if $T \in \mathcal{I}_1(H)$, then $||T||_2 \leq ||T||_1$.*

Proof This follows from comparing (A.9) with (A.8). □

Define $\mathcal{I}_1'(H) := \{T \in \mathcal{I}_2(H); ||T||_1' < \infty\}$, where the norm

$$||T||_1' := \sup\{\langle T, X \rangle_2, X \in \mathcal{L}(H), ||X|| \leq 1\}.$$

Theorem A.0.9 *$\mathcal{I}_1'(H) = \mathcal{I}_1(H)$ and $|| \cdot ||_1 = || \cdot ||_1'$.*

Proof We follow Walter [213]. Let $T \in \mathcal{I}_1(H)$ with $T = \sum_{n=1}^{\infty} c_n \langle \cdot, e_n \rangle f_n$. Then if $X \in \mathcal{L}(H)$ with $||X|| \leq 1$ we have

$$
\begin{aligned}
|\langle T, X\rangle_2| &= \left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_n \langle e_m, e_n\rangle\langle f_n, Xe_m\rangle\right| \\
&\le \sum_{n=1}^{\infty} |c_n| ||e_n||.||X||.||f_n|| \\
&\le \sum_{n=1}^{\infty} |c_n|,
\end{aligned}
$$

and so $T \in \mathcal{I}_1'(H)$ with $||T||_1' \le ||T||_1$. Conversely, if $T \in \mathcal{I}_1'(H)$ choose $d_n \in \mathbb{C}$ so that $|d_n| = 1$ and $|c_n| = c_n d_n$ for all $n \in \mathbb{N}$ and consider the finite-rank operator $X_N := \sum_{n=1}^{N} d_n \langle \cdot, e_n\rangle f_n$. We easily check that $||X_N|| \le 1$, and then

$$
\begin{aligned}
||T||_1' &\ge |\langle T, X_N\rangle_2| \\
&= \left|\sum_{n=1}^{\infty}\langle Te_n, X_N e_n\rangle\right| \\
&= \left|\sum_{n=1}^{N} c_n d_n\right| \\
&= \sum_{n=1}^{N} |c_n| \\
&\to ||T||_1 \text{ as } N \to \infty,
\end{aligned}
$$

where we have used (A.9). Hence $T \in \mathcal{I}_1(H)$ and the required result follows. □

The main purpose of introducing the norm $||\cdot||_1'$ is for ease of proving the next theorem.

Theorem A.0.10 *$T \in \mathcal{I}_1(H)$ if and only if there exist $S_1, S_2 \in \mathcal{I}_2(H)$ so that $T = S_1 S_2$.*

Proof We again follow Walter [213].
(Necessity) If T has the form (A.6), then just choose $S_1 = \sum_{n=1}^{\infty} \sqrt{c_n}\langle \cdot, e_n\rangle f_n$ and $S_2 = \sum_{n=1}^{\infty} \sqrt{c_n}\langle \cdot, e_n\rangle e_n$.
(Sufficiency) Let $S_1, S_2 \in \mathcal{I}_2(H)$ and $X \in \mathcal{I}_2(H)$ with $||X|| \le 1$. Then

$$
\begin{aligned}
|\langle S_1 S_2, X\rangle_2| &= |\langle S_2, S_1^* X\rangle_2| \\
&\le ||S_2||_2 ||S_1^* X||_2 \\
&= ||S_2||_2 ||X^* S_1||_2 \\
&\le ||S_2||_2 ||X||.||S_1||_2 \\
&\le ||S_1||_2.||S_2||_2.
\end{aligned}
$$

We then conclude that $S_1 S_2 \in \mathcal{I}_1(H)$ with $||S_1 S_2||_1 \le ||S_1||_2.||S_2||_2$. □

A.7 Appendix 7: Semigroups of Linear Operators

In this brief appendix we describe some of the key basic ideas of the analytic theory of semigroups. Standard references are Davies [53] which is revised and updated within Chaps. 6–8 of Davies [54], and Pazy [158].

A C_0-*semigroup* acting in a (real, or complex) Banach space E is a family $(T_t, t \geq 0)$ of bounded linear operators acting on E satisfying the conditions:

(i) (Semigroup condition) $T_{s+t} = T_s T_t$ for all $s, t \geq 0$,
(ii) $T_0 = I$,
(iii) (Strong continuity) The mapping $t \to T_t\psi$ is continuous from $\mathbb{R}^+$ to E for all $\psi \in E$.

A C_0 semigroup is said to be a *contraction semigroup* if T_t is a contraction for all $t \geq 0$. We define a subset $\mathcal{D}_L$ of E by the prescription

$$\mathcal{D}_L := \left\{ \psi \in E, \text{there exists } \phi_\psi \in E \text{ so that } \lim_{t\to 0} \left|\left| \frac{T_t\psi - \psi}{t} - \phi_\psi \right|\right| = 0 \right\}.$$

Then $\mathcal{D}_L$ is a linear subspace of E and the mapping $L : \mathcal{D}_L \to E$ defined by $L\psi = \phi_\psi$ is well-defined and linear. We call L the *infinitesimal generator* of the semigroup $(T_t, t \geq 0)$. Then $\mathcal{D}_L$ is the domain of L, and we have

$$L\psi = \lim_{t\to 0} \frac{T_t\psi - \psi}{t},$$

for all $\psi \in \mathcal{D}_L$, i.e. $L\psi = \left.\dfrac{dT_t\psi}{dt}\right|_{t=0}$ in the sense of the strong derivative on Banach space.

Theorem A.0.11 *The infinitesimal generator of a C_0-semigroup is densely defined.*

Proof We must show that D_L is dense in E. Let $\psi \in E$ be arbitrary and fix $u > 0$. Define $\phi_u = \int_0^u T_t\psi dt$. Then $\phi_u \in \mathcal{D}_L$ since for all $s > 0$,

$$\begin{aligned}
\frac{T_s\phi_u - \phi_u}{s} &= \frac{1}{s}\left\{ \int_0^u T_{s+t}\psi dt - \int_0^u T_t\psi dt \right\} \\
&= \frac{1}{s}\left\{ \int_s^{s+u} T_t\psi dt - \int_0^u T_t\psi dt \right\} \\
&= \frac{1}{s}\left\{ \int_u^{s+u} T_t\psi dt - \int_0^s T_t\psi dt \right\} \\
&\to T_u\psi - \psi, \text{ as } s \to 0.
\end{aligned}$$

Now $\psi = \lim_{u\to 0} \frac{1}{u} \int_0^u T_t\psi dt$ and the result follows. □

Theorem A.0.12 *For all* $t \geq 0$, $T_t\mathcal{D}_L \subseteq \mathcal{D}_L$ *and if* $\psi \in \mathcal{D}_L$

(i) $T_t L\psi = LT_t\psi$,

(ii) $\dfrac{d}{dt}T_t\psi = LT_t\psi$,

(iii) $T_t\psi - \psi = \int_0^t T_s L\psi ds$.

Proof (Sketch) Everything follows from the following

$$\begin{aligned} LT_t\psi &= \lim_{s\to 0}\frac{T_{s+t}\psi - T_t\psi}{s} \\ &= T_t \lim_{s\to 0}\frac{T_s\psi - \psi}{s} \end{aligned}$$

□

We can interpret Theorem A.7.2 (ii) as telling us that $u(t) = T_t\psi$ is the unique solution of the abstract E-valued differential equation $\dfrac{du(t)}{dt} = Lu(t)$ with initial condition $u(0) = \psi$. Many examples of this equation where $E = C_0(G, \mathbb{R})$ are studied in Chap. 5, where G is a Lie group and $\mathcal{L}$ is the Hunt generator of a convolution semigroup. Another much-studied example is the *heat semigroup* on $C_0(M, \mathbb{R})$, where M is a Riemannian manifold and L is the Laplace-Beltrami operator Δ. If M is compact, there is a *heat kernel* $k \in C^\infty((0, \infty) \times M \times M, \mathbb{R})$ (i.e. a fundamental solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u$) so that

$$T_t f(x) = \int_M f(y)k_t(x, u)\mu(dy),$$

for all $t > 0$, $f \in C(M, \mathbb{R})$, $x \in M$, where μ is the volume measure on M. Another important class of semigroups are those generated by the Schrödinger operators $L = -\Delta + V$, where V is a suitable multiplication operator acting on $L^2(M, \mu)$.

Theorem A.0.13 *L is a closed operator.*

Proof Let $(\phi_n, n \in \mathbb{N})$ be a sequence of vectors in E with $\phi_n \in D_L$ for all $n \in \mathbb{N}$ which converges to $\phi \in E$ and suppose that we also have $L\phi_n \to \psi \in E$. We are done if we can show that $\phi \in D_L$ and that $\psi = L\phi$. To establish this, observe that by Theorem A.7.2 (iii), for all $n \in \mathbb{N}$, $t > 0$,

$$\frac{1}{t}(T_t\phi_n - \phi_n) = \frac{1}{t}\int_0^t T_s L\phi_n ds.$$

Now take limits as $n \to \infty$ to obtain

$$\frac{1}{t}(T_t\phi - \phi) = \frac{1}{t}\int_0^t T_s\psi ds.$$

Finally we take limits as $t \to 0$ to find that

$$\lim_{t \to 0} \frac{1}{t}(T_t \phi - \phi) = \psi,$$

and the result follows. □

A C_0-semigroup becomes a *one-parameter group* of operators if it can be extended to a family of bounded operators $(T_t, t \in \mathbb{R})$ on E so that $T_{s+t} = T_s T_t$ for all $s, t \in \mathbb{R}$. An example of great importance arises when E is a complex Hilbert space (which we henceforth denote as $\mathcal{H}$), and each T_t is a unitary operator (which from now on, we write as U_t) acting on $\mathcal{H}$. We then call $(U_t, t \in \mathbb{R})$ a *one-parameter unitary group*. A deep insight into these groups is given by the following:

Theorem A.0.14 (Stone's theorem) *If* $(U_t, t \in \mathbb{R})$ *is a one-parameter group acting in a complex, separable Hilbert space* $\mathcal{H}$*, then there exists a self-adjoint operator* H *acting in* $\mathcal{H}$ *so that for all* $t \in \mathbb{R}$*,*

$$U_t = e^{itH}. \tag{A.10}$$

For a proof, see e.g. Reed and Simon [166] Theorem VIII.8, pp. 266–268. Note that the operator H is typically unbounded (unless $\mathcal{H}$ is finite-dimensional), and the exponential in (A.10) is defined via spectral theory.

Stone's theorem plays an important role in quantum mechanics. Fix $\psi \in \mathcal{H}$ and define $\psi(t) = U(t)\psi$, for $t \in \mathbb{R}$. Then the mapping $t \to \psi(t)$ is strongly differentiable, and formally differentiating in (A.10) yields an abstract form of *Schrödinger's equation*:

$$\frac{d\psi(t)}{dt} = iH\psi(t),$$

in which the operator H is playing the role of the Hamiltonian. Stone's theorem may also be utilised to give a delightful proof of Bochner's theorem, see Reed and Simon [166] Theorem IX.9, pp. 330–331.

A.8 Appendix 8: Cores of Closed Linear Operators on Banach Spaces

Let E be a real or complex separable Banach space and $T : E \to E$ be a densely defined closed linear operator with domain Dom(T). A linear manifold $\mathcal{D} \subset$ Dom(T) is said to be a *core* for T if $\overline{T|_{\mathcal{D}}} = T$. Equivalently, given any $g \in$ Dom(T) there exists a sequence $(g_n, n \in \mathbb{N})$, where $g_n \in \mathcal{D}$ for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} g_n = g$ and $\lim_{n\to\infty} Tg_n = Tg$ (see e.g. Davies [54]). Since T is closed, its *resolvent set* $\rho_T := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}$ is a non-empty open set in $\mathbb{C}$. For each

$\lambda \in \rho(T)$, the *resolvent* $R_\lambda := (\lambda I - T)^{-1}$ is a bounded linear operator from E onto Dom(T). Hence $\lambda I - T$ maps Dom(T) onto E.

Our main result is the following (see also the first part of the proof of Proposition 1.1 in Breuillard [35]):

Theorem A.0.15 *The following are equivalent.*

1. *$\mathcal{D}$ is a core for T.*
2. *$\lambda I - T$ maps $\mathcal{D}$ to a dense linear manifold in E for some $\lambda \in \rho_T$.*
3. *$\lambda I - T$ maps $\mathcal{D}$ to a dense linear manifold in E for all $\lambda \in \rho_T$.*

Proof (2) ⇒ (1). Let $g \in$ Dom(T) be arbitrary. By assumption there exists $\lambda \in \rho_T$ and a sequence $(g_n, n \in \mathbb{N})$ where $g_n \in \mathcal{D}$ for all n such that $\lim_{n\to\infty}(\lambda g_n - T g_n) = \lambda g - T g$. Applying the bounded operator R_λ, we find that $\lim_{n\to\infty} g_n = g$. Then $\lim_{n\to\infty} T g_n = T g$ follows easily.

(1) ⇒ (3) Conversely, let $\mathcal{D}$ be a core for T. Given any $\lambda \in \rho_T$ we know that $\lambda I - T$ maps Dom(T) onto E. Hence given any $f \in E$ we can find $g \in$ Dom(T) so that $f = \lambda g - T g$. But by the core property, given any $\epsilon > 0$ there exists $g' \in \mathcal{D}$ so that $||(\lambda I - T)(g - g')|| < \epsilon$. The result follows.

(3) ⇒ (2) is obvious. □

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