Part IV
Persistence

Exercises

Question 1. ($20 = 6 + 7 + 7$ points). Let S be the set of points $A = (-1, 0, 0)$, $B = (1, 0, 0)$, $C = (0, -1, 0)$, $D = (0, 1, 0)$, $E = (0, 0, -1)$, $F(0, 0, 1)$ in $\mathbb{R}^3$. Note that the convex hull of S is a regular octahedron.

(a) Draw the simplicial complex, K, whose vertices are the six points in S, that contain the edge connecting ($\pm$1, 0, 0), and whose underlying space is the convex hull of S?

(b) List the simplices of K such that σ precedes τ either because $\dim \sigma < \dim \tau$ or because the dimensions are equal and the ordering of the vertices, sorted by name, is lexicographically smaller for σ.

(c) Draw the barcode and the persistence diagram of the filtration defined by the filter in (b).

Question 2. ($20 = 6 + 7 + 7$ points). Let S be a set of n points in general position in $\mathbb{R}^2$.

(a) Show that the number of different alpha complexes of S is less than $5n$.

(b) Show that the number of different Vietoris–Rips complexes of S is less than n^2.

(c) Show that the number of different Čech complexes of S is less than n^3.

Question 3. ($20 = 10 + 10$ points). The *dunce cap* is a two-dimensional topological space obtained by gluing the three edges of a triangle, as shown in Fig. 1.

(a) Draw a triangulation of the dunce cap (making sure that the identification of vertices and edges on the boundary of the triangle does not violate basic properties of a simplicial complex).

(b) Compute the Betti numbers of the dunce cap.

Question 4. ($20 = 6 + 7 + 7$ points). Let A, B, C three vertices of a triangle, Pick two numbers S, $T \in$ [0, 1], independently and uniformly at random from the unit

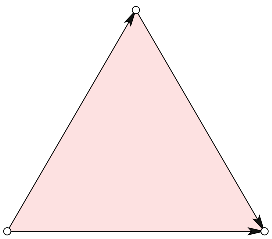

Fig. 1 The dunce cap. Gluing the *left* to the *bottom* edge, we get a cone with a rim and a seam. Gluing the rim to the seam, we get the dunce cap, which is perhaps similar to a snail house

interval, and set $s = \min\{S, T\}$ and $t = \max\{S, T\}$. Finally, define $\alpha = s$, $\beta = t - s$, $\gamma = 1 - t$, interpreting them as the barycentric coordinates of the point $x = \alpha A + \beta B + \gamma C$.

(a) What is the probability that S and T are both smaller than or equal to $\frac{1}{2}$? Draw the corresponding region of points in the triangle?
(b) What are the expected values of α, β, γ?
(c) Prove that x is chosen uniformly at random from the triangle.

Question 5. ($20 = 10 + 10$ points). Let $X = (b, d)$ and $X' = (b', d')$ be points in the plane. The L_1 -and L_∞ -distances between them are $\| X - X' \|_1 = | b - b' | + | d - d' |$; $\| X - X' \|_\infty = \max\{| b - b' |, | d - d' |\}$.

(a) Draw the unit disk (the points at distance at most 1 from the origin) under both definitions of distance.
(b) Define $A = (b + d, d - b)$ and $A' = (b' + d', d' - b')$. Prove $\| A - A' \|_1 = 2\| X - X' \|_\infty$.

Question 6. (20 points). Recall the two matrix reduction algorithms for persistent homology given in Chap. 13. Construct a boundary matrix such that the two algorithms produce different reduced matrices.