Part III
Homology

Exercises

Question 1. $(20 = 10 + 10$ points). Recall the conditions (i) and (ii) in the definition of a simplicial complex.

(a) Give a small collection of simplices that violates (i) but not (ii).
(b) Give a small collection of simplices that violates (ii) but not (i).

Question 2. $(20 = 10 + 10$ points). Let K be a triangulation of a compact surface, and let L be another triangulation obtained by refining K. In other words, each simplex in L is contained in a simplex in K.

(a) Determine the Euler characteristic of a simplicial complex that triangulates $\mathbb{B}^2$.
(b) Use the result in (a) to prove that K and L have the same Euler characteristic.

Question 3. $(20 = 10 + 10$ points). Consider the one-dimensional simplicial complex with vertices A, B, C, D, E and edges AB, AC, AD, BE, CE, DE.

(a) What are the ranks of C_1, Z_1 and B_0?
(b) How many 1-chains have the same boundary? In other words, how many elements of C_1 map to the same element in B_0?

Question 4. $(20 = 10 + 10$ points). Recall the four 2-manifolds depicted in Figs. 9.2 and 9.3.

(a) What are the Betti numbers of the projective plane?
(b) What are the Betti numbers of the double torus, $T_2 = \mathbb{T}^2 \# \mathbb{T}^2$?

Question 5. ($20 = 7 + 7 + 6$ points). Let A_n^2 be a two-dimensional abstract simplicial complex with n vertices and $\binom{n}{3}$ triangles. Determine the largest value of n such that A_n^2 has a geometric realization

  (a) in $\mathbb{R}^2$;
  (b) in $\mathbb{R}^3$;
  (c) in $\mathbb{R}^5$.

Question 6. ($20 = 6 + 7 + 7$ points). Let A be an abstract simplicial complex, and v, w vertices not in A. The *suspension* of A consists of all simplices of A together with $\{v\}$, $\{w\}$, and all simplices of the form $\alpha \cup \{v\}$ and $\alpha \cup \{w\}$, for $\alpha \in A$.

  (a) Verify that the suspension of A is an abstract simplicial complex.
  (b) Describe the space triangulated by the suspension of $\mathbb{S}^1$.
  (c) What are the Betti numbers of the suspension of the letter B?