Part II
Complexes

Exercises

Question 1. ($20 = 10 + 10$ points). Let S be a finite set of points in $\mathbb{R}^2$, s and t two points in S, and α positive real number. Recall that $D_t(\alpha)$ is the α-disk centered at $s \in S$, V_s is the Voronoi region of s, and $R_S(\alpha) = D_s(\alpha) \cap V_S$.

(a) Prove that every point $x \in V_s \cap D_t(\alpha)$ is contained in $D_S(\alpha)$,
(b) Prove $\cup_{s \in S} D_s(\alpha) = \cup_{s \in S} R_s(\alpha)$.

Question 2. ($20 = 10 + 10$ points). Let S be a finite set of sites in general position in $\mathbb{R}^2$. Let *stu* and *stv* be two triangles in the Delaunay triangulation, and denote the angles at u and v inside these triangles by φ and ψ. Recall that an *acute* angle is smaller than 90° and an *obtuse* angle is larger than 90°.

(a) Show that at most one of φ and ψ is obtuse.
(b) Is it possible that both φ and ψ are obtuse if the sites are weighted and we are talking about the weighted Delaunay triangulation?

Question 3. ($20 = 10 + 10$ points). Consider the 26 uppercase letters of the Roman alphabet, each drawn in a simple font, without any fancy extras.

(a) Classify each uppercase letter in terms of its symmetries.
HINT a *symmetry* is a transformation (here: rotation, or reflection) that leaves the letter unchanged. For example, “A” has a vertical symmetry axis, while “C” has a horizontal symmetry axis, and “S” has neither but is centrally symmetric.
(b) Which of the uppercase letters have no hole but develop holes as they get thickened?

Question 4. ($20 = 10 + 10$ points). Let S be a finite set of points in $\mathbb{R}^2$, consider the union of disks with radius α centered at the points, and write $L(\alpha)$ for the total length of the boundary.

(a) Prove $L(2\alpha) \leq 2L(\alpha)$.
(b) Is it true that $L(\alpha) \leq L(2\alpha)$ for all choices of S and α?

Question 5. (20 points). Let S be a finite set of points in $\mathbb{R}^2$ and α positive real number. Prove that the common intersection of the disks with radius α centered at the points in S is nonempty iff there exists a disk of radius α (not necessarily centered at a point in S) that contains all points in S.