BRIEF COMMENTS ON THE LITERATURE

Chapter 1. The finite-dimensional aspects of linear functional analysis are treated in detail in [21]. All facts from general functional analysis used in this monograph can be found, for example, in [70].

Formula (1.1.17) belongs to R. L. Dobrushin [15], though it has been subsequently rediscovered, in particular, in [53]. The bound (1.1.18) that one derives from it yields the corresponding bound for the spectral radius $\rho(A|H)$, found by Hopf [27].

A detailed treatment of the finite-dimensional spectral theory is available in any textbook on linear algebra or matrix theory (see, among others, [18] and [19]).

Fekete's Theorem can be found in [64]. Theorems 1.3.2 and 1.3.3 were established in [48] as the basis of an effective method for computing the spectral radius. From the vast literature devoted to the theory of one-parameter semigroups in Banach space and its connections with the Cauchy problem we mention [25], [51], and [39].

The Appendix to Chapter 1 is based on the note [50]. Formula (1.A.4) goes back to Turing, as a definition for that special case. Generally, matrix norms are widely used in modern computational mathematics (see [16],[28]).

Chapter 2. The theory of iterative processes and their applications in computational mathematics are discussed in [62]. Formula (2.1.3) was obtained in [57] (unlike the known estimate $\ell_T \leq \ell'_T$). Therein was also derived formula (2.1.8). Inequality (2.1.9) was obtained, as a necessary condition, by Kesten [33], in a rather tedious way (the first simplification of his approach has been proposed in [53]). The works [35], [36] are devoted to the further development (resting on the method of extreme points) of this theme.

Quadratic maps and their iterations play an important role in mathematical genetics (see [53], [57]). The example considered here of a map which satisfies the condition $\max \rho(T'(x)) < 1$, but is not a contraction in any norm, has been proposed by M. Yu. Lyubich. The Ergodic Theorem 2.1.6 is the prototype of a large number of results of the same kind which are part of modern ergodic theory (see [24], [14]).

The theory of stability is usually constructed for differential equations (see, for example, [11]). However, it can be (and usefully so) developed in parallel for discrete dynamical systems generated by maps. The condition $\rho(T'(0)) < 1$ is necessary for asymptotic stability in the complex-analytic setting [56].

The classes of dissipative and conservative operators in Banach space were introduced in [45] and respectively in [44], [49] (our definitions differ from the generally-accepted ones by the factor i). Theorem 2.3.3 (in Banach space) belongs to V. E. Katsnel'son [31] (it has been obtained somewhat later by other authors, among them Sinclair [74]). A far-reaching analysis of the operatorial meaning of the inequality of S. N. Bernshtein and of other related inequalities was carried out by E. A. Gorin [22]. The classical proof of Bernshtein's inequality can be found in [1].

In §4 we discuss the work [52] (cf. [37]), where, in particular, the Boundary Spectrum Splitting-Off Theorem is established. The Sushkevich kernel emerged for the first time in the fundamental paper [76], devoted to finite semigroups. The kernel is however present in every compact semigroup (see [41], where it is used to split-off the boundary spectrum in the weakly-compact operatorial setting). Theorem 2.4.7 leads to the so-called Masur's conjecture (see [78]). The progress mentioned in the end of §4 towards the solution of "Problem 1" is due to V. N. Kalyuzhnyi [29], [30].

The notion of critical exponent was introduced by Marik and Ptak [60]. They obtaine Theorem 2.8.5 by a method which, despite its similarity with the one discussed in §8, differs from the latter (and, in our opinion, is more complicated). Theorem 2.6.2 is due to Ptak [66]. The state in which the problem of critical exponents was in 1965 is described in the surveys [63] and [66]. The general theorem 2.6.1 and its corollaries 2.6.1-2.6.3 were obtained by V. M.

Kirzhner and M. I. Tabachnikov [34].

With the basic notions of graph theory one can get acquainted in [12]. Subharmonic functions on graphs were introduced in [59], where the maximum principle for such functions was established (not only for finite, but also for a certain class of infinite graphs, which proved useful in applications (see [77])). The notion of boundary vertex of a graph is equivalent to that of essential state known from the theory of Markov chains.

Block decompositions are systematically used in the theory of nonnegative matrices (see [18]), wherefrom we essentially borrowed Theorem 2.7.3 and its corollaries. The theory of nonnegative matrices as a whole was developed in the beginning of this century by Perron and Frobenius. In this area the machinery of graphs has made its appearance (though in a somewhat disguised form) in [65] and [68].

§8 is based on the papers [58], [59]. The Wielandt graph is associated with the matrix that he indicated in [79]. Apparently it is in this particular paper that Theorem 2.9.1 has been stated for the first time (without a proof), as a replacement for the rather crude bound $W_n \leq 2n^2-2n$ found by Frobenius. The proof that we give belongs to Sedlaček [71]. A number of variations on this theme can be found in [26].

M. G. Krein and M. A. Rutman have generalized the Perron-Frobenius theory to operators nonnegative relative to a given cone (in Banach space) [38].

An exposition of the theory of Markov chains can be found in practically every textbook on probability theory (see, for example, [17]), as well as in specialized monographs (among which we mention [32] and [69]). However, the simple approach presented in §10 has been proposed only recently [53]. The asymptotics of $\ln M(n)$ was obtained by I. V. Ostrovskii (see [47]). The works [47], [42], and [23], the point of departure of which is Wielandt's bound, are devoted to estimates of the number of states of a finite automaton syntesized from a given description of its operation. Theorem 2.4.10 is connected with the notion of entropy of a topological Markov chain (see [13]).

§11 gives an exposition of the content of [55]. The general form of stochastic projectors was found in [54] (cf. [43]) in connection with a problem from mathematical genetics.

Chapter 3. §1 uses the papers [3] and [6]. Its concluding part is a finite-dimensional adaptation of Gelfand's theory of Banach algebras (see [20] and [70]).

The characterization of operator norms as minimal elements in N (Theorem 3.2.2) was obtained by Yu. I. Lyubich in the paper [46], on which §2 is based. In particular, it is in [46] that the first example of a unit-preserving ring norm which is not an operator norm was given (see Corollary 3.2.6). Somewhat later the aforementioned characterization of operator norms was obtained by Stoer [75]. The paper [46] has served as point of departure for the investigations of G. R. Belitskii [2-10]. His works [5], [4], and [9] are discussed, with certain supplements, in §§ 3,4, and 5, respectively. The interpolation theorem 3.3.3 is the finite-dimensional analogue of one of the central results of the theory of interpolation of linear operators (see [40]).

The foundations of the theory of cross-norms were laid by Schatten[73]. Orthogonally-invariant norms were studied by von Neumann [61]. Theorem 3.6.2 belongs to Schatten[73].

Chapter 4. §1 gives an exposition of the paper [2]. The basic theorem 4.4.1 on the automorphisms of the order structure on the set of ring norms was established in [7] (a detailed account is given in [8]).

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