Appendix
Convergence of the Proposed Analytical Approach

Although the convergence of the analytical approach developed in Chap. 3 for any n variables cannot be guaranteed, upon rigorous testings with more than 100 variables, we found that the algorithm always converges within eight iterations, suggesting that the algorithm is robust.

To prove the convergence of an iterative algorithm, we first need to express the iteration in the following format:

$$\mathbf{Q}_{z+1} = \mathbf{A}\mathbf{Q}_z + \mathbf{b} \tag{A.1}$$

where $\mathbf{Q}_z$ is a column vector of the variables in the system (reactive power output of the DERs), and z indicates the iteration number. If the maximum absolute value of the eigenvalues (they are all in unit circle) of matrix $\mathbf{A}$ is less than 1 [1], then the algorithm is convergent. First, the convergence of the proposed analytical approach will be proven for the case of two variables.

For simplicity, consider two PVs located at node k and l in an N-bus system, where $k < l$. Assuming $P^{\text{PV}} = 0$, $Q^{\text{PV}}_{k,z}$ and $Q^{\text{PV}}_{l,z}$ can be expressed as:

$$Q^{\text{PV}}_{k,z} = \frac{B_{k,z} + D_{k,z} - E + c^{\text{Qgrid}}/2}{A_k + C_k + F} \tag{A.2}$$

$$Q^{\text{PV}}_{l,z} = \frac{B_{l,z} + D_{l,z} - E + c^{\text{Qgrid}}/2}{A_l + C_l + F} \tag{A.3}$$

Only values of B and D are updated at each iteration, as they contain the term Q_i. Values of E and F are the same for DERs with the same inverter parameters. Updating the values of B and D yield:

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$$
\begin{aligned}
B_{k,z+1} &= \left[\sum_{i=1}^{k-1} \frac{r_i\left(Q_i - Q_{k,z}^{\mathrm{PV}} - Q_{l,z}^{\mathrm{PV}}\right)}{\left|\overline{V_i}\right|^2}\right] c^{\mathrm{Pgrid}} \eta^{\mathrm{P},1} \\
&= B_{k,z} - A_l\left(Q_{k,z}^{\mathrm{PV}} + Q_{l,z}^{\mathrm{PV}}\right) \\
B_{l,z+1} &= \left[\sum_{i=1}^{k-1} \frac{r_i\left(Q_i - Q_{k,z}^{\mathrm{PV}} - Q_{l,z}^{\mathrm{PV}}\right)}{\left|\overline{V_i}\right|^2} + \sum_{i=k}^{l-1} \frac{r_i\left(Q_i - Q_{l,z}^{\mathrm{PV}}\right)}{\left|\overline{V_i}\right|^2}\right] c^{\mathrm{Pgrid}} \eta^{\mathrm{P},1} \\
&= B_{l,z} - A_k Q_{k,z}^{\mathrm{PV}} - A_l Q_{l,z}^{\mathrm{PV}}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
D_{k,z+1} &= D_{k,z} - C_l\left(Q_{k,z}^{\mathrm{PV}} + Q_{l,z}^{\mathrm{PV}}\right) \\
D_{l,z+1} &= D_{l,z} - C_k Q_{k,z}^{\mathrm{PV}} - C_l Q_{l,z}^{\mathrm{PV}}
\end{aligned}
$$

Now we can obtain the expression for $Q_{k,z+1}^{\mathrm{PV}}$:

$$
\begin{aligned}
Q_{k,z+1}^{\mathrm{PV}} &= \frac{B_{k,z+1} + D_{k,z+1} - E + c^{\mathrm{Qgrid}}/2}{A_k + C_k + F} \\
&= Q_{k,z}^{\mathrm{PV}}\left(1 - \frac{A_k + C_k}{A_k + C_k + F}\right) - Q_{l,z}^{\mathrm{PV}} \frac{A_k + C_k}{A_k + C_k + F}
\end{aligned} \tag{A.4}
$$

Similarly,

$$
Q_{l,z+1}^{\mathrm{PV}} = Q_{l,z}^{\mathrm{PV}}\left(1 - \frac{A_l + C_l}{A_l + C_l + F}\right) - Q_{k,z}^{\mathrm{PV}} \frac{A_k + C_k}{A_l + C_l + F} \tag{A.5}
$$

Let $J_{kk} = \frac{A_k+C_k}{A_k+C_k+F}$, $J_{kl} = \frac{A_k+C_k}{A_l+C_l+F}$, $J_{ll} = \frac{A_l+C_l}{A_l+C_l+F}$. From Sect. 3.3, looking at the definitions of A_k and C_k, it follows that $A_k < A_l$ and $C_k < C_l$ for $k < l$. Therefore, $J_{ll} > J_{kk} > J_{kl}$. Any J_{kk} is positive for most, if not all, practical applications, as the values of r_i and x_i are mostly positive. Even if there is some negative x_i in the system (e.g. due to a capacitor), J_{kk} will most likely still be positive because of the summation of all x_i for C_k and r_i for A_k.

Expressing the variables in the format of Eq. (A.1),

$$
\begin{pmatrix} Q_{k,z+1}^{\mathrm{PV}} \\ Q_{l,z+1}^{\mathrm{PV}} \end{pmatrix} = \begin{pmatrix} 1 - J_{kk} & -J_{kk} \\ -J_{kl} & 1 - J_{ll} \end{pmatrix} \begin{pmatrix} Q_{k,z}^{\mathrm{PV}} \\ Q_{l,z}^{\mathrm{PV}} \end{pmatrix} \tag{A.6}
$$

where $\mathbf{A} = \begin{pmatrix} 1 - J_{kk} & -J_{kk} \\ -J_{kl} & 1 - J_{ll} \end{pmatrix}$ and $\mathbf{b} = 0$. The next step is to find λ, the eigenvalues of $\mathbf{A}$, by having $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$:

$$\left\| \begin{matrix} 1 - J_{kk} - \lambda & -J_{kk} \\ -J_{kl} & 1 - J_{ll} - \lambda \end{matrix} \right\| = 0 \tag{A.7}$$

which yields

$$\lambda^2 + \lambda \left(J_{kk} + J_{ll} - 2 \right) + \left(1 - J_{kk} \right) \left(1 - J_{ll} \right) - J_{kk} J_{kl} = 0$$

$$\lambda = \frac{1}{2} \left[2 - J_{kk} - J_{ll} \pm \sqrt{\left(J_{kk} - J_{ll} \right)^2 + 4 J_{kk} J_{kl}} \right] \tag{A.8}$$

As J_{kk}, J_{ll}, and J_{kl} are all positive, λ is real. The larger λ is then

$$\lambda = \frac{1}{2} \left[2 - J_{kk} - J_{ll} + \sqrt{\left(J_{kk} - J_{ll} \right)^2 + 4 J_{kk} J_{kl}} \right] \tag{A.9}$$

which has to be less than one for the algorithm to converge. Therefore we need to prove that:

$$2 - J_{kk} - J_{ll} + \sqrt{\left(J_{kk} - J_{ll} \right)^2 + 4 J_{kk} J_{kl}} < 2 \tag{A.10}$$

By rearranging the variables, we obtain:

$$\begin{aligned} J_{kk} + J_{ll} &> \sqrt{\left(J_{kk} - J_{ll} \right)^2 + 4 J_{kk} J_{kl}} \\ \left(J_{kk} + J_{ll} \right)^2 &> \left(J_{kk} - J_{ll} \right)^2 + 4 J_{kk} J_{kl} \\ J_{ll} &> J_{kl} \end{aligned} \tag{A.11}$$

which is always true for $k < l$[1]. Hence, the convergence of the algorithm for two variables when $P^{\mathrm{X}} = 0$ has been proven.

To prove the convergence for n variables, the maximum absolute value of the eigenvalues of $n \times n$ matrix **A** has to be less than one. This means that the absolute value of the sum of each **A**'s row and column must be less than one. The components of **A**, a_{kl} can be expressed as:

$$a_{kl} = \begin{cases} 1 - J_{kl} & \forall k = l \\ -J_{kl} & \forall k < l \\ -J_{kk} & \forall l < k \\ 0 & \text{otherwise} \end{cases}$$

[1] $k < l$ means that the power flowing to node l has to go through node k. If node k and node l are on a different lateral and the power flowing through one node does not have to go through the other, then the "value" of k and l cannot be compared. In this case, A_{kl}, the element of **A** connecting $Q_{k,z+1}$ and $Q_{l,z}$ will be 0.

Even though for small number of DERs in the system, the absolute value of the sum of each row and column of **A** will be less than one, this is not the case when there is a large number of variables in the system (more than 20 in the test systems used in this thesis). Nevertheless, upon rigorous testing, even with more than 100 variables, the proposed approach always converges by the eighth iteration and provide results that are better than the benchmark approaches. Therefore, although it has not been proven that the proposed approach will always converge for any n variables and for when $P^{\mathrm{X}} \neq 0$, the author is confident that it will work in practical applications.

Reference

1. Strong DM (2005) Iterative methods for solving [i]ax[/i] = [i]b[/i] - convergence analysis of iterative methods. Convergence. https://www.maa.org/press/periodicals/loci/joma/iterativemethods-for-solving-iaxi-ibi-convergence-analysis-of-iterativemethods