Appendix A: Proofs of Results from Chapter 2

A.1 Proofs of Results from Sect. 2.2

Lemma 2.1 *Let assumptions* ***A****1*–***A****3 be valid. Then, for all nonnegative values of* ν_i, $i = 0, 1$, *supply values* q_i *are strictly positive (i.e.,* $q_i > 0$, $i = 0, 1$*) at any exterior equilibrium if and only if* $p > p_0$.

Proof If $p > p_0 = b_1$, then, the inequalities $p \le -\beta \nu_0 q_1 + b_0$ and $p \le b_1$, from the optimality conditions (2.11) and (2.13), respectively, never apply, which implies that no (equilibrium) value q_i, $i = 0, 1$, can vanish. Conversely, if all the equilibrium outputs are positive, i.e., $q_i > 0, i = 0, 1$, then, the optimality condition (2.13) directly entails $p = q_1\nu_1 + a_1 q_1 + b_1 > b_1$. Hence, $p > p_0 = b_1$, and the proof is complete. ■

Theorem 2.1 *Under assumptions* ***A****1*–***A****3, for any* $\beta \in (0, 1]$, $D \ge 0$ *and* $\nu_i \ge 0$, $i = 0, 1$, *there exists uniquely the exterior equilibrium* (p, q_0, q_1) *depending continuously upon the parameters* (D, ν_0, ν_1). *The equilibrium price p as a function of these parameters is continuously differentiable with respect to D and* ν_i, $i = 0, 1$. *Moreover* $p(D, \nu_0, \nu_1) > p_0$ *and*

$$\frac{\partial p}{\partial D} = \frac{1}{\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\left(\dfrac{1}{\nu_1 + a_1}\right) - G'(p)}. \tag{A.1}$$

Proof Let $\nu_0, \nu_1 \ge 0$. By using the optimality conditions (2.11) and (2.13), we can find the output volume functions $q_i = q_i(p, \nu_0, \nu_1)$, $i = 0, 1$, defined on the interval $[p_0, +\infty)$. These functions are differentiable with respect to p and ν_i, $i = 0, 1$, and they are given by:

J. G. Flores Muñiz et al., *Public Interest and Private Enterprize: New Developments*, Lecture Notes in Networks and Systems 138,
https://doi.org/10.1007/978-3-030-58349-1

$$q_0 = \frac{p - b_0}{(1-\beta)\nu_0 + a_0} + \frac{\beta \nu_0}{(1-\beta)\nu_0 + a_0}\left(\frac{p - b_1}{\nu_1 + a_1}\right), \tag{A.2}$$

$$q_1 = \frac{p - b_1}{\nu_1 + a_1}. \tag{A.3}$$

Now we introduce the following function:

$$\begin{aligned}
Q(p, \nu_0, \nu_1) =& q_0(p, \nu_0, \nu_1) + q_1(p, \nu_0, \nu_1) \\
=& \frac{p - b_0}{(1-\beta)\nu_0 + a_0} + \frac{\beta \nu_0}{(1-\beta)\nu_0 + a_0}\left(\frac{p - b_1}{\nu_1 + a_1}\right) + \frac{p - b_1}{\nu_1 + a_1} \\
=& p\left[\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\beta \nu_0}{(1-\beta)\nu_0 + a_0}\left(\frac{1}{\nu_1 + a_1}\right) + \frac{1}{\nu_1 + a_1}\right] \\
& - \left[\frac{b_0}{(1-\beta)\nu_0 + a_0} + \frac{\beta \nu_0}{(1-\beta)\nu_0 + a_0}\left(\frac{b_1}{\nu_1 + a_1}\right) + \frac{b_1}{\nu_1 + a_1}\right] \\
=& p\left[\frac{1}{(1-\beta)\nu_0 + a_0} + \left(\frac{\beta \nu_0}{(1-\beta)\nu_0 + a_0} + 1\right)\left(\frac{1}{\nu_1 + a_1}\right)\right] \\
& - \left[\frac{b_0}{(1-\beta)\nu_0 + a_0} + \left(\frac{\beta \nu_0}{(1-\beta)\nu_0 + a_0} + 1\right)\left(\frac{b_1}{\nu_1 + a_1}\right)\right] \\
=& p\left[\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\beta \nu_0 + (1-\beta)\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\left(\frac{1}{\nu_1 + a_1}\right)\right] \\
& - \left[\frac{b_0}{(1-\beta)\nu_0 + a_0} + \frac{\beta \nu_0 + (1-\beta)\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\left(\frac{b_1}{\nu_1 + a_1}\right)\right] \\
=& p\left[\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\left(\frac{1}{\nu_1 + a_1}\right)\right] \\
& - \left[\frac{b_0}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0}\left(\frac{b_1}{\nu_1 + a_1}\right)\right].
\end{aligned} \tag{A.4}$$

As we can see from (A.4), the function Q is linear in p with positive slope. Therefore, $Q(p, \nu_0, \nu_1)$ strictly increases with respect to p, and tends to $+\infty$ when $p \to +\infty$. By assumption **A**3, one has that for all $\nu_i \geq 0$, $i = 0, 1$,

$$\begin{aligned}
Q(p_0, \nu_0, \nu_1) =& q_0(p_0, \nu_0, \nu_1) + q_1(p_0, \nu_0, \nu_1) \\
=& \frac{p_0 - b_0}{(1-\beta)\nu_0 + a_0} \leq \frac{p_0 - b_0}{a_0} < G(p_0) \leq G(p_0) + D.
\end{aligned} \tag{A.5}$$

Hence, $Q(p, \nu_0, \nu_1)$ strictly increases with respect to p, the function $G(p)$ is non-increasing by p and D is constant, so by inequality (A.5), there exists a unique value $p^* > p_0$ such that

$$Q(p^*, \nu_0, \nu_1) = G(p^*) + D. \tag{A.6}$$

For this value p^*, using (A.2) and (A.3), we compute uniquely the equilibrium output volumes $q_i^* = q_i(p^*, \nu_0, \nu_1)$, $i = 0, 1$. So we have established the existence and uniqueness of the exterior equilibrium (p^*, q_0^*, q_1^*) for any $D \geq 0$ and $\nu_i \geq 0$, $i = 0, 1$.

Now we are going to show that the equilibrium price p^* of the exterior equilibrium is differentiable with respect to the parameters (D, ν_0, ν_1). From (A.6) we get the following relationships:

$$Q(p^*, \nu_0, \nu_1) - G(p^*) - D = 0, \tag{A.7}$$

and we introduce the following function:

$$\begin{aligned} \Gamma(p^*, D, \nu_0, \nu_1) =& Q(p^*, \nu_0, \nu_1) - G(p^*) - D \\ =& p^* \left[\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) \right] \\ & - \left[\frac{b_0}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{b_1}{\nu_1 + a_1} \right) \right] \\ & - G(p^*) - D. \end{aligned} \tag{A.8}$$

Thus, we can rewrite (A.7) as a functional equation

$$\Gamma(p^*, D, \nu_0, \nu_1) = 0 \tag{A.9}$$

and compute its partial derivative with respect to p^*:

$$\begin{aligned} \frac{\partial \Gamma}{\partial p^*} =& \frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) - G'(p^*) \\ \geq& \frac{1}{(1-\beta)\nu_0 + a_0} > 0. \end{aligned} \tag{A.10}$$

From (A.10) we can see that the partial derivative of Γ with respect to p^* is positive. Because of that, Implicit Function Theorem implies that the equilibrium price p^* can be considered as a function $p^* = p^*(D, \nu_0, \nu_1)$, which is differentiable with respect to D and ν_i, $i = 0, 1$. Moreover, the partial derivative of the price p^* with respect to D can be found from the equation

$$\frac{\partial \Gamma}{\partial p^*} \frac{\partial p^*}{\partial D} + \frac{\partial \Gamma}{\partial D} = 0. \tag{A.11}$$

The latter leads to

$$\frac{\partial p^*}{\partial D} = -\frac{\dfrac{\partial \Gamma}{\partial D}}{\dfrac{\partial \Gamma}{\partial p^*}} = \frac{1}{\dfrac{1}{(1-\beta)\nu_0 + a_0} + \dfrac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\dfrac{1}{\nu_1 + a_1} \right) - G'(p^*)}. \tag{A.12}$$

Finally, since the function p^* depends upon (D, ν_0, ν_1) and is differentiable with respect to D and ν_i, $i = 0, 1$, the functions q_i^*, $i = 0, 1$, also depend on (D, ν_0, ν_1) and are differentiable with respect to D and ν_i, $i = 0, 1$. Therefore, the equilibrium (p^*, q_0^*, q_1^*) continuously depends on the parameters (D, ν_0, ν_1). The proof of the theorem is complete ■

A.2 Proofs of Results from Sect. 2.3

Proposition 2.1 *For all $\tau \le 0$, there exists a unique solution $\nu_i = \nu_i(\tau)$, $i = 0, 1$, of system (2.20) and (2.21), which continuously depends upon τ. In addition, $\nu_i(\tau) \to 0$ whenever $\tau \to -\infty$, and $\nu_i(\tau)$ strictly grows and tends to $\nu_i(0)$ as $\tau \to 0$, $i = 0, 1$.*

Proof The variables ν_i, $i = 0, 1$, given by (2.20) and (2.21) are considered on their domains: $\nu_i \ge 0$, $a_i > 0$, $i = 0, 1$, $\beta \in (0, 1]$, and $\tau \in (-\infty, 0]$.

Substituting (2.21) in (2.20) we get the following equation:

$$\begin{aligned}
\nu_0 =& \frac{1}{\dfrac{1}{\left(\dfrac{1}{\dfrac{1}{(1-\beta)\nu_0 + a_0} - \tau}\right) + a_1} - \tau} \\
=& \frac{1}{\dfrac{1}{\left(\dfrac{(1-\beta)\nu_0 + a_0}{1 - [(1-\beta)\nu_0 + a_0]\tau}\right) + a_1} - \tau} \\
=& \frac{1}{\dfrac{1}{\left(\dfrac{(1-\beta)\nu_0 + a_0}{-(1-\beta)\tau\nu_0 + (1 - a_0\tau)}\right) + a_1} - \tau} \\
=& \frac{1}{\dfrac{-(1-\beta)\tau\nu_0 + (1 - a_0\tau)}{(1-\beta)\nu_0 + a_0 + a_1\left[-(1-\beta)\tau\nu_0 + (1 - a_0\tau)\right]} - \tau} \\
=& \frac{1}{\dfrac{-(1-\beta)\tau\nu_0 + (1 - a_0\tau)}{(1-\beta)(1 - a_1\tau)\nu_0 + (a_0 + a_1 - a_0 a_1\tau)} - \tau} \\
=& \frac{(1-\beta)(1 - a_1\tau)\nu_0 + (a_0 + a_1 - a_0 a_1\tau)}{-(1-\beta)\tau\nu_0 + (1 - a_0\tau) - \left[(1-\beta)(1 - a_1\tau)\nu_0 + (a_0 + a_1 - a_0 a_1\tau)\right]\tau} \\
=& \frac{(1-\beta)(1 - a_1\tau)\nu_0 + (a_0 + a_1 - a_0 a_1\tau)}{(1-\beta)\left(-2\tau + a_1\tau^2\right)\nu_0 + \left(1 - 2a_0\tau - a_1\tau + a_0 a_1\tau^2\right)}.
\end{aligned} \tag{A.13}$$

Then, we can multiply (A.13) by $\left[(1-\beta)\left(-2\tau+a_1\tau^2\right)\nu_0+(1-2a_0\tau-a_1\tau +a_0a_1\tau^2)\right]$ to obtain

$$\begin{aligned}\left[(1-\beta)\left(-2\tau+a_1\tau^2\right)\nu_0+\left(1-2a_0\tau-a_1\tau+a_0a_1\tau^2\right)\right]\nu_0 \\ =(1-\beta)\left(1-a_1\tau\right)\nu_0+\left(a_0+a_1-a_0a_1\tau\right).\end{aligned} \tag{A.14}$$

Move all the terms of (A.14) to the left-hand side and get

$$\begin{aligned}\left[(1-\beta)\left(-2\tau+a_1\tau^2\right)\nu_0+\left(1-2a_0\tau-a_1\tau+a_0a_1\tau^2\right)\right]\nu_0 \\ -(1-\beta)\left(1-a_1\tau\right)\nu_0-\left(a_0+a_1-a_0a_1\tau\right)=0.\end{aligned} \tag{A.15}$$

By extracting a common factor from (A.15) we obtain the following quadratic equation for ν_0:

$$(1-\beta)\left(-2\tau+a_1\tau^2\right)\nu_0^2+\left(\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2\right)\nu_0-\left(a_0+a_1-a_0a_1\tau\right)=0. \tag{A.16}$$

Now, in order to simplify the notation, we rewrite (A.16) as follows:

$$A\nu_0^2+B\nu_0-C=0, \tag{A.17}$$

where

$$A=A(\tau)=(1-\beta)\left(-2\tau+a_1\tau^2\right)\geq 0, \tag{A.18}$$

$$B=B(\tau)=\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2>0, \tag{A.19}$$

$$C=C(\tau)=a_0+a_1-a_0a_1\tau>0. \tag{A.20}$$

If $\tau=0$ or $\beta=1$, then, $A=0$ and (A.17) is linear, so we can find the unique solution for ν_0 given by:

$$\nu_0(\tau)=\frac{C}{B}=\begin{cases}\dfrac{a_0+a_1}{\beta} & \text{if } \tau=0,\\ \dfrac{a_0+a_1-a_0a_1\tau}{1-2a_0\tau-a_1\tau+a_0a_1\tau^2} & \text{if } \beta=1.\end{cases} \tag{A.21}$$

If $\beta\in(0,1)$ and $\tau<0$, then, $A\neq 0$ and we can find both roots of (A.17), which are:

$$\nu_0(\tau)=\frac{-B+\sqrt{B^2+4AC}}{2A}, \tag{A.22}$$

$$\nu_0(\tau)=\frac{-B-\sqrt{B^2+4AC}}{2A}. \tag{A.23}$$

However, since $\nu_0 \geq 0$, the root (A.23) is impossible; that is, (A.22) is the unique solution of (A.17).

Moreover, (A.21) and (A.22) can be combined in a single equation for all $\beta \in (0, 1]$ and $\tau \in (-\infty, 0]$ as follows:

$$\begin{aligned}\nu_0(\tau) =& \nu_0 = \frac{2C}{B + \sqrt{B^2 + 4AC}} \\ =& \frac{2(a_0+a_1-a_0a_1\tau)}{(\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2)+\sqrt{(\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2)^2+4(1-\beta)(-2\tau+a_1\tau^2)(a_0+a_1-a_0a_1\tau)}},\end{aligned} \tag{A.24}$$

where

$$B + \sqrt{B^2 + 4AC} > 0, \tag{A.25}$$

and so (A.24) is the unique solution for ν_0.

We can see that the solution (A.24) for any parameter $\beta \in (0, 1]$ satisfies the condition $\nu_0 \to 0$ as $\tau \to -\infty$. Hence, there exits a positive value $\overline{\nu_0}(\beta)$ such that $\nu_0(\tau) \leq \overline{\nu_0}(\beta)$ for all $\tau \leq 0$.

From (2.21) and (A.24), we can see that ν_1 also has a unique solution, which is given by

$$\nu_1(\tau) = \nu_1 = \frac{1}{\dfrac{1}{(1-\beta)\nu_0(\tau) + a_0} - \tau}. \tag{A.26}$$

For any parameter $\beta \in (0, 1]$, the conditions $\nu_1 \to 0$ as $\tau \to -\infty$ and $\nu_1(\tau) \leq a_0 + (1-\beta)\overline{\nu_0}(\beta)$ for all $\tau \leq 0$, are satisfied.

Now, it is apparent that the functions (A.18)–(A.20) are continuously differentiable with respect to τ, $\tau \in (-\infty, 0]$, and

$$A' = (1-\beta)(-2 + 2a_1\tau) \leq 0, \tag{A.27}$$

$$B' = -2a_0 - \beta a_1 + 2a_0a_1\tau < 0, \tag{A.28}$$

$$C' = -a_0a_1 < 0. \tag{A.29}$$

Thus, from (A.24), we have that $\nu_0(\tau)$ is continuously differentiable and

$$\begin{aligned}\nu_0' =& \frac{2C'\left(B + \sqrt{B^2+4AC}\right) - 2C\left(B' + \frac{2BB'+4A'C+4AC'}{2\sqrt{B^2+4AC}}\right)}{\left(B+\sqrt{B^2+4AC}\right)^2} \\ =& \frac{2C'\left(B + \sqrt{B^2+4AC}\right)\sqrt{B^2+4AC} - 2C\left(B'\sqrt{B^2+4AC} + \frac{2BB'+4A'C+4AC'}{2}\right)}{\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}} \\ =& \frac{2C'\left(B\sqrt{B^2+4AC} + B^2 + 4AC\right) - 2C\left(B'\sqrt{B^2+4AC} + BB' + 2A'C + 2AC'\right)}{\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}}\end{aligned}$$

$$
\begin{aligned}
&=\frac{2C'B\sqrt{B^2+4AC}+2C'B^2+4ACC'-2CB'\sqrt{B^2+4AC}-2CBB'-4A'C^2}{\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}}\\
&=\frac{2\left(C'B-CB'\right)\sqrt{B^2+4AC}+2\left(C'B-CB'\right)B+4\left(AC'-A'C\right)C}{\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}}\\
&=\frac{2\left(C'B-CB'\right)\left(B+\sqrt{B^2+4AC}\right)+4\left(AC'-A'C\right)C}{\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}}.
\end{aligned}
\tag{A.30}
$$

Now, we estimate the values of (A.30) in order to reveal the behavior of $\nu_0(\tau)$ as the function of τ.

From (A.18)–(A.20), it is evident that the denominator of (A.30) is positive:

$$
\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}>0. \tag{A.31}
$$

Thus, plugging (A.18)–(A.20) and (A.27)–(A.29) in (A.30), we can find that

$$
\begin{aligned}
C'B-CB'=&\left(-a_0a_1\right)\left(\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2\right)\\
&-\left(a_0+a_1-a_0a_1\tau\right)\left(-2a_0-\beta a_1+2a_0a_1\tau\right)\\
=&\left(-a_0a_1\right)\left(\beta-a_0a_1\tau^2\right)+\left(-a_0a_1\right)\left(-2a_0\tau-\beta a_1\tau+2a_0a_1\tau^2\right)\\
&-\left[\left(a_0+a_1\right)\left(-2a_0-\beta a_1+2a_0a_1\tau\right)+\left(-a_0a_1\tau\right)\left(-2a_0-\beta a_1+2a_0a_1\tau\right)\right]\\
=&a_0a_1\left(-\beta+a_0a_1\tau^2\right)-a_0a_1\left(-2a_0\tau-\beta a_1\tau+2a_0a_1\tau^2\right)\\
&+a_0a_1\left(-2a_0\tau-\beta a_1\tau+2a_0a_1\tau^2\right)+\left(a_0+a_1\right)\left(2a_0+\beta a_1-2a_0a_1\tau\right)\\
=&a_0a_1\left(-\beta+a_0a_1\tau^2\right)+\left(a_0+a_1\right)\left(2a_0+\beta a_1-2a_0a_1\tau\right)\\
=&a_0a_1\left(-\beta\right)+a_0a_1\left(a_0a_1\tau^2\right)+\left(a_0+a_1\right)\left(2a_0-2a_0a_1\tau\right)+\left(a_0+a_1\right)\left(\beta a_1\right)\\
=&-a_0\left(\beta a_1\right)+\left(a_0+a_1\right)\left(\beta a_1\right)+a_0a_1\left(a_0a_1\tau^2\right)+\left(a_0+a_1\right)\left(2a_0-2a_0a_1\tau\right)\\
=&\beta a_1^2+a_0^2a_1^2\tau^2+\left(a_0+a_1\right)\left(2a_0-2a_0a_1\tau\right)>0,
\end{aligned}
\tag{A.32}
$$

and

$$
\begin{aligned}
AC'-A'C=&(1-\beta)\left(-2\tau+a_1\tau^2\right)\left(-a_0a_1\right)\\
&-(1-\beta)\left(-2+2a_1\tau\right)\left(a_0+a_1-a_0a_1\tau\right)\\
=&(1-\beta)\left(-2\tau+2a_1\tau^2\right)\left(-a_0a_1\right)+(1-\beta)\left(-a_1\tau^2\right)\left(-a_0a_1\right)\\
&-(1-\beta)\left(-2+2a_1\tau\right)\left(a_0+a_1\right)-(1-\beta)\left(-2+2a_1\tau\right)\left(-a_0a_1\tau\right)\\
=&(1-\beta)a_0a_1^2\tau^2-(1-\beta)\left(-2\tau+2a_1\tau^2\right)\left(a_0a_1\right)\\
&+(1-\beta)\left(-2\tau+2a_1\tau^2\right)\left(a_0a_1\right)+(1-\beta)\left(2-2a_1\tau\right)\left(a_0+a_1\right)\\
=&(1-\beta)a_0a_1^2\tau^2+(1-\beta)\left(2-2a_1\tau\right)\left(a_0+a_1\right)\geq 0.
\end{aligned}
\tag{A.33}
$$

Therefore, given the values of (A.18)–(A.20), (A.25) and (A.31)–(A.33), we can conclude that

$$\begin{aligned}\nu_0' =& \frac{2\left(C'B - CB'\right)\left(B + \sqrt{B^2 + 4AC}\right) + 4\left(AC' - A'C\right)C}{\left(B + \sqrt{B^2 + 4AC}\right)^2 \sqrt{B^2 + 4AC}} \\ \geq& \frac{2\left(C'B - CB'\right)\left(B + \sqrt{B^2 + 4AC}\right)}{\left(B + \sqrt{B^2 + 4AC}\right)^2 \sqrt{B^2 + 4AC}} > 0.\end{aligned} \tag{A.34}$$

Therefore, $\nu_0(\tau)$ is strictly increasing with respect to τ, $\tau \in (-\infty, 0]$. Since the function $\nu_0 = \nu_0(\tau)$ is continuous, it tends to $\nu_0(0)$ as τ goes to 0.

Now, from (A.26) we have

$$\nu_1 = \frac{1}{\dfrac{1}{(1-\beta)\nu_0 + a_0} - \tau}. \tag{A.35}$$

Since, $\nu_0(\tau)$ is continuously differentiable with respect to τ, the same is true for $\nu_1(\tau)$, and

$$\begin{aligned}\nu_1' =& -\frac{1}{\left(\dfrac{1}{(1-\beta)\nu_0 + a_0} - \tau\right)^2}\left(-\frac{1}{[(1-\beta)\nu_0 + a_0]^2}(1-\beta)\nu_0' - 1\right) \\ =& \nu_1^2\left(\frac{(1-\beta)\nu_0'}{[(1-\beta)\nu_0 + a_0]^2} + 1\right),\end{aligned} \tag{A.36}$$

where $\nu_0' > 0$, On account of that,

$$\nu_1' = \nu_1^2\left(\frac{(1-\beta)\nu_0'}{[(1-\beta)\nu_0 + a_0]^2} + 1\right) \geq \nu_1^2 > 0. \tag{A.37}$$

Therefore, $\nu_1(\tau)$ strictly increases with respect to τ, $\tau \in (-\infty, 0]$. Since the function $\nu_1 = \nu_1(\tau)$ is continuous, it tends to $\nu_1(0)$ as τ goes to 0. The proof of the theorem is complete. ■

Theorem 2.2 *Under assumptions **A**1–**A**3, there exists the interior equilibrium.*

Proof We are going to show that there exist $\nu_i^* \geq 0, q_i^* \geq 0$, $i = 0, 1$, and $p^* > p_0$ such that the vector (p^*, q_0^*, q_1^*) is the exterior equilibrium, and the influence coefficients (ν_0^*, ν_1^*) are consistent, i.e., Eqs. (2.18) and (2.19) hold.

As it was proved in Proposition 2.1, ν_0 and ν_1 solve uniquely Eqs. (2.20) and (2.21), and continuously depend on $\tau = G'(p)$. Moreover, $G'(p)$ continuously depends on p, hence, the functions ν_0 and ν_1 are continuous with respect to p.

Recall the function (A.4) introduced when proving Theorem 2.1:

$$\begin{aligned} Q(p,\nu_0(p),\nu_1(p)) &= Q(p) = q_0(p,\nu_0(p),\nu_1(p)) + q_1(p,\nu_0(p),\nu_1(p)) \\ &= \frac{p-b_0}{(1-\beta)\nu_0(p)+a_0} + \frac{\beta\nu_0(p)}{(1-\beta)\nu_0(p)+a_0}\left(\frac{p-b_1}{\nu_1(p)+a_1}\right) + \frac{p-b_1}{\nu_1(p)+a_1} \\ &= p\left[\frac{1}{(1-\beta)\nu_0(p)+a_0} + \frac{\nu_0(p)+a_0}{(1-\beta)\nu_0(p)+a_0}\left(\frac{1}{\nu_1(p)+a_1}\right)\right] \\ &\quad -\left[\frac{b_0}{(1-\beta)\nu_0(p)+a_0} + \frac{\nu_0(p)+a_0}{(1-\beta)\nu_0(p)+a_0}\left(\frac{b_1}{\nu_1(p)+a_1}\right)\right]. \end{aligned} \tag{A.38}$$

which continuously depends on p and tends to $+\infty$ as $p \to +\infty$ since $\nu_0(p)$ and $\nu_1(p)$ are bounded. Thus, by assumption **A**3, we have that

$$\begin{aligned} Q(p_0) =& q_0(p_0,\nu_0(p_0),\nu_1(p_0)) + q_1(p_0,\nu_0(p_0),\nu_1(p_0)) \\ =& \frac{p_0-b_0}{(1-\beta)\nu_0(p_0)+a_0} \le \frac{p_0-b_0}{a_0} < G(p_0) \le G(p_0)+D. \end{aligned} \tag{A.39}$$

Therefore, there exists the value $p^* > p_0$ such that

$$Q(p^*) = G(p^*) + D. \tag{A.40}$$

For this value p^*, we compute the influence coefficients $\nu_i^* = \nu_i(G'(p^*))$, $i = 0, 1$, using (A.24) and (A.26), as well as the output volumes $q_i^* = q_i(p^*, \nu_0^*, \nu_1^*)$, $i = 0, 1$, given by (A.2) and (A.3). Thus, ν_0^* and ν_1^* satisfy (2.18) and (2.19), whereas the vector (p^*, q_0^*, q_1^*) is the exterior equilibrium. As a consequence, the extended vector $(p^*, q_0^*, q_1^*, \nu_0^*, \nu_1^*)$ is the interior equilibrium. The proof of the theorem is complete. ■

A.3 Proofs of Results from Sect. 2.4

Corollary 2.1 *Under assumptions **A**1–**A**3, for all $\beta \in (0, 1]$, the demand function of type (2.22) implies the uniqueness of the interior equilibrium.*

Proof Consider an arbitrary $\beta \in (0, 1]$. Since $G'(p) = -K$, then, by Proposition 2.1, for $\tau = -K$ there exists a unique solution (ν_0^*, ν_1^*) of Eqs. (2.20) and (2.21):

$$\nu_0^* = \frac{2(a_0+a_1+a_0a_1K)}{(\beta+2a_0K+\beta a_1K+a_0a_1K^2)+\sqrt{(\beta+2a_0K+\beta a_1K+a_0a_1K^2)^2+4(1-\beta)(2K+a_1K^2)(a_0+a_1+a_0a_1K)}}, \tag{A.41}$$

and

$$\nu_1^* = \frac{1}{\dfrac{1}{(1-\beta)\nu_0^*+a_0}+K} = \frac{(1-\beta)\nu_0^*+a_0}{1+\left[(1-\beta)\nu_0^*+a_0\right]K}. \tag{A.42}$$

Moreover, from (2.20), we can rewrite (A.41) as follows:

$$\nu_0^* = \frac{1}{\dfrac{1}{\nu_1^* + a_1} + K}. \tag{A.43}$$

It is not difficult to see that the influence coefficients ν_0^* and ν_1^* don't depend on p, therefore, by Theorem 2.1, there exists the unique exterior equilibrium (p^*, q_0^*, q_1^*) with the influence coefficients (ν_0^*, ν_1^*). Hence, the vector

$$(p^*, q_0^*, q_1^*, \nu_0^*, \nu_1^*) = (p^*(\beta), q_0^*(\beta), q_1^*(\beta), \nu_0^*(\beta), \nu_1^*(\beta))$$

is the unique interior equilibrium for $\beta \in (0, 1]$. The proof of the corollary is complete. ■

Theorem 2.3 *For the affine demand function $G(p)$ from (2.22), the price $p^*(\beta)$, the supply outputs $q_i^*(\beta)$, $i = 0, 1$, and the influence coefficients $\nu_i^*(\beta)$, $i = 0, 1$, characterizing the interior equilibrium, together with total market supply $G^*(\beta) = q_0^*(\beta) + q_1^*(\beta)$, are continuously differentiable by $\beta \in (0, 1]$. Furthermore, $q_0^*(\beta)$ and $G^*(\beta)$ strictly increase, whereas $p^*(\beta)$, $\nu_0^*(\beta)$, $\nu_1^*(\beta)$ and $q_1^*(\beta)$ strictly decrease.*

Proof First, we are going to show that the functions $\nu_i^*(\beta)$, $i = 0, 1$, are continuously differentiable and strictly decreasing with respect to β. Let us consider the functions

$$\mathscr{A} = \mathscr{A}(\beta) = (1-\beta)\left(2K + a_1K^2\right) \ge 0, \tag{A.44}$$

$$\mathscr{B} = \mathscr{B}(\beta) = \beta + 2a_0K + \beta a_1 K + a_0 a_1 K^2 > 0, \tag{A.45}$$

$$\mathscr{C} = \mathscr{C}(\beta) = a_0 + a_1 + a_0 a_1 K > 0, \tag{A.46}$$

which are continuously differentiable with respect to β, with

$$\mathscr{A}' = -\left(2K + a_1K^2\right) < 0, \tag{A.47}$$

$$\mathscr{B}' = 1 + a_1 K > 0, \tag{A.48}$$

$$\mathscr{C}' = 0. \tag{A.49}$$

Using (A.44)–(A.46) we rewrite (A.41) as follows:

$$\nu_0^*(\beta) = \nu_0^* = \frac{2\mathscr{C}}{\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}}, \tag{A.50}$$

where

$$\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}} > 0. \tag{A.51}$$

Then, $\nu_0^*(\beta)$ is continuously differentiable with respect to β and, similarly to (A.30),

$$\nu_0^{*\prime} = \frac{2\left(\mathscr{C}'\mathscr{B} - \mathscr{C}\mathscr{B}'\right)\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right) + 4\left(\mathscr{A}\mathscr{C}' - \mathscr{A}'\mathscr{C}\right)\mathscr{C}}{\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right)^2 \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}}. \quad \text{(A.52)}$$

Since $\mathscr{C}' = 0$, then,

$$\begin{aligned} \nu_0^{*\prime} &= \frac{2\left(-\mathscr{C}\mathscr{B}'\right)\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right) + 4\left(-\mathscr{A}'\mathscr{C}\right)\mathscr{C}}{\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right)^2 \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}} \\ &= \frac{-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right) + 2\mathscr{A}'\mathscr{C}\right]}{\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right)^2 \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}}. \end{aligned} \quad \text{(A.53)}$$

Now we are going to estimate the value of (A.53) in order to describe the behavior of $\nu_0^*(\beta)$ as a function of β.

From (A.44)–(A.46), it is evident that the denominator of (A.53) is positive:

$$\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right)^2 \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}} > 0. \quad \text{(A.54)}$$

Suppose that the numerator of (A.53) is nonnegative for some $\beta_0 \in (0, 1]$, i.e,

$$-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right) + 2\mathscr{A}'\mathscr{C}\right] \geq 0. \quad \text{(A.55)}$$

Since $\mathscr{C} > 0$, by (A.46), we have that

$$\mathscr{B}'\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right) + 2\mathscr{A}'\mathscr{C} \leq 0. \quad \text{(A.56)}$$

Moreover, $B' > 0$, by (A.48), therefore,

$$\sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}} \leq \frac{-2\mathscr{A}'\mathscr{C}}{\mathscr{B}'} - \mathscr{B} \quad \text{(A.57)}$$

where $\sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}} > 0$. Now squaring both sides of (A.57) we have

$$\mathscr{B}^2 + 4\mathscr{A}\mathscr{C} \leq \frac{4\mathscr{A}'^2\mathscr{C}^2}{\mathscr{B}'^2} + \frac{4\mathscr{A}'\mathscr{C}\mathscr{B}}{\mathscr{B}'} + \mathscr{B}^2. \quad \text{(A.58)}$$

Solving (A.58) for $\mathscr{A}$ we get

$$\mathscr{A} \leq \frac{\mathscr{A}'^2\mathscr{C}}{\mathscr{B}'^2} + \frac{\mathscr{A}'\mathscr{B}}{\mathscr{B}'}. \quad \text{(A.59)}$$

Multiplying both sides of (A.59) by $\mathscr{B}^2$ we deduce

$$\mathscr{A}\mathscr{B}'^2 \leq \mathscr{A}'^2\mathscr{C} + \mathscr{A}'\mathscr{B}\mathscr{B}' = \mathscr{A}'\left(\mathscr{A}'\mathscr{C} + \mathscr{B}\mathscr{B}'\right). \tag{A.60}$$

Now, we substitute the values of A and A' given by (A.44) and (A.47) in (A.60) to obtain:

$$(1-\beta)\left(2K + a_1K^2\right)\mathscr{B}'^2 \leq -\left(2K + a_1K^2\right)\left[-\left(2K + a_1K^2\right)\mathscr{C} + \mathscr{B}\mathscr{B}'\right], \tag{A.61}$$

and since $\left(2K + a_1K^2\right) > 0$, we have that

$$(1-\beta)\mathscr{B}'^2 \leq -\left[-\left(2K + a_1K^2\right)\mathscr{C} + \mathscr{B}\mathscr{B}'\right] = \left(2K + a_1K^2\right)\mathscr{C} - \mathscr{B}\mathscr{B}'. \tag{A.62}$$

The latter implies

$$(1-\beta)\mathscr{B}'^2 + \mathscr{B}\mathscr{B}' - \left(2K + a_1K^2\right)\mathscr{C} = \left[(1-\beta)\mathscr{B}' + \mathscr{B}\right]\mathscr{B}' - \left(2K + a_1K^2\right)\mathscr{C} \leq 0. \tag{A.63}$$

Plugging equations (A.45), (A.46) and (A.48) in (A.63) we yield

$$\begin{aligned}
&\left[(1-\beta)\mathscr{B}' + \mathscr{B}\right]\mathscr{B}' - \left(2K + a_1K^2\right)\mathscr{C} = \\
&= \left[(1-\beta)\left(1 + a_1K\right) + \left(\beta + 2a_0K + \beta a_1K + a_0a_1K^2\right)\right]\left(1 + a_1K\right) \\
&\quad - \left(2K + a_1K^2\right)\left(a_0 + a_1 + a_0a_1K\right) \\
&= \left(1 + 2a_0K + a_1K + a_0a_1K^2\right)\left(1 + a_1K\right) \\
&\quad - \left(2 + a_1K\right)\left(a_0K + a_1K + a_0a_1K^2\right) \\
&= 1 + \left(a_1K\right) + \left(2a_0K + a_1K + a_0a_1K^2\right) + \left(2a_0K + a_1K + a_0a_1K^2\right)\left(a_1K\right) \\
&\quad - \left(2 + a_1K\right)\left(a_0K + a_1K + a_0a_1K^2\right) \\
&= 1 + 2\left(a_0K + a_1K + a_0a_1K^2\right) + \left(a_0K + a_1K + a_0a_1K^2\right)\left(a_1K\right) \\
&\quad - \left(2 + a_1K\right)\left(a_0K + a_1K + a_0a_1K^2\right) \\
&= 1 + \left(2 + a_1K\right)\left(a_0K + a_1K + a_0a_1K^2\right) \\
&\quad - \left(2 + a_1K\right)\left(a_0K + a_1K + a_0a_1K^2\right) \\
&= 1 > 0,
\end{aligned} \tag{A.64}$$

which contradicts (A.63). Hence, (A.55) cannot hold for any $\beta \in (0, 1]$, which implies

$$-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right) + 2\mathscr{A}'\mathscr{C}\right] < 0 \tag{A.65}$$

for all $\beta \in (0, 1]$.

Therefore, from (A.54) and (A.65), we conclude that

$$\nu_0^{*\prime} = \frac{-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\right]}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}} < 0 \tag{A.66}$$

for all $\beta \in (0, 1]$. On account of the latter, $\nu_0^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in (0, 1]$.

From (A.42), it is clear that ν_1^* is continuously differentiable with respect to ν_0^* and, since $\nu_0^*(\beta)$, on its own, is also smooth as a function of β, then, $\nu_1^*(\beta)$, is continuously differentiable by β.

Differentiating (A.43) with respect to β we get

$$\nu_0^{*\prime} = \frac{1}{\left(\frac{1}{\nu_1^*+a_1}+K\right)^2}\left(\frac{1}{\left(\nu_1^*+a_1\right)^2}\nu_1^{*\prime}\right) = \nu_0^{*2}\left(\frac{\nu_1^{*\prime}}{\left(\nu_1^*+a_1\right)^2}\right) = \left(\frac{\nu_0^*}{\nu_1^*+a_1}\right)^2\nu_1^{*\prime}. \tag{A.67}$$

Since $\nu_0^{*\prime} < 0$, then, (A.67) implies that $\nu_1^{*\prime} < 0$, for all $\beta \in (0, 1]$. Thus, $\nu_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in (0, 1]$. Before continuing the proof, we are going to establish the following inequality:

$$\nu_0^* + \beta\nu_0^{*\prime} > 0. \tag{A.68}$$

Substituting (A.50) and (A.53) in (A.68), we get

$$\begin{aligned}\nu_0^* + \beta\nu_0^{*\prime} =& \frac{2\mathscr{C}}{\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}} + \beta\frac{-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\right]}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}}\\ =& \frac{2\mathscr{C}}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}}\\ &\cdot\left\{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}} - \beta\left[\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\right]\right\}\\ =& \frac{2\mathscr{C}}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}}\\ &\cdot\left[\left(-\beta\mathscr{B}'+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right) - 2\beta\mathscr{A}'\mathscr{C}\right].\end{aligned} \tag{A.69}$$

By (A.46), (A.47) and (A.54),

$$\frac{2\mathscr{C}}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}} > 0 \tag{A.70}$$

and

$$-2\beta\mathscr{A}'\mathscr{C} > 0. \tag{A.71}$$

Then, to prove inequality (A.68), it suffices to show that

$$\left(-\beta\mathscr{B}' + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right)\left(\mathscr{B} + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}}\right) > 0, \tag{A.72}$$

which, by (A.51), is equivalent to show that

$$-\beta\mathscr{B}' + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}} > 0. \tag{A.73}$$

Suppose, on the contrary, that

$$-\beta\mathscr{B}' + \sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}} \leq 0. \tag{A.74}$$

Then,

$$\sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}} \leq \beta\mathscr{B}' \tag{A.75}$$

where $\sqrt{\mathscr{B}^2 + 4\mathscr{A}\mathscr{C}} > 0$. Hence, by squaring both sides of (A.75) we have

$$\mathscr{B}^2 + 4\mathscr{A}\mathscr{C} \leq \beta^2\mathscr{B}'^2. \tag{A.76}$$

Plugging (A.45) and (A.48) in (A.76) yields

$$\left(\beta + 2a_0K + \beta a_1K + a_0a_1K^2\right)^2 + 4\mathscr{A}\mathscr{C} \leq \beta^2\left(1 + a_1K\right)^2, \tag{A.77}$$

which implies

$$\left[(\beta + \beta a_1K) + 2a_0K + a_0a_1K^2\right]^2 + 4\mathscr{A}\mathscr{C} \leq (\beta + \beta a_1K)^2. \tag{A.78}$$

However, by (A.44) and (A.46),

$$4\mathscr{A}\mathscr{C} \geq 0, \tag{A.79}$$

that is,

$$\left[(\beta + \beta a_1K) + 2a_0K + a_0a_1K^2\right]^2 \leq (\beta + \beta a_1K)^2. \tag{A.80}$$

On the other hand,

$$2a_0K + a_0a_1K^2 > 0, \tag{A.81}$$

whence

$$(\beta + \beta a_1K) < (\beta + \beta a_1K) + 2a_0K + a_0a_1K^2 \tag{A.82}$$

where $(\beta + \beta a_1K) > 0$. Now by squaring both sides of (A.75) we have

$$(\beta + \beta a_1 K)^2 < \left[(\beta + \beta a_1 K) + 2a_0 K + a_0 a_1 K^2\right]^2. \tag{A.83}$$

Nevertheless, inequality (A.83) contradicts (A.80), which means that (A.73) must hold and thus prove (A.68).

Now, coming back to the proof of the theorem, we are going to show that the equilibrium price $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β. Consider again the function (A.4) and by plugging it in $G(p^*) = -Kp^* + T$ get the following relationships:

$$\begin{aligned} &Q(p^*, \nu_0^*, \nu_1^*) - G(p^*) - D = \\ &= p^* \left[\frac{1}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0}\left(\frac{1}{\nu_1^* + a_1}\right)\right] \\ &\quad - \left[\frac{b_0}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0}\left(\frac{b_1}{\nu_1^* + a_1}\right)\right] \\ &\quad + Kp^* - T - D = 0. \end{aligned} \tag{A.84}$$

Consider the function

$$\begin{aligned} \mathscr{F}(p^*, \beta) = &p^* \left[\frac{1}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0}\left(\frac{1}{\nu_1^* + a_1}\right)\right] \\ &- \left[\frac{b_0}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0}\left(\frac{b_1}{\nu_1^* + a_1}\right)\right] \\ &+ Kp^* - T - D, \end{aligned} \tag{A.85}$$

having in mind that ν_0^* and ν_1^* depend on β, but not on p^*. Now, we rewrite (A.84) using (A.85) as a functional equation:

$$\mathscr{F}(p^*, \beta) = 0. \tag{A.86}$$

Now we are in a position to estimate the value of the partial derivative of the function $\mathscr{F}(p^*, \beta)$ with respect to p^*:

$$\frac{\partial \mathscr{F}}{\partial p^*} = \frac{1}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0}\left(\frac{1}{\nu_1^* + a_1}\right) + K \geq K > 0. \tag{A.87}$$

We observe that the partial derivative $\mathscr{F}$ with respect to p^* is positive. Hence, by the Implicit Function Theorem, the function $p^* = p^*(\beta)$ is differentiable with respect to β, and its partial derivative with respect to β can be found from the equation

$$\frac{\partial \mathscr{F}}{\partial p^*}\frac{dp^*}{d\beta} + \frac{\partial \mathscr{F}}{\partial \beta} = 0, \tag{A.88}$$

which leads to

$$\frac{dp^*}{d\beta} = -\frac{\dfrac{\partial \mathscr{F}}{\partial \beta}}{\dfrac{\partial \mathscr{F}}{\partial p^*}}. \tag{A.89}$$

From (A.87), we have

$$\frac{\partial \mathscr{F}}{\partial p^*} > 0. \tag{A.90}$$

Therefore, to prove that p^* is strictly increasing, we have to show that

$$\frac{\partial \mathscr{F}}{\partial \beta} > 0. \tag{A.91}$$

Indeed,

$$\begin{aligned}
\frac{\partial \mathscr{F}}{\partial \beta} =& \frac{\partial}{\partial \beta}\left(Q(p^*, \nu_0^*, \nu_1^*) - G(p^*) - D\right) = \frac{\partial}{\partial \beta} Q(p^*, \nu_0^*, \nu_1^*) \\
=& \frac{\partial}{\partial \beta}\left(q_0(p^*, \nu_0^*, \nu_1^*) + q_1(p^*, \nu_0^*, \nu_1^*)\right) \\
=& \frac{\partial}{\partial \beta}\left[\frac{p^* - b_0}{(1-\beta)\nu_0^* + a_0} + \frac{\beta \nu_0^*}{(1-\beta)\nu_0^* + a_0}\left(\frac{p^* - b_1}{\nu_1^* + a_1}\right) + \frac{p^* - b_1}{\nu_1^* + a_1}\right] \\
=& -\frac{p^* - b_0}{\left[(1-\beta)\nu_0^* + a_0\right]^2}\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right] \\
&+ \frac{\left(\nu_0^* + \beta \nu_0^{*\prime}\right)\left[(1-\beta)\nu_0^* + a_0\right] - \beta \nu_0^*\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right]}{\left[(1-\beta)\nu_0^* + a_0\right]^2}\left(\frac{p^* - b_1}{\nu_1^* + a_1}\right) \\
&+ \frac{\beta \nu_0^*}{(1-\beta)\nu_0^* + a_0}\left(-\frac{p^* - b_1}{\left(\nu_1^* + a_1\right)^2}\nu_1^{*\prime}\right) + \left(-\frac{p^* - b_1}{\left(\nu_1^* + a_1\right)^2}\nu_1^{*\prime}\right) \\
=& \frac{p^* - b_0}{\left[(1-\beta)\nu_0^* + a_0\right]^2}\left[\nu_0^* + (1-\beta)\left(-\nu_0^{*\prime}\right)\right] \\
&+ \frac{\left(\nu_0^* + \beta \nu_0^{*\prime}\right)\left[(1-\beta)\nu_0^* + a_0\right] + \beta \nu_0^*\left[\nu_0^* + (1-\beta)\left(-\nu_0^{*\prime}\right)\right]}{\left[(1-\beta)\nu_0^* + a_0\right]^2}\left(\frac{p^* - b_1}{\nu_1^* + a_1}\right) \\
&+ \frac{\beta \nu_0^*}{(1-\beta)\nu_0^* + a_0}\left[\frac{p^* - b_1}{\left(\nu_1^* + a_1\right)^2}\left(-\nu_1^{*\prime}\right)\right] + \left[\frac{p^* - b_1}{\left(\nu_1^* + a_1\right)^2}\left(-\nu_1^{*\prime}\right)\right].
\end{aligned} \tag{A.92}$$

Given the values of a_0, a_1, b_0, b_1, β, ν_0^*, ν_1^*, $\nu_0^{*\prime}$, $\nu_1^{*\prime}$, p^* and Eq. (A.68), it isn't difficult to see that (A.92) is nonnegative. Moreover,

$$\begin{aligned}\frac{\partial \mathscr{F}}{\partial \beta} =& \frac{p^* - b_0}{\left[(1-\beta)\nu_0^* + a_0\right]^2}\left[\nu_0^* + (1-\beta)\left(-\nu_0^{*\prime}\right)\right] \\ &+ \frac{\left(\nu_0^* + \beta\nu_0^{*\prime}\right)\left[(1-\beta)\nu_0^* + a_0\right] + \beta\nu_0^*\left[\nu_0^* + (1-\beta)\left(-\nu_0^{*\prime}\right)\right]}{\left[(1-\beta)\nu_0^* + a_0\right]^2}\left(\frac{p^* - b_1}{\nu_1^* + a_1}\right) \\ &+ \frac{\beta\nu_0^*}{(1-\beta)\nu_0^* + a_0}\left[\frac{p^* - b_1}{\left(\nu_1^* + a_1\right)^2}\left(-\nu_1^{*\prime}\right)\right] + \left[\frac{p^* - b_1}{\left(\nu_1^* + a_1\right)^2}\left(-\nu_1^{*\prime}\right)\right] \\ \geq& \frac{p^* - b_1}{\left(\nu_1^* + a_1\right)^2}\left(-\nu_1^{*\prime}\right) > 0,\end{aligned} \tag{A.93}$$

which proves (A.91). On account of that,

$$\frac{dp^*}{d\beta} = -\frac{\dfrac{\partial \mathscr{F}}{\partial \beta}}{\dfrac{\partial \mathscr{F}}{\partial p^*}} < 0, \tag{A.94}$$

where $\dfrac{\partial \mathscr{F}}{\partial \beta}$ and $\dfrac{\partial \mathscr{F}}{\partial p^*}$ are continuous with respect to β. Hence $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in (0, 1]$.

Now, since

$$G^*(\beta) = G(p^*(\beta)) = -Kp^*(\beta) + T, \tag{A.95}$$

and $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, and K and T are positive constants, then, $G^*(\beta)$ is continuously differentiable and strictly increasing with respect to β, $\beta \in (0, 1]$.

Now, we are going to show that $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β. To do that, we first solve Eq. (A.84) for p^* to obtain the following equality:

$$\begin{aligned}p^* =& \frac{\dfrac{b_0}{(1-\beta)\nu_0^* + a_0} + \dfrac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0}\left(\dfrac{b_1}{\nu_1^* + a_1}\right) + T + D}{\dfrac{1}{(1-\beta)\nu_0^* + a_0} + \dfrac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0}\left(\dfrac{1}{\nu_1^* + a_1}\right) + K} \\ =& \frac{b_0 + \left(\nu_0^* + a_0\right)\left(\dfrac{b_1}{\nu_1^* + a_1}\right) + \left[(1-\beta)\nu_0^* + a_0\right](T + D)}{1 + \left(\nu_0^* + a_0\right)\left(\dfrac{1}{\nu_1^* + a_1}\right) + \left[(1-\beta)\nu_0^* + a_0\right]K} \\ =& \frac{\left(\nu_0^* + a_0\right)b_1 + \left(\nu_1^* + a_1\right)b_0 + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)(T + D)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)K}.\end{aligned} \tag{A.96}$$

We substitute (A.96) in $q_1^* = q_1(p^*, \nu_0^*, \nu_1^*)$, to deduce

$$
\begin{aligned}
q_1^* =& \frac{p^* - b_1}{\nu_1^* + a_1} \\
=& \frac{\dfrac{\left(\nu_0^* + a_0\right) b_1 + \left(\nu_1^* + a_1\right) b_0 + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)(T + D)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K} - b_1}{\nu_1^* + a_1} \\
=& \frac{\left(\nu_0^* + a_0\right) b_1 + \left(\nu_1^* + a_1\right) b_0 + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)(T + D)}{\left(\nu_1^* + a_1\right)\left\{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K\right\}} \\
& - \frac{\left\{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K\right\} b_1}{\left(\nu_1^* + a_1\right)\left\{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K\right\}} \\
=& \frac{\left(\nu_1^* + a_1\right) b_0 + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)(T + D)}{\left(\nu_1^* + a_1\right)\left\{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K\right\}} \\
& - \frac{\left(\nu_1^* + a_1\right) b_1 + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K b_1}{\left(\nu_1^* + a_1\right)\left\{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K\right\}} \\
=& \frac{b_0 + \left[(1-\beta)\nu_0^* + a_0\right](T + D)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K} \\
& - \frac{b_1 + \left[(1-\beta)\nu_0^* + a_0\right] K b_1}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K} \\
=& \frac{-b_1 + b_0 + \left[(1-\beta)\nu_0^* + a_0\right](-K b_1 + T + D)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K} \\
=& \frac{-(b_1 - b_0) + \left[(1-\beta)\nu_0^* + a_0\right](G(b_1) + D)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right)\left\{1 + \left[(1-\beta)\nu_0^* + a_0\right] K\right\}}.
\end{aligned}
\tag{A.97}
$$

By plugging (A.42) in (A.97) we have that

$$
\begin{aligned}
q_1^* =& \frac{-(b_1 - b_0) + \left[(1-\beta)\nu_0^* + a_0\right](G(b_1) + D)}{\left(\nu_0^* + a_0\right) + \left[\dfrac{(1-\beta)\nu_0^* + a_0}{1 + \left[(1-\beta)\nu_0^* + a_0\right] K} + a_1\right]\left\{1 + \left[(1-\beta)\nu_0^* + a_0\right] K\right\}} \\
=& \frac{-(b_1 - b_0) + \left[(1-\beta)\nu_0^* + a_0\right](G(b_1) + D)}{\left(\nu_0^* + a_0\right) + \left[(1-\beta)\nu_0^* + a_0\right] + a_1\left\{1 + \left[(1-\beta)\nu_0^* + a_0\right] K\right\}} \\
=& \frac{-(b_1 - b_0) + \left[(1-\beta)\nu_0^* + a_0\right](G(b_1) + D)}{\left(\nu_0^* + a_0 + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right](1 + a_1 K)} = \frac{M}{N},
\end{aligned}
\tag{A.98}
$$

where

$$
M = M(\beta) = -(b_1 - b_0) + \left[(1-\beta)\nu_0^* + a_0\right](G(b_1) + D) \tag{A.99}
$$

and

$$N = N(\beta) = \left(\nu_0^* + a_0 + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right](1 + a_1 K)\,. \tag{A.100}$$

It is easy to see that M and N are continuously differentiable with respect to β with

$$M' = \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1) + D)\,, \tag{A.101}$$
$$N' = \nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\,. \tag{A.102}$$

Moreover, $N > 0$, so q_1^* is continuously differentiable with respect to β and

$$q_1^{*\prime} = \frac{M'N - MN'}{N^2}. \tag{A.103}$$

Thus, to find the value of $q_1^{*\prime}$ it suffices to estimate the value of the numerator of (A.103).

$$\begin{aligned}
&M'N - MN' = \\
&= \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1)+D)\left\{\left(\nu_0^* + a_0 + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right](1 + a_1 K)\right\} \\
&\quad - \left\{-(b_1 - b_0) + \left[(1-\beta)\nu_0^* + a_0\right](G(b_1)+D)\right\}\left\{\nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\right\} \\
&= \left(\nu_0^* + a_0 + a_1\right)\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1)+D) \\
&\quad + \left[(1-\beta)\nu_0^* + a_0\right]\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)(G(b_1)+D) \\
&\quad + \left\{\nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\right\}(b_1 - b_0) \\
&\quad - \left\{\nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\right\}\left[(1-\beta)\nu_0^* + a_0\right](G(b_1)+D) \\
&= \left(\nu_0^* + a_0\right)\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1)+D) \\
&\quad + a_1\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1)+D) \\
&\quad + \left[(1-\beta)\nu_0^* + a_0\right]\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)(G(b_1)+D) \\
&\quad + \left\{\nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\right\}(b_1 - b_0) \\
&\quad - \nu_0^{*\prime}\left[(1-\beta)\nu_0^* + a_0\right](G(b_1)+D) \\
&\quad - \left[(1-\beta)\nu_0^* + a_0\right]\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)(G(b_1)+D) \\
&= \left(\nu_0^* + a_0\right)\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1)+D) \\
&\quad + a_1\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1)+D) \\
&\quad + \left\{\nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\right\}(b_1 - b_0) \\
&\quad - \nu_0^{*\prime}\left[(1-\beta)\nu_0^* + a_0\right](G(b_1)+D) \\
&= \left\{\left(\nu_0^* + a_0\right)\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right] - \nu_0^{*\prime}\left[(1-\beta)\nu_0^* + a_0\right]\right\}(G(b_1)+D) \\
&\quad + a_1\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1)+D) \\
&\quad + \left\{\nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\right\}(b_1 - b_0) \\
&= \left[\left(\nu_0^* + a_0\right)\left(-\nu_0^* - \beta\nu_0^{*\prime}\right) + \beta\nu_0^*\nu_0^{*\prime}\right](G(b_1)+D) \\
&\quad + a_1\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1)+D) \\
&\quad + \left\{\nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\right\}(b_1 - b_0).
\end{aligned} \tag{A.104}$$

Given the values of a_0, a_1, b_0, b_1, β, ν_0^*, $\nu_0^{*\prime}$, $G(p)$, D and Eq. (A.68), it is clear that (A.104) is non-positive. Moreover,

$$\begin{aligned} M'N - MN' =& \left[\left(\nu_0^* + a_0\right)\left(-\nu_0^* - \beta\nu_0^{*\prime}\right) + \beta\nu_0^*\nu_0^{*\prime}\right](G(b_1) + D) \\ &+ a_1\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1) + D) \\ &+ \left\{\nu_0^{*\prime} + \left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](1 + a_1 K)\right\}(b_1 - b_0) \\ \leq& a_1\left[-\nu_0^* + (1-\beta)\nu_0^{*\prime}\right](G(b_1) + D) < 0. \end{aligned} \tag{A.105}$$

Thus,

$$M'N - MN' < 0, \tag{A.106}$$

which proves that $q_1^{*\prime} < 0$, so $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in (0, 1]$.

Finally, since

$$q_0^*(\beta) + q_1^*(\beta) = G^*(\beta) + D, \tag{A.107}$$

then,

$$q_0^*(\beta) = -q_1^*(\beta) + G^*(\beta) + D. \tag{A.108}$$

And since $G^*(\beta)$ is continuously differentiable and strictly increasing with respect to β, the function $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β. Because D is constant, we have that $q_0^*(\beta)$ is continuously differentiable and strictly increasing with respect to β, $\beta \in (0, 1]$ The proof of the theorem is complete ■

Theorem 2.4 *For the affine demand function $G(p)$ described in (2.22), the price $p^c(\beta)$ and the supply values $q_i^c(\beta)$, $i = 0, 1$, from the Cournot-Nash equilibrium, are continuously differentiable with respect to $\beta \in (0, 1]$. Moreover, $p^c(\beta)$ and $q_1^c(\beta)$ strictly decrease, whereas $q_0^c(\beta)$ strictly increase.*

Proof Let's consider the exterior equilibrium (p^c, q_0^c, q_1^c), i.e., such a vector that the following equalities hold:

$$q_0^c + q_1^c = G(p^c) + D, \tag{A.109}$$

$$q_0^c = \frac{p^c - b_0}{(1-\beta)\dfrac{1}{K} + a_0} + \frac{\beta\dfrac{1}{K}}{(1-\beta)\dfrac{1}{K} + a_0}\left(\frac{p^c - b_1}{\dfrac{1}{K} + a_1}\right), \tag{A.110}$$

$$q_1^c = \frac{p^c - b_1}{\dfrac{1}{K} + a_1}, \tag{A.111}$$

where

$$G(p^c) = -Kp^c + T. \tag{A.112}$$

From Eq. (A.109) one has

$$q_0^c + q_1^c - G(p^c) - D = 0. \tag{A.113}$$

By substituting (A.110), (A.111) and (A.112) in (A.113), similarly to (A.4), we have

$$\begin{aligned}
q_0^c + q_1^c - G(p^c) - D =& \frac{p^c - b_0}{(1-\beta)\frac{1}{K} + a_0} + \frac{\beta \frac{1}{K}}{(1-\beta)\frac{1}{K} + a_0} \left(\frac{p^c - b_1}{\frac{1}{K} + a_1} \right) \\
&+ \frac{p^c - b_1}{\frac{1}{K} + a_1} + Kp^c - T - D \\
=& p^c \left[\frac{1}{(1-\beta)\frac{1}{K} + a_0} + \frac{\frac{1}{K} + a_0}{(1-\beta)\frac{1}{K} + a_0} \left(\frac{1}{\frac{1}{K} + a_1} \right) \right] \\
&- \left[\frac{b_0}{(1-\beta)\frac{1}{K} + a_0} + \frac{\frac{1}{K} + a_0}{(1-\beta)\frac{1}{K} + a_0} \left(\frac{b_1}{\frac{1}{K} + a_1} \right) \right] \\
&+ Kp^c - T - D = 0.
\end{aligned} \tag{A.114}$$

Solving (A.114) for p^c, similarly to (A.96), we get the equation

$$\begin{aligned}
p^c =& \frac{\frac{b_0}{(1-\beta)\frac{1}{K} + a_0} + \frac{\frac{1}{K} + a_0}{(1-\beta)\frac{1}{K} + a_0} \left(\frac{b_1}{\frac{1}{K} + a_1} \right) + T + D}{\frac{1}{(1-\beta)\frac{1}{K} + a_0} + \frac{\frac{1}{K} + a_0}{(1-\beta)\frac{1}{K} + a_0} \left(\frac{1}{\frac{1}{K} + a_1} \right) + K} \\
=& \frac{\left(\frac{1}{K} + a_0 \right) b_1 + \left(\frac{1}{K} + a_1 \right) b_0 + \left[(1-\beta)\frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) (T + D)}{\left(\frac{1}{K} + a_0 \right) + \left(\frac{1}{K} + a_1 \right) + \left[(1-\beta)\frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) K} \\
=& \frac{X}{Y},
\end{aligned} \tag{A.115}$$

where

$$X(\beta) = \left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\frac{1}{K} + a_1\right)(T + D) \tag{A.116}$$

and

$$Y(\beta) = \left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\frac{1}{K} + a_1\right) K. \tag{A.117}$$

It's easy to see that X and Y are continuously differentiable with respect to β with

$$X' = -\frac{1}{K}\left(\frac{1}{K} + a_1\right)(T + D), \tag{A.118}$$

$$Y' = -\left(\frac{1}{K} + a_1\right). \tag{A.119}$$

Moreover, $Y > 0$, whence p^c is continuously differentiable with respect to β with

$$p^{c\prime} = \frac{X'Y - XY'}{Y^2}. \tag{A.120}$$

To compute the value of $p^{c\prime}$ it is sufficient to calculate the value of the numerator of (A.120):

$$\begin{aligned}
&X'Y - XY' = \\
&= -\tfrac{1}{K}\left(\tfrac{1}{K} + a_1\right)(T + D)\left\{\left(\tfrac{1}{K} + a_0\right) + \left(\tfrac{1}{K} + a_1\right) + \left[(1-\beta)\tfrac{1}{K} + a_0\right]\left(\tfrac{1}{K} + a_1\right) K\right\} \\
&\quad - \left\{\left(\tfrac{1}{K} + a_0\right) b_1 + \left(\tfrac{1}{K} + a_1\right) b_0 + \left[(1-\beta)\tfrac{1}{K} + a_0\right]\left(\tfrac{1}{K} + a_1\right)(T + D)\right\}\left[-\left(\tfrac{1}{K} + a_1\right)\right] \\
&= -\tfrac{1}{K}\left(\tfrac{1}{K} + a_1\right)(T + D)\left\{\left(\tfrac{1}{K} + a_0\right) + \left(\tfrac{1}{K} + a_1\right) + \left[(1-\beta)\tfrac{1}{K} + a_0\right]\left(\tfrac{1}{K} + a_1\right) K\right\} \\
&\quad + \left(\tfrac{1}{K} + a_1\right)\left\{\left(\tfrac{1}{K} + a_0\right) b_1 + \left(\tfrac{1}{K} + a_1\right) b_0 + \left[(1-\beta)\tfrac{1}{K} + a_0\right]\left(\tfrac{1}{K} + a_1\right)(T + D)\right\} \\
&= -\tfrac{1}{K}\left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right)(T + D) \\
&\quad - \tfrac{1}{K}\left(\tfrac{1}{K} + a_1\right)^2 (T + D) \\
&\quad - \left[(1-\beta)\tfrac{1}{K} + a_0\right]\left(\tfrac{1}{K} + a_1\right)^2 (T + D) \\
&\quad + \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right) b_1 \\
&\quad + \left(\tfrac{1}{K} + a_1\right)^2 b_0 \\
&\quad + \left[(1-\beta)\tfrac{1}{K} + a_0\right]\left(\tfrac{1}{K} + a_1\right)^2 (T + D) \\
&= \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right) b_1 - \tfrac{1}{K}\left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right)(T + D) \\
&\quad + \left(\tfrac{1}{K} + a_1\right)^2 b_0 - \tfrac{1}{K}\left(\tfrac{1}{K} + a_1\right)^2 (T + D) \\
&= -\tfrac{1}{K}\left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right)(-K b_1 + T + D) \\
&\quad - \tfrac{1}{K}\left(\tfrac{1}{K} + a_1\right)^2 (-K b_0 + T + D) \\
&= -\tfrac{1}{K}\left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right)(G(b_1) + D) \\
&\quad - \tfrac{1}{K}\left(\tfrac{1}{K} + a_1\right)^2 (G(b_0) + D).
\end{aligned} \tag{A.121}$$

Given the values of $a_0, a_1, K, G(p)$ and D, it is clear that (A.121) is non-positive. Moreover,

$$\begin{aligned} X'Y - XY' = &- \frac{1}{K}\left(\frac{1}{K} + a_0\right)\left(\frac{1}{K} + a_1\right)(G(b_1) + D) \\ &- \frac{1}{K}\left(\frac{1}{K} + a_1\right)^2 (G(b_0) + D) \\ \leq &- \frac{1}{K}\left(\frac{1}{K} + a_0\right)\left(\frac{1}{K} + a_1\right)(G(b_1) + D) < 0. \end{aligned} \tag{A.122}$$

Then,

$$X'Y - XY' < 0, \tag{A.123}$$

which proves that $p^{c'} < 0$, so $p^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in (0, 1]$.

Since

$$q_1^c(\beta) = \frac{p^c(\beta) - b_1}{\dfrac{1}{K} + a_1}, \tag{A.124}$$

and $p^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β, and a_1, b_1 and K re positive constants, then, $q_1^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in (0, 1]$.

Finally, since

$$q_0^c(\beta) + q_1^c(\beta) = G(p^c(\beta)) + D = -Kp^c(\beta) + T + D, \tag{A.125}$$

then,

$$q_0^c(\beta) = -q_1^c(\beta) - Kp^c(\beta) + T + D. \tag{A.126}$$

And as $q_1^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β, the function $p^c(\beta)$ also has the same property, and K, T and D are non-negative constants, then, $q_0^*(\beta)$ is continuously differentiable and strictly increasing with respect to β, $\beta \in (0, 1]$. The proof of the theorem is complete ■

Theorem 2.5 *For the affine demand function $G(p)$ described in (2.22), the price $p^t(\beta)$ and the output volumes $q_i^t(\beta)$, $i = 0, 1$, related to the perfect competition equilibrium, are invariant for all $\beta \in (0, 1]$ and are described by the clear-cut expressions:*

$$p^t = \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)}{a_0 + a_1 + a_0 a_1 K}, \tag{A.127}$$

$$q_0^t = \frac{a_1 \left(G(b_0) + D\right) + (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}, \tag{A.128}$$

$$q_1^t = \frac{a_0 \left(G(b_1) + D\right) - (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}. \tag{A.129}$$

Proof Let us consider the exterior equilibrium (p^t, q_0^t, q_1^t), i.e., such a vector that the following equalities hold:

$$q_0^t + q_1^t = G(p^t) + D, \tag{A.130}$$

$$q_0^t = \frac{p^t - b_0}{a_0}, \tag{A.131}$$

$$q_1^t = \frac{p^t - b_1}{a_1}, \tag{A.132}$$

where

$$G(p^t) = -K p^t + T. \tag{A.133}$$

From (A.109) one gets that

$$q_0^t + q_1^t - G(p^t) - D = 0. \tag{A.134}$$

Next, by plugging (A.131), (A.132) and (A.133) in (A.134), we deduce that

$$\begin{aligned} q_0^t + q_1^t - G(p^t) - D =& \frac{p^t - b_0}{a_0} + \frac{p^t - b_1}{a_1} + K p^t - T - D \\ =& p^t \left(\frac{1}{a_0} + \frac{1}{a_1}\right) - \left(\frac{b_1}{a_1} + \frac{b_0}{a_0}\right) + K p^c - T - D = 0. \end{aligned} \tag{A.135}$$

By solving Eq. (A.135) for p^t, we obtain the equality

$$p^t = \frac{\frac{b_1}{a_1} + \frac{b_0}{a_0} + T + D}{\frac{1}{a_0} + \frac{1}{a_1} + K} = \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)}{a_0 + a_1 + a_0 a_1 K}, \tag{A.136}$$

showing that the function $p^t(\beta)$ is constant for all $\beta \in (0, 1]$.

Moreover, since

$$q_0^t = \frac{p^t - b_0}{a_0} = \frac{\dfrac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)}{a_0 + a_1 + a_0 a_1 K} - b_0}{a_0}$$
$$= \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D) - (a_0 + a_1 + a_0 a_1 K)\, b_0}{a_0\,(a_0 + a_1 + a_0 a_1 K)} \tag{A.137}$$
$$= \frac{a_0\,(b_1 - b_0) + a_0 a_1\,(-K b_0 + T + D)}{a_0\,(a_0 + a_1 + a_0 a_1 K)}$$
$$= \frac{a_1\,(G(b_0) + D) + (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K},$$

and

$$q_1^t = \frac{p^t - b_1}{a_1} = \frac{\dfrac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)}{a_0 + a_1 + a_0 a_1 K} - b_1}{a_1}$$
$$= \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D) - (a_0 + a_1 + a_0 a_1 K)\, b_1}{a_1\,(a_0 + a_1 + a_0 a_1 K)} \tag{A.138}$$
$$= \frac{-a_1\,(b_1 - b_0) + a_0 a_1\,(-K b_1 + T + D)}{a_1\,(a_0 + a_1 + a_0 a_1 K)}$$
$$= \frac{a_0\,(G(b_1) + D) - (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K},$$

the functions $q_0^t(\beta)$ and $q_1^t(\beta)$ are constant for all $\beta \in (0, 1]$, too. The proof of the theorem is complete ■

Theorem 2.6 *For the affine demand function $G(p)$ from (2.22), the price functions in the CCVE, $p^*(\beta)$, the Cournot-Nash equilibrium, $p^c(\beta)$, and the perfect competition equilibrium, p^t, satisfy the following inequalities:*

$$p^t < \lim_{\beta \to 0} p^*(\beta), \tag{A.139}$$

and

$$p^*(\beta) < p^c(\beta), \quad \forall \beta \in (0, 1]. \tag{A.140}$$

Proof First, we prove inequality (A.139):

$$p^t < \lim_{\beta \to 0} p^*(\beta).$$

Introduce the following notation:

$$\widehat{\nu_0^*} = \lim_{\beta \to 0} \nu_0^*(\beta)$$
$$= \frac{2(a_0 + a_1 + a_0 a_1 K)}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} > 0, \tag{A.141}$$

$$\widehat{\nu_1^*} = \lim_{\beta \to 0} \nu_1^*(\beta) = \lim_{\beta \to 0} \frac{(1-\beta)\nu_0^* + a_0}{1 + \left[(1-\beta)\nu_0^* + a_0\right] K} = \frac{\widehat{\nu_0^*} + a_0}{1 + \left(\widehat{\nu_0^*} + a_0\right) K} > 0. \tag{A.142}$$

Therefore,

$$\begin{aligned}\lim_{\beta \to 0} p^*(\beta) &= \lim_{\beta \to 0} \frac{\left(\nu_0^* + a_0\right) b_1 + \left(\nu_1^* + a_1\right) b_0 + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)(T+D)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K} \\ &= \frac{\left(\widehat{\nu_0^*} + a_0\right) b_1 + \left(\widehat{\nu_1^*} + a_1\right) b_0 + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right)(T+D)}{\left(\widehat{\nu_0^*} + a_0\right) + \left(\widehat{\nu_1^*} + a_1\right) + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right) K}.\end{aligned} \tag{A.143}$$

Now, we compute the difference

$$\begin{aligned}&\lim_{\beta \to 0} p^*(\beta) - p^t = \\ &= \frac{\left(\widehat{\nu_0^*} + a_0\right) b_1 + \left(\widehat{\nu_1^*} + a_1\right) b_0 + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right)(T+D)}{\left(\widehat{\nu_0^*} + a_0\right) + \left(\widehat{\nu_1^*} + a_1\right) + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right) K} \\ &\quad - \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T+D)}{a_0 + a_1 + a_0 a_1 K} \\ &= \frac{\left[\left(\widehat{\nu_0^*} + a_0\right) b_1 + \left(\widehat{\nu_1^*} + a_1\right) b_0 + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right)(T+D)\right](a_0 + a_1 + a_0 a_1 K)}{\left[\left(\widehat{\nu_0^*} + a_0\right) + \left(\widehat{\nu_1^*} + a_1\right) + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right) K\right](a_0 + a_1 + a_0 a_1 K)} \\ &\quad - \frac{\left[\left(\widehat{\nu_0^*} + a_0\right) + \left(\widehat{\nu_1^*} + a_1\right) + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right) K\right][a_0 b_1 + a_1 b_0 + a_0 a_1 (T+D)]}{\left[\left(\widehat{\nu_0^*} + a_0\right) + \left(\widehat{\nu_1^*} + a_1\right) + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right) K\right](a_0 + a_1 + a_0 a_1 K)} \\ &= \frac{R_1}{R_2},\end{aligned} \tag{A.144}$$

where

$$\begin{aligned}R_1 = &\left[\left(\widehat{\nu_0^*} + a_0\right) b_1 + \left(\widehat{\nu_1^*} + a_1\right) b_0 + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right)(T+D)\right](a_0 + a_1 + a_0 a_1 K) \\ &- \left[\left(\widehat{\nu_0^*} + a_0\right) + \left(\widehat{\nu_1^*} + a_1\right) + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right) K\right][a_0 b_1 + a_1 b_0 + a_0 a_1 (T+D)]\end{aligned} \tag{A.145}$$

and

$$R_2 = \left[\left(\widehat{\nu_0^*} + a_0\right) + \left(\widehat{\nu_1^*} + a_1\right) + \left(\widehat{\nu_0^*} + a_0\right)\left(\widehat{\nu_1^*} + a_1\right) K\right](a_0 + a_1 + a_0 a_1 K). \tag{A.146}$$

Given the values of a_0, a_1, $\widehat{\nu_0^*}$, $\widehat{\nu_1^*}$, and K, it is easy to see that $R_2 > 0$. Hence, to calculate the value of (A.144), it is enough to estimate the value of (A.145). That is,

$$
\begin{aligned}
R_1 =& \left[\left(\widehat{\nu_0^*}+a_0\right)b_1+\left(\widehat{\nu_1^*}+a_1\right)b_0+\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)(T+D)\right](a_0+a_1+a_0a_1K)\\
&-\left[\left(\widehat{\nu_0^*}+a_0\right)+\left(\widehat{\nu_1^*}+a_1\right)+\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)K\right][a_0b_1+a_1b_0+a_0a_1(T+D)]\\
=&(a_0+a_1+a_0a_1K)\left(\widehat{\nu_0^*}+a_0\right)b_1\\
&+(a_0+a_1+a_0a_1K)\left(\widehat{\nu_1^*}+a_1\right)b_0\\
&+(a_0+a_1+a_0a_1K)\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)(T+D)\\
&-a_0\left[\left(\widehat{\nu_0^*}+a_0\right)+\left(\widehat{\nu_1^*}+a_1\right)+\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)K\right]b_1\\
&-a_1\left[\left(\widehat{\nu_0^*}+a_0\right)+\left(\widehat{\nu_1^*}+a_1\right)+\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)K\right]b_0\\
&-a_0a_1\left[\left(\widehat{\nu_0^*}+a_0\right)+\left(\widehat{\nu_1^*}+a_1\right)+\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)K\right](T+D)\\
=&\left\{(a_1+a_0a_1K)\left(\widehat{\nu_0^*}+a_0\right)-a_0\left[\left(\widehat{\nu_1^*}+a_1\right)+\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)K\right]\right\}b_1\\
&+\left\{(a_0+a_0a_1K)\left(\widehat{\nu_1^*}+a_1\right)-a_1\left[\left(\widehat{\nu_0^*}+a_0\right)+\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)K\right]\right\}b_0\\
&+\left\{(a_0+a_1)\left(\widehat{\nu_0^*}+a_0\right)\left(\widehat{\nu_1^*}+a_1\right)-a_0a_1\left[\left(\widehat{\nu_0^*}+a_0\right)+\left(\widehat{\nu_1^*}+a_1\right)\right]\right\}(T+D)\\
=&\left[a_1\widehat{\nu_0^*}-a_0\widehat{\nu_1^*}-a_0\widehat{\nu_1^*}\left(\widehat{\nu_0^*}+a_0\right)K\right]b_1\\
&+\left[a_0\widehat{\nu_1^*}-a_1\widehat{\nu_0^*}-a_1\widehat{\nu_0^*}\left(\widehat{\nu_1^*}+a_1\right)K\right]b_0\\
&+\left[a_0\widehat{\nu_1^*}\left(\widehat{\nu_0^*}+a_0\right)+a_1\widehat{\nu_0^*}\left(\widehat{\nu_1^*}+a_1\right)\right](T+D)\\
=&-a_0\widehat{\nu_1^*}\left(\widehat{\nu_0^*}+a_0\right)Kb_1+a_0\widehat{\nu_1^*}\left(\widehat{\nu_0^*}+a_0\right)(T+D)\\
&-a_1\widehat{\nu_0^*}\left(\widehat{\nu_1^*}+a_1\right)Kb_0+a_1\widehat{\nu_0^*}\left(\widehat{\nu_1^*}+a_1\right)(T+D)\\
&+\left(a_1\widehat{\nu_0^*}-a_0\widehat{\nu_1^*}\right)b_1+\left(a_0\widehat{\nu_1^*}-a_1\widehat{\nu_0^*}\right)b_0\\
=&a_0\widehat{\nu_1^*}\left(\widehat{\nu_0^*}+a_0\right)(-Kb_1+T+D)\\
&+a_1\widehat{\nu_0^*}\left(\widehat{\nu_1^*}+a_1\right)(-Kb_0+T+D)\\
&+\left(a_1\widehat{\nu_0^*}-a_0\widehat{\nu_1^*}\right)(b_1-b_0)\\
=&a_0\widehat{\nu_1^*}\left(\widehat{\nu_0^*}+a_0\right)(G(b_1)+D)-a_0\widehat{\nu_1^*}(b_1-b_0)\\
&+a_1\widehat{\nu_0^*}\left(\widehat{\nu_1^*}+a_1\right)(G(b_0)+D)+a_1\widehat{\nu_0^*}(b_1-b_0)\\
=&a_0\widehat{\nu_1^*}\left[\left(\widehat{\nu_0^*}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]\\
&+a_1\widehat{\nu_0^*}\left[\left(\widehat{\nu_1^*}+a_1\right)(G(b_0)+D)+(b_1-b_0)\right].
\end{aligned}
\tag{A.147}
$$

Given the values of $a_0, a_1, b_0, b_1, \widehat{\nu_0^*}, \widehat{\nu_1^*}, G(p), D$ and assumption **A**3, it is trivial that (A.147) is nonnegative. Moreover,

$$\begin{aligned} R_1 =& a_0 \widehat{\nu_1^*} \left[\left(\widehat{\nu_0^*} + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right] \\ &+ a_1 \widehat{\nu_0^*} \left[\left(\widehat{\nu_1^*} + a_1 \right) (G(b_0) + D) + (b_1 - b_0) \right] \\ \geq& a_1 \widehat{\nu_0^*} \left[\left(\widehat{\nu_1^*} + a_1 \right) (G(b_0) + D) + (b_1 - b_0) \right] \geq a_1 \widehat{\nu_0^*} \left(\widehat{\nu_1^*} + a_1 \right) (G(b_0) + D) \\ \geq& a_1^2 \widehat{\nu_0^*} (G(b_0) + D) \geq a_1^2 \widehat{\nu_0^*} G(b_0) > 0. \end{aligned} \tag{A.148}$$

And since $R_1 > 0$, by (A.148), then,

$$\lim_{\beta \to 0} p^*(\beta) - p^t > 0, \tag{A.149}$$

which proves inequality (A.139).

Now, we establish inequality (A.140):

$$p^*(\beta) < p^c(\beta) \text{ para todo } \beta \in (0, 1].$$

In order to do that, we introduce the following notation:

$$\nu_i^* = \nu_i^*(\beta), i = 0, 1.$$

From Eqs. (A.42) and (A.43) it's easy to see that the following inequality hold for all $\beta \in (0, 1]$:

$$\nu_i^* < \frac{1}{K}, i = 0, 1. \tag{A.150}$$

Now, we compute the difference

$$\begin{aligned} &\left(p^c - p^*\right)(\beta) = \\ &= \frac{\left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\frac{1}{K} + a_1\right)(T + D)}{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\frac{1}{K} + a_1\right) K} \\ &\quad - \frac{\left(\nu_0^* + a_0\right) b_1 + \left(\nu_1^* + a_1\right) b_0 + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)(T + D)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K} \\ &= \left\langle \left\{ \left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\frac{1}{K} + a_1\right)(T + D) \right\} \times \right. \\ &\quad \left\{ \left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K \right\} \\ &\quad - \left\{ \left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\frac{1}{K} + a_1\right) K \right\} \times \\ &\quad \left. \left\{ \left(\nu_0^* + a_0\right) b_1 + \left(\nu_1^* + a_1\right) b_0 + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)(T + D) \right\} \right\rangle \Big/ \\ &\quad \left\langle \left\{ \left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\frac{1}{K} + a_1\right) K \right\} \times \right. \\ &\quad \left. \left\{ \left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right) K \right\} \right\rangle \\ &= \frac{S_1}{S_2}, \end{aligned} \tag{A.151}$$

where

$$S_1 = \left\{\left(\frac{1}{K}+a_0\right)b_1+\left(\frac{1}{K}+a_1\right)b_0+\left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)(T+D)\right\}\times \\ \left\{\left(\nu_0^*+a_0\right)+\left(\nu_1^*+a_1\right)+\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)K\right\} \\ -\left\{\left(\frac{1}{K}+a_0\right)+\left(\frac{1}{K}+a_1\right)+\left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K\right\}\times \\ \left\{\left(\nu_0^*+a_0\right)b_1+\left(\nu_1^*+a_1\right)b_0+\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)(T+D)\right\} \tag{A.152}$$

and

$$S_2 = \left\{\left(\tfrac{1}{K}+a_0\right)+\left(\tfrac{1}{K}+a_1\right)+\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\tfrac{1}{K}+a_1\right)K\right\}\times \\ \left\{\left(\nu_0^*+a_0\right)+\left(\nu_1^*+a_1\right)+\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)K\right\}. \tag{A.153}$$

For any fixed values of $a_0, a_1, \beta, \nu_0^*, \nu_1^*$, and K, it is apparent that $S_2 > 0$. Because of that, in order to find the value of (A.151), it suffices to calculate the value of (A.152). So,

$$\begin{aligned} S_1 =& \left\{\left(\nu_0^*+a_0\right)+\left(\nu_1^*+a_1\right)+\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)K\right\}\left(\tfrac{1}{K}+a_0\right)b_1 \\ &+\left\{\left(\nu_0^*+a_0\right)+\left(\nu_1^*+a_1\right)+\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)K\right\}\left(\tfrac{1}{K}+a_1\right)b_0 \\ &+\left\{\left(\nu_0^*+a_0\right)+\left(\nu_1^*+a_1\right)+\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)K\right\}\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\tfrac{1}{K}+a_1\right)(T+D) \\ &-\left\{\left(\tfrac{1}{K}+a_0\right)+\left(\tfrac{1}{K}+a_1\right)+\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\tfrac{1}{K}+a_1\right)K\right\}\left(\nu_0^*+a_0\right)b_1 \\ &-\left\{\left(\tfrac{1}{K}+a_0\right)+\left(\tfrac{1}{K}+a_1\right)+\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\tfrac{1}{K}+a_1\right)K\right\}\left(\nu_1^*+a_1\right)b_0 \\ &-\left\{\left(\tfrac{1}{K}+a_0\right)+\left(\tfrac{1}{K}+a_1\right)+\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\tfrac{1}{K}+a_1\right)K\right\}\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)(T+D) \\ =&\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right)(-Kb_1+T+D) \\ &-\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_0\right)(-Kb_1+T+D) \\ &+\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_1\right)(-Kb_0+T+D) \\ &-\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_1\right)(-Kb_0+T+D) \\ &+\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_0\right)(b_1-b_0)-\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right)(b_1-b_0) \\ =&\left\{\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right)-\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_0\right)\right\}(-Kb_1+T+D) \\ &+\left\{\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_1\right)-\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_1\right)\right\}(-Kb_0+T+D) \\ &+\left[\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_0\right)-\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right)\right](b_1-b_0) \\ =&\left\{\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right)-\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_0\right)\right\}(G(b_1)+D) \\ &+\left\{\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_1\right)-\left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_1\right)\right\}(G(b_0)+D) \\ &+\left[\left(\nu_1^*+a_1\right)\left(\tfrac{1}{K}+a_0\right)-\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right)\right](b_1-b_0) \\ =&X_1(G(b_1)+D)+X_2(G(b_0)+D)+X_3(b_1-b_0), \end{aligned} \tag{A.154}$$

where

$$X_1 = \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\nu_0^* + a_0\right)\left(\frac{1}{K} + a_1\right) - \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)\left(\frac{1}{K} + a_0\right), \tag{A.155}$$

$$X_2 = \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\nu_1^* + a_1\right)\left(\frac{1}{K} + a_1\right) - \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)\left(\frac{1}{K} + a_1\right) \tag{A.156}$$

and

$$X_3 = \left(\nu_1^* + a_1\right)\left(\frac{1}{K} + a_0\right) - \left(\nu_0^* + a_0\right)\left(\frac{1}{K} + a_1\right). \tag{A.157}$$

Now, for any given values of a_0, a_1, β, ν_0^*, ν_1^*, K and (A.150), one finds

$$\begin{aligned} X_2 =& \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\nu_1^* + a_1\right)\left(\frac{1}{K} + a_1\right) - \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)\left(\frac{1}{K} + a_1\right) \\ =&(1-\beta)\left(\frac{1}{K} - \nu_0^*\right)\left(\nu_1^* + a_1\right)\left(\frac{1}{K} + a_1\right) \geq 0 \end{aligned} \tag{A.158}$$

for all $\beta \in (0, 1]$.

Now, we are going to show that $X_1 > 0$ for all $\beta \in (0, 1]$. Plugging (A.42) in X_1, we obtain

$$\begin{aligned} X_1 =& \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\nu_0^* + a_0\right)\left(\frac{1}{K} + a_1\right) \\ & - \left[(1-\beta)\nu_0^* + a_0\right]\left(\frac{(1-\beta)\nu_0^* + a_0}{1 + \left[(1-\beta)\nu_0^* + a_0\right]K} + a_1\right)\left(\frac{1}{K} + a_0\right) \\ =& \frac{\left[(1-\beta)\frac{1}{K} + a_0\right]\left(\nu_0^* + a_0\right)\left(\frac{1}{K} + a_1\right)\left\{1 + \left[(1-\beta)\nu_0^* + a_0\right]K\right\}}{1 + \left[(1-\beta)\nu_0^* + a_0\right]K} \\ & - \frac{\left[(1-\beta)\nu_0^* + a_0\right]\left[(1-\beta)\nu_0^* + a_0 + a_1\left\{1 + \left[(1-\beta)\nu_0^* + a_0\right]K\right\}\right]\left(\frac{1}{K} + a_0\right)}{1 + \left[(1-\beta)\nu_0^* + a_0\right]K} \\ =& \frac{T_1}{T_2}, \end{aligned} \tag{A.159}$$

where

$$\begin{aligned} T_1 =& \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\nu_0^* + a_0\right)\left(\frac{1}{K} + a_1\right)\left\{1 + \left[(1-\beta)\nu_0^* + a_0\right]K\right\} \\ & - \left[(1-\beta)\nu_0^* + a_0\right]\left[(1-\beta)\nu_0^* + a_0 + a_1\left\{1 + \left[(1-\beta)\nu_0^* + a_0\right]K\right\}\right]\left(\frac{1}{K} + a_0\right) \end{aligned} \tag{A.160}$$

and

$$T_2 = 1 + \left[(1-\beta)\nu_0^* + a_0\right]K. \tag{A.161}$$

For any fixed values of a_0, β, ν_0^*, and K, it is clear that $T_2 > 0$. Therefore, to compute the value of (A.159), we need to calculate the value of T_1.

$$
\begin{aligned}
T_1 =& \left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right)\left\{1+\left[(1-\beta)\nu_0^*+a_0\right]K\right\} \\
&-\left[(1-\beta)\nu_0^*+a_0\right]\left[(1-\beta)\nu_0^*+a_0+a_1\left\{1+\left[(1-\beta)\nu_0^*+a_0\right]K\right\}\right]\left(\tfrac{1}{K}+a_0\right) \\
=& \left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right) \\
&+\left[(1-\beta)^2\tfrac{1}{K}\nu_0^*+(1-\beta)a_0\left(\tfrac{1}{K}+\nu_0^*\right)+a_0^2\right]\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)K \\
&-a_1\left[(1-\beta)\nu_0^*+a_0\right]\left(\tfrac{1}{K}+a_0\right) \\
&-\left[(1-\beta)^2\nu_0^{*2}+2(1-\beta)a_0\nu_0^*+a_0^2\right]\left(1+a_1K\right)\left(\tfrac{1}{K}+a_0\right) \\
=&(1-\beta)^2\nu_0^*\left[\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)-\left(\tfrac{1}{K}+a_0\right)\left(1+a_1K\right)\nu_0^*\right] \\
&+(1-\beta)\left[\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)\tfrac{1}{K}+\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+\nu_0^*\right)a_0K\right. \\
&\qquad\left.-\left(\tfrac{1}{K}+a_0\right)a_1\nu_0^*-2\left(\tfrac{1}{K}+a_0\right)\left(1+a_1K\right)a_0\nu_0^*\right] \\
&+a_0\left[\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)a_0K+\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)\right. \\
&\qquad\left.-\left(\tfrac{1}{K}+a_0\right)a_1-\left(\tfrac{1}{K}+a_0\right)\left(1+a_1K\right)a_0\right] \\
=&(1-\beta)^2\left(\tfrac{1}{K}+a_1\right)\left(\tfrac{1}{K}-\nu_0^*\right)a_0K\nu_0^* \\
&+(1-\beta)\left[\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)\tfrac{1}{K}+\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+\nu_0^*\right)a_0K\right. \\
&\qquad\left.-\left(\tfrac{1}{K}+a_0\right)a_1\nu_0^*-2\left(\tfrac{1}{K}+a_0\right)\left(1+a_1K\right)a_0\nu_0^*\right] \\
&+\left(\tfrac{1}{K}+a_0\right)a_0\left[\left(\tfrac{1}{K}+a_1\right)K\nu_0^*-a_1\right] \\
=&(1-\beta)^2Y_1+(1-\beta)Y_2+\left(\tfrac{1}{K}+a_0\right)a_0Y_3,
\end{aligned}
\tag{A.162}
$$

where

$$Y_1 = \left(\tfrac{1}{K}+a_1\right)\left(\tfrac{1}{K}-\nu_0^*\right)a_0K\nu_0^*, \tag{A.163}$$

$$
\begin{aligned}
Y_2 =& \left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)\tfrac{1}{K}+\left(\tfrac{1}{K}+a_1\right)\left(\nu_0^*+a_0\right)\left(\tfrac{1}{K}+\nu_0^*\right)a_0K \\
&-\left(\tfrac{1}{K}+a_0\right)a_1\nu_0^*-2\left(\tfrac{1}{K}+a_0\right)\left(1+a_1K\right)a_0\nu_0^*
\end{aligned}
\tag{A.164}
$$

and

$$Y_3 = \left(\tfrac{1}{K}+a_1\right)K\nu_0^*-a_1. \tag{A.165}$$

For any fixed values of a_0, a_1, ν_0^*, K, and (A.150), one concludes that $Y_1 > 0$, and $Y_3 = Y_3(\beta)$ strictly decreases by β, since $\nu_0^* = \nu_0^*(\beta)$ is strictly decreasing with respect to β, and $\left(\frac{1}{K}+a_1\right)K > 0$. Thus,

$$\begin{aligned}
Y_3 =& Y_3(\beta) \geq Y_3(1) = \left(\tfrac{1}{K} + a_1\right) K \nu_0^*(1) - a_1 \\
=& \left(\tfrac{1}{K} + a_1\right) K \frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} - a_1 \\
=& \frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} (1 + a_1 K) - a_1 \\
=& \frac{(a_0 + a_1 + a_0 a_1 K)(1 + a_1 K) - \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) a_1}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{[(1 + a_1 K) a_0 + a_1](1 + a_1 K) - [(1 + a_1 K) + (2 + a_1 K) a_0 K] a_1}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{(1 + a_1 K)^2 a_0 + (1 + a_1 K) a_1 - (1 + a_1 K) a_1 - (2 + a_1 K) a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{\left[(1 + a_1 K)^2 - (2 + a_1 K) a_1 K\right] a_0}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{a_0}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} > 0.
\end{aligned} \tag{A.166}$$

Then, $Y_3 > 0$ for all $\beta \in (0, 1]$.

Now, we are going to show that $Y_2 > 0$ for all $\beta \in (0, 1]$:

$$\begin{aligned}
Y_2 =& \left(\tfrac{1}{K} + a_1\right)\left(\nu_0^* + a_0\right)\tfrac{1}{K} + \left(\tfrac{1}{K} + a_1\right)\left(\nu_0^* + a_0\right)\left(\tfrac{1}{K} + \nu_0^*\right) a_0 K \\
&- \left(\tfrac{1}{K} + a_0\right) a_1 \nu_0^* - 2\left(\tfrac{1}{K} + a_0\right)(1 + a_1 K) a_0 \nu_0^* \\
=& \left(\tfrac{1}{K} + a_1\right) \nu_0^* \tfrac{1}{K} + \left(\tfrac{1}{K} + a_1\right) a_0 \tfrac{1}{K} + \left(\tfrac{1}{K} + a_1\right) a_0 K \left[\nu_0^{*2} + \left(\tfrac{1}{K} + a_0\right)\nu_0^* + a_0 \tfrac{1}{K}\right] \\
&- \left(\tfrac{1}{K} + a_0\right) a_1 \nu_0^* - 2\left(\tfrac{1}{K} + a_0\right)(1 + a_1 K) a_0 \nu_0^* \\
=& \left(\tfrac{1}{K} + a_1\right) \nu_0^* \tfrac{1}{K} + \left(\tfrac{1}{K} + a_1\right) a_0 \tfrac{1}{K} \\
&+ \left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^{*2} + \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^* + \left(\tfrac{1}{K} + a_1\right) a_0^2 \\
&- \left(\tfrac{1}{K} + a_0\right) a_1 \nu_0^* - 2\left(\tfrac{1}{K} + a_0\right)(1 + a_1 K) a_0 \nu_0^* \\
=& \left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^{*2} + \left(\tfrac{1}{K} + a_1\right) \nu_0^* \tfrac{1}{K} - \left(\tfrac{1}{K} + a_0\right) a_1 \nu_0^* \\
&+ \left(\tfrac{1}{K} + a_1\right) a_0 \tfrac{1}{K} + \left(\tfrac{1}{K} + a_1\right) a_0^2 \\
&+ \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^* - 2\left(\tfrac{1}{K} + a_0\right)(1 + a_1 K) a_0 \nu_0^* \\
=& \left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^{*2} + \left[\left(\tfrac{1}{K} + a_1\right)\tfrac{1}{K} - \left(\tfrac{1}{K} + a_0\right) a_1\right] \nu_0^* \\
&+ \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right) a_0 - \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^* \\
=& \left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^{*2} + \left(\tfrac{1}{K^2} - a_0 a_1\right) \nu_0^* + \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right) a_0 K \left(\tfrac{1}{K} - \nu_0^*\right) \\
=& \left[\left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^* + \left(\tfrac{1}{K^2} - a_0 a_1\right)\right] \nu_0^* + \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right)\left(\tfrac{1}{K} - \nu_0^*\right) a_0 K \\
=& Z_1 \nu_0^* + Z_2,
\end{aligned} \tag{A.167}$$

where

$$Z_1 = \left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^* + \left(\tfrac{1}{K^2} - a_0 a_1\right) \tag{A.168}$$

and

$$Z_2 = \left(\tfrac{1}{K} + a_0\right)\left(\tfrac{1}{K} + a_1\right)\left(\tfrac{1}{K} - \nu_0^*\right) a_0 K. \tag{A.169}$$

Given the values of a_0, a_1, ν_0^*, K, and (A.150), one has that $Z_2 > 0$ for all $\beta \in (0, 1]$, and $Z_1 = Z_1(\beta)$ is strictly decreasing with respect to β, because $\nu_0^*(\beta)$ strictly decreases by β, and $(a_1 + \frac{1}{K})a_0 K > 0$. Thus,

$$\begin{aligned}
Z_1 =& Z_1(\beta) \geq Z_1(1) = \left(\tfrac{1}{K} + a_1\right) a_0 K \nu_0^*(1) + \left(\tfrac{1}{K^2} - a_0 a_1\right) \\
=& \left(\tfrac{1}{K} + a_1\right) a_0 K \frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} + \left(\tfrac{1}{K^2} - a_0 a_1\right) \\
=& \frac{(a_0 + a_1 + a_0 a_1 K)\left(\tfrac{1}{K} + a_1\right) a_0 K + \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right)\left(\tfrac{1}{K^2} - a_0 a_1\right)}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{(a_0 + a_1 + a_0 a_1 K)(1 + a_1 K) a_0 + \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right)\left(\tfrac{1}{K^2} - a_0 a_1\right)}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{\left(a_0 + a_1 + 2a_0 a_1 K + a_1^2 K + a_0 a_1^2 K^2\right) a_0}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
&+ \frac{\left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right)\left(\tfrac{1}{K^2} - a_0 a_1\right)}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{a_0^2 + \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) a_0 a_1}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
&+ \frac{\left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right)\tfrac{1}{K^2} - \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) a_0 a_1}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{a_0^2 + \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right)\tfrac{1}{K^2}}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
=& \frac{a_0^2}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} + \frac{1}{K^2} > 0.
\end{aligned} \tag{A.170}$$

Then, $Z_1 > 0$ for all $\beta \in (0, 1]$, which proves that $Y_2 = \nu_0^* Z_1 + Z_2 > 0$ for all $\beta \in (0, 1]$.

Now, since $Y_1, Y_2, Y_3 > 0$, we have that

$$T_1 = (1 - \beta)^2 Y_1 + (1 - \beta) Y_2 + \left(\tfrac{1}{K} + a_0\right) a_0 Y_3 > 0, \tag{A.171}$$

which proves that

$$X_1 = \frac{T_1}{T_2} > 0. \tag{A.172}$$

Since $X_1 > 0$ and $X_2 \geq 0$, then, if $X_3 \geq 0$ for $\beta_0 \in (0, 1]$, we have that

$$S_1 = X_1(G(b_1) + D) + X_2(G(b_0) + D) + X_3(b_1 - b_0) > 0, \tag{A.173}$$

for $\beta_0 \in (0, 1]$.

On the other hand, if $X_3 < 0$, for $\beta_0 \in (0, 1]$, then,

$$\begin{aligned}
S_1 =& X_1(G(b_1) + D) + X_2(G(b_0) + D) + X_3(b_1 - b_0)\\
=& \left[(1-\beta)\tfrac{1}{K} + a_0\right]\left(\nu_0^* + a_0\right)\left(\tfrac{1}{K} + a_1\right)(G(b_1) + D)\\
&- \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)\left(\tfrac{1}{K} + a_0\right)(G(b_1) + D)\\
&+ X_2(G(b_0) + D) + X_3(b_1 - b_0)\\
=&(1-\beta)\left[\tfrac{1}{K}\left(\nu_0^* + a_0\right)\left(\tfrac{1}{K} + a_1\right) - \nu_0^*\left(\nu_1^* + a_1\right)\left(\tfrac{1}{K} + a_0\right)\right](G(b_1) + D)\\
&- a_0\left[\left(\nu_1^* + a_1\right)\left(\tfrac{1}{K} + a_0\right) - \left(\nu_0^* + a_0\right)\left(\tfrac{1}{K} + a_1\right)\right](G(b_1) + D)\\
&+ X_2(G(b_0) + D) + X_3(b_1 - b_0)\\
=&(1-\beta)X_4(G(b_1) + D) - a_0X_3(G(b_1) + D) + X_2(G(b_0) + D) + X_3(b_1 - b_0)\\
=&(1-\beta)X_4(G(b_1) + D) - X_3\left[a_0(G(b_1) + D) - (b_1 - b_0)\right] + X_2(G(b_0) + D),
\end{aligned} \tag{A.174}$$

where

$$X_4 = \tfrac{1}{K}\left(\nu_0^* + a_0\right)\left(\tfrac{1}{K} + a_1\right) - \nu_0^*\left(\nu_1^* + a_1\right)\left(\tfrac{1}{K} + a_0\right). \tag{A.175}$$

Applying inequalities (A.150)–(A.175), we see that

$$\begin{aligned}
X_4 =& \tfrac{1}{K}\left(\nu_0^* + a_0\right)\left(\tfrac{1}{K} + a_1\right) - \nu_0^*\left(\nu_1^* + a_1\right)\left(\tfrac{1}{K} + a_0\right)\\
>& \tfrac{1}{K}\left(\nu_0^* + a_0\right)\left(\nu_1^* + a_1\right) - \nu_0^*\left(\nu_1^* + a_1\right)\left(\tfrac{1}{K} + a_0\right)\\
=& \left(\nu_1^* + a_1\right)\left[\tfrac{1}{K}\left(\nu_0^* + a_0\right) - \nu_0^*\left(\tfrac{1}{K} + a_0\right)\right]\\
=& \left(\nu_1^* + a_1\right)\left(a_0\tfrac{1}{K} - a_0\nu_0^*\right)\\
=& a_0\left(\nu_1^* + a_1\right)\left(\tfrac{1}{K} - \nu_0^*\right) > 0.
\end{aligned} \tag{A.176}$$

Thus, $X_4 > 0$ for $\beta_0 \in (0, 1]$, and since $X_2 \geq 0$, $X_3 < 0$ and assumption **A3**, we have that

$$\begin{aligned}
S_1 =& (1-\beta)X_4(G(b_1) + D) - X_3\left[a_0(G(b_1) + D) - (b_1 - b_0)\right] + X_2(G(b_0) + D)\\
\geq& - X_3\left[a_0(G(b_1) + D) - (b_1 - b_0)\right] > 0,
\end{aligned} \tag{A.177}$$

for $\beta_0 \in (0, 1]$.

Therefore, $S_1 > 0$ for all $\beta \in (0, 1]$, that is,

$$\left(p^c - p^*\right)(\beta) = \frac{S_1}{S_2} > 0, \tag{A.178}$$

which finally proves (A.140). The proof of the theorem is complete. ■

Theorem 2.7 *The functions* $\pi_1^*(\beta)$ *and* $\pi_1^c(\beta)$ *are strictly decreasing with respect to* $\beta \in (0, 1]$. *Moreover, the following inequalities hold:*

$$\pi_1^*(1) > \pi_1^c(1) \tag{A.179}$$

and

$$\lim_{\beta\to 0} \pi_1^*(\beta) < \lim_{\beta\to 0} \pi_1^c(\beta). \tag{A.180}$$

Proof First, we are going to show that π_1^* and π_1^c strictly decrease by β.

The function π_1^* is differentiable with respect to β and

$$\begin{aligned}
\pi_1^{*\prime} &= \left(p^* q_1^* - \frac{1}{2} a_1 q_1^{*2} - b_1 q_1^*\right)' = p^{*\prime} q_1^* + p^* q_1^{*\prime} - a_1 q_1^* q_1^{*\prime} - b_1 q_1^{*\prime} \\
&= p^{*\prime} q_1^* + \left(p^* - a_1 q_1^* - b_1\right) q_1^{*\prime} \\
&= p^{*\prime} q_1^* + \left(p^* - b_1 - a_1 \frac{p^* - b_1}{\nu_1^* + a_1}\right) q_1^{*\prime} \\
&= p^{*\prime} q_1^* + \left(1 - \frac{a_1}{\nu_1^* + a_1}\right) \left(p^* - b_1\right) q_1^{*\prime} \\
&= p^{*\prime} q_1^* + \frac{\nu_1^*}{\nu_1^* + a_1} \left(p^* - b_1\right) q_1^{*\prime}.
\end{aligned} \tag{A.181}$$

Given the values of a_1, b_1, ν_1^*, p^*, q_1^*, $p^{*\prime}$ and $q_1^{*\prime}$, it's easy to see that

$$\pi_1^{*\prime} = p^{*\prime} q_1^* + \frac{\nu_1^*}{\nu_1^* + a_1} \left(p^* - b_1\right) q_1^{*\prime} < 0. \tag{A.182}$$

Similarly,

$$\pi_1^{c\prime} = p^{c\prime} q_1^c + \frac{\frac{1}{K}}{\frac{1}{K} + a_1} \left(p^c - b_1\right) q_1^{c\prime} < 0. \tag{A.183}$$

Because of that, π_1^* and π_1^c strictly decrease with respect to $\beta \in (0, 1]$.

Now consider the difference of the functions π_1^* and π_1^c as follows:

$$\begin{aligned}
\pi_1^c - \pi_1^* &= \left(p^c q_1^c - \frac{1}{2} a_1 q_1^{c2} - b_1 q_1^c\right) - \left(p^* q_1^* - \frac{1}{2} a_1 q_1^{*2} - b_1 q_1^*\right) \\
&= \left(p^c - b_1 - \frac{1}{2} a_1 q_1^c\right) q_1^c - \left(p^* - b_1 - \frac{1}{2} a_1 q_1^*\right) q_1^* \\
&= \left[\left(\frac{1}{K} + a_1\right) \frac{p^c - b_1}{\frac{1}{K} + a_1} - \frac{1}{2} a_1 q_1^c\right] q_1^c - \left[\left(\nu_1^* + a_1\right) \frac{p^* - b_1}{\nu_1^* + a_1} - \frac{1}{2} a_1 q_1^*\right] q_1^* \\
&= \left[\left(\frac{1}{K} + a_1\right) q_1^c - \frac{1}{2} a_1 q_1^c\right] q_1^c - \left[\left(\nu_1^* + a_1\right) q_1^* - \frac{1}{2} a_1 q_1^*\right] q_1^* \\
&= \left(\frac{1}{K} + \frac{1}{2} a_1\right) q_1^{c2} - \left(\nu_1^* + \frac{1}{2} a_1\right) q_1^{*2}.
\end{aligned} \tag{A.184}$$

From (A.97) we have that

$$q_1^* = \frac{-(b_1 - b_0) + \left[(1-\beta)\nu_0^* + a_0\right](G(b_1) + D)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right)\left\{1 + \left[(1-\beta)\nu_0^* + a_0\right]K\right\}} = \frac{\left[(1-\beta)\nu_0^* + a_0\right](G(b_1) + D) - (b_1 - b_0)}{\left(\nu_0^* + a_0\right) + \left(\nu_1^* + a_1\right) + \left[(1-\beta)\nu_0^* + a_0\right]\left(\nu_1^* + a_1\right)K}, \tag{A.185}$$

and similarly to (A.97) and (A.185),

$$\begin{aligned} q_1^c &= \frac{p^c - b_1}{\frac{1}{K} + a_1} = \frac{\frac{\left(\frac{1}{K}+a_0\right)b_1 + \left(\frac{1}{K}+a_1\right)b_0 + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)(T+D)}{\left(\frac{1}{K}+a_0\right) + \left(\frac{1}{K}+a_1\right) + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K} - b_1}{\frac{1}{K}+a_1} \\ &= \frac{\left(\frac{1}{K}+a_0\right)b_1 + \left(\frac{1}{K}+a_1\right)b_0 + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)(T+D)}{\left(\frac{1}{K}+a_1\right)\left[\left(\frac{1}{K}+a_0\right) + \left(\frac{1}{K}+a_1\right) + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K\right]} \\ &\quad - \frac{\left[\left(\frac{1}{K}+a_0\right) + \left(\frac{1}{K}+a_1\right) + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K\right]b_1}{\left(\frac{1}{K}+a_1\right)\left[\left(\frac{1}{K}+a_0\right) + \left(\frac{1}{K}+a_1\right) + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K\right]} \\ &= \frac{-\left(\frac{1}{K}+a_1\right)(b_1-b_0) + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)(-Kb_1+T+D)}{\left(\frac{1}{K}+a_1\right)\left[\left(\frac{1}{K}+a_0\right) + \left(\frac{1}{K}+a_1\right) + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K\right]} \\ &= \frac{-(b_1-b_0) + \left[(1-\beta)\frac{1}{K}+a_0\right](-Kb_1+T+D)}{\left(\frac{1}{K}+a_0\right) + \left(\frac{1}{K}+a_1\right) + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K} \\ &= \frac{\left[(1-\beta)\frac{1}{K}+a_0\right](G(b_1)+D) - (b_1-b_0)}{\left(\frac{1}{K}+a_0\right) + \left(\frac{1}{K}+a_1\right) + \left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K}. \end{aligned} \tag{A.186}$$

By substituting the expression of ν_1^* given by (A.42) in Eq. (A.185) we have that

$$\begin{aligned} q_1^* &= \frac{\left[(1-\beta)\nu_0^* + a_0\right](G(b_1)+D) - (b_1-b_0)}{\left(\nu_0^*+a_0\right) + \left(\nu_1^*+a_1\right) + \left[(1-\beta)\nu_0^*+a_0\right]\left(\nu_1^*+a_1\right)K} \\ &= \frac{\left[(1-\beta)\nu_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(\nu_0^*+a_0\right)+\left(\frac{(1-\beta)\nu_0^*+a_0}{1+\left[(1-\beta)\nu_0^*+a_0\right]K}+a_1\right)+\left[(1-\beta)\nu_0^*+a_0\right]\left(\frac{(1-\beta)\nu_0^*+a_0}{1+\left[(1-\beta)\nu_0^*+a_0\right]K}+a_1\right)K} \\ &= \frac{\left(1+\left[(1-\beta)\nu_0^*+a_0\right]K\right)\left\{\left[(1-\beta)\nu_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)\right\}}{\left(1+\left[(1-\beta)\nu_0^*+a_0\right]K\right)\left\{\left(\nu_0^*+a_0\right)+a_1+(1+a_1K)\left[(1-\beta)\nu_0^*+a_0\right]\right\}} \\ &= \frac{\left[(1-\beta)\nu_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(\nu_0^*+a_0+a_1\right)+(1+a_1K)\left[(1-\beta)\nu_0^*+a_0\right]}. \end{aligned} \tag{A.187}$$

By Eq. (A.42),

$$\nu_1^* = \frac{(1-\beta)\nu_0^* + a_0}{1 + \left[(1-\beta)\nu_0^* + a_0\right]K},$$

therefore,

$$
\begin{aligned}
\nu_1^* + \frac{1}{2}a_1 =& \frac{(1-\beta)\nu_0^* + a_0}{1 + \left[(1-\beta)\nu_0^* + a_0\right]K} + \frac{1}{2}a_1 \\
=& \frac{(1-\beta)\nu_0^* + a_0 + \frac{1}{2}a_1\left(1 + \left[(1-\beta)\nu_0^* + a_0\right]K\right)}{1 + \left[(1-\beta)\nu_0^* + a_0\right]K} \\
=& \frac{\frac{1}{2}a_1 + \left(1 + \frac{1}{2}a_1 K\right)\left[(1-\beta)\nu_0^* + a_0\right]}{1 + \left[(1-\beta)\nu_0^* + a_0\right]K}.
\end{aligned}
\tag{A.188}
$$

On the other hand, from the expression for q_1^c obtained from (A.186) we have that

$$
\begin{aligned}
q_1^c =& \frac{\left[(1-\beta)\frac{1}{K} + a_0\right](G(b_1) + D) - (b_1 - b_0)}{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1-\beta)\frac{1}{K} + a_0\right]\left(\frac{1}{K} + a_1\right)K} \\
=& \frac{\left[(1-\beta)\frac{1}{K} + a_0\right](G(b_1) + D) - (b_1 - b_0)}{\frac{1}{K}(1 + a_0 K) + \frac{1}{K}(1 + a_1 K) + \frac{1}{K}\left[(1-\beta) + a_0 K\right](1 + a_1 K)} \\
=& \frac{\left\{\left[(1-\beta)\frac{1}{K} + a_0\right](G(b_1) + D) - (b_1 - b_0)\right\}K}{(1 + a_0 K) + (1 + a_1 K)\left[(2-\beta) + a_0 K\right]}.
\end{aligned}
\tag{A.189}
$$

Plugging Eqs. (A.187), (A.188) and (A.189) in Eq. (A.184) we deduce

$$
\begin{aligned}
\pi_1^c - \pi_1^* =& \left(\tfrac{1}{K} + \tfrac{1}{2}a_1\right) q_1^{c2} - \left(\nu_1^* + \tfrac{1}{2}a_1\right) q_1^{*2} \\
=& \left(\tfrac{1}{K} + \tfrac{1}{2}a_1\right)\left(\frac{\left\{\left[(1-\beta)\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)\right\}K}{(1+a_0K)+(1+a_1K)[(2-\beta)+a_0K]}\right)^2 \\
&- \left(\frac{\frac{1}{2}a_1+\left(1+\frac{1}{2}a_1K\right)\left[(1-\beta)\nu_0^*+a_0\right]}{1+\left[(1-\beta)\nu_0^*+a_0\right]K}\right)\left(\frac{\left[(1-\beta)\nu_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(\nu_0^*+a_0+a_1\right)+(1+a_1K)\left[(1-\beta)\nu_0^*+a_0\right]}\right)^2 \\
=& \tfrac{1}{2}K\,(2 + a_1 K)\left(\frac{\left[(1-\beta)\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)}{(1+a_0K)+(1+a_1K)[(2-\beta)+a_0K]}\right)^2 \\
&- \tfrac{1}{2}\left(\frac{a_1+(2+a_1K)\left[(1-\beta)\nu_0^*+a_0\right]}{1+\left[(1-\beta)\nu_0^*+a_0\right]K}\right)\left(\frac{\left[(1-\beta)\nu_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(\nu_0^*+a_0+a_1\right)+(1+a_1K)\left[(1-\beta)\nu_0^*+a_0\right]}\right)^2.
\end{aligned}
\tag{A.190}
$$

Then, to prove the inequalities (A.5) and (A.6) the following conditions has to be met:

$$\pi_1^c(1) - \pi_1^*(1) = (\pi_1^c - \pi_1^*)(1) < 0 \tag{A.191}$$

and

$$\lim_{\beta\to 0}\pi_1^c(\beta) - \lim_{\beta\to 0}\pi_1^*(\beta) = \lim_{\beta\to 0}(\pi_1^c - \pi_1^*)(\beta) > 0. \tag{A.192}$$

Evaluating the expression of ν_0^*, given by (A.41), for $\beta = 1$ and using the notation $\overline{\nu_0^*} = \nu_0^*(1)$, one has

$$\overline{\nu_0^*} = \nu_0^*(1) = \frac{2\left(a_0 + a_1 + a_0 a_1 K\right)}{\left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) + \sqrt{\left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right)^2}}$$
$$= \frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2}. \tag{A.193}$$

Now, we evaluate (A.190) for $\beta = 1$ to obtain

$$\begin{aligned}
(\pi_1^c - \pi_1^*)(1) =& \frac{1}{2} K\left(2 + a_1 K\right)\left(\frac{[a_0]\left(G(b_1) + D\right) - (b_1 - b_0)}{(1 + a_0 K) + (1 + a_1 K)\left[1 + a_0 K\right]}\right)^2 \\
&- \frac{1}{2}\left(\frac{a_1 + (2 + a_1 K)\left[a_0\right]}{1 + [a_0]\, K}\right)\left(\frac{[a_0]\left(G(b_1) + D\right) - (b_1 - b_0)}{\left(\overline{\nu_0^*} + a_0 + a_1\right) + (1 + a_1 K)\left[a_0\right]}\right)^2 \\
=& \frac{1}{2} K\left(2 + a_1 K\right)\left(\frac{a_0\left(G(b_1) + D\right) - (b_1 - b_0)}{(1 + a_0 K)\left(2 + a_1 K\right)}\right)^2 \\
&- \frac{1}{2}\left(\frac{a_1 + a_0\left(2 + a_1 K\right)}{1 + a_0 K}\right)\left(\frac{a_0\left(G(b_1) + D\right) - (b_1 - b_0)}{\left(\overline{\nu_0^*} + a_0 + a_1\right) + a_0\left(1 + a_1 K\right)}\right)^2 \\
=& \frac{1}{2}\frac{\left[a_0\left(G(b_1) + D\right) - (b_1 - b_0)\right]^2}{1 + a_0 K}\frac{K}{(1 + a_0 K)\left(2 + a_1 K\right)} \\
&- \frac{1}{2}\frac{\left[a_0\left(G(b_1) + D\right) - (b_1 - b_0)\right]^2}{1 + a_0 K}\frac{a_1 + a_0\left(2 + a_1 K\right)}{\left[\left(\overline{\nu_0^*} + a_0 + a_1\right) + a_0\left(1 + a_1 K\right)\right]^2} \\
=& \frac{1}{2}\frac{\left[a_0\left(G(b_1) + D\right) - (b_1 - b_0)\right]^2}{1 + a_0 K}\frac{K}{2 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
&- \frac{1}{2}\frac{\left[a_0\left(G(b_1) + D\right) - (b_1 - b_0)\right]^2}{1 + a_0 K}\frac{2a_0 + a_1 + a_0 a_1 K}{\left(\overline{\nu_0^*} + 2a_0 + a_1 + a_0 a_1 K\right)^2} \\
=& U_1 \frac{V_1}{W_1},
\end{aligned} \tag{A.194}$$

where

$$U_1 = \frac{1}{2}\frac{\left[a_0\left(G(b_1) + D\right) - (b_1 - b_0)\right]^2}{1 + a_0 K}, \tag{A.195}$$

$$V_1 = K\left(\overline{\nu_0^*} + 2a_0 + a_1 + a_0 a_1 K\right)^2 - \left(2a_0 + a_1 + a_0 a_1 K\right)\left(2 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) \tag{A.196}$$

and

$$W_1 = \left(2 + 2a_0 K + a_1 K + a_0 a_1 K^2\right)\left(\overline{\nu_0^*} + 2a_0 + a_1 + a_0 a_1 K\right)^2. \tag{A.197}$$

Given the values of a_0, a_1, $\overline{\nu_0^*}$ and K, it isn't difficult to see that $U_1 > 0$ and $W_1 > 0$. Hence, to prove (A.191) it is enough to show that $V_1 > 0$. Indeed, plugging the expression of $\overline{\nu_0^*}$ given by (A.193) in (A.196), we have that

$$
\begin{aligned}
V_1 =& K\left(\overline{\nu_0^*} + 2a_0 + a_1 + a_0a_1K\right)^2 - (2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) \\
=& K\left(\frac{a_0 + a_1 + a_0a_1K}{1 + 2a_0K + a_1K + a_0a_1K^2} + 2a_0 + a_1 + a_0a_1K\right)^2 \\
& - (2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) \\
=& K\left[a_0 + a_1 + a_0a_1K + (2a_0 + a_1 + a_0a_1K)\left(1 + 2a_0K + a_1K + a_0a_1K^2\right)\right]^2 \\
& - (2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right)\left(1 + 2a_0K + a_1K + a_0a_1K^2\right)^2 \\
<& K\left[(2a_0 + a_1 + a_0a_1K) + (2a_0 + a_1 + a_0a_1K)\left(1 + 2a_0K + a_1K + a_0a_1K^2\right)\right]^2 \\
& - (2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right)\left(1 + 2a_0K + a_1K + a_0a_1K^2\right)^2 \\
=& K\left[(2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right)\right]^2 \\
& - (2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right)\left(1 + 2a_0K + a_1K + a_0a_1K^2\right)^2 \\
=& \Big[K\,(2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) \\
& - \left(1 + 2a_0K + a_1K + a_0a_1K^2\right)^2\Big](2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) \\
=& \Big[\left(2a_0K + a_1K + a_0a_1K^2\right)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) \\
& - \left(1 + 2a_0K + a_1K + a_0a_1K^2\right)^2\Big](2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) \\
=& \Big[\left(1 + 2a_0K + a_1K + a_0a_1K^2 - 1\right)\left(1 + 2a_0K + a_1K + a_0a_1K^2 + 1\right) \\
& - \left(1 + 2a_0K + a_1K + a_0a_1K^2\right)^2\Big](2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) \\
=& \Big[\left(1 + 2a_0K + a_1K + a_0a_1K^2\right)^2 - 1 \\
& - \left(1 + 2a_0K + a_1K + a_0a_1K^2\right)^2\Big](2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) \\
=& -(2a_0 + a_1 + a_0a_1K)\left(2 + 2a_0K + a_1K + a_0a_1K^2\right) < 0.
\end{aligned}
\tag{A.198}
$$

Therefore $V_1 < 0$. Then, since $U_1 > 0$ and $W_1 > 0$, we have that

$$(\pi_1^c - \pi_1^*)(1) = U_1\frac{V_1}{W_1} < 0, \tag{A.199}$$

which proves (A.191).

Now, we need only to prove (A.192). Using the notation $\widehat{\nu_0^*} = \lim_{\beta\to 0} \nu_0^*(\beta)$ given by (A.141), from (A.190) we have that

$$
\begin{aligned}
&\lim_{\beta\to 0}(\pi_1^c - \pi_1^*)(\beta) = \\
&= \frac{1}{2}K\,(2+a_1K)\left(\frac{\left[\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)}{(1+a_0K)+(1+a_1K)\,[2+a_0K]}\right)^2 \\
&\quad -\frac{1}{2}\left(\frac{a_1+(2+a_1K)\left[\widehat{\nu_0^*}+a_0\right]}{1+\left[\widehat{\nu_0^*}+a_0\right]K}\right)\left(\frac{\left[\widehat{\nu_0^*}+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(\widehat{\nu_0^*}+a_0+a_1\right)+(1+a_1K)\left[\widehat{\nu_0^*}+a_0\right]}\right)^2 \\
&= \frac{1}{2}K\,(2+a_1K)\left(\frac{\left(\frac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)}{(1+a_0K)+(1+a_1K)+(1+a_0K)\,(1+a_1K)}\right)^2 \\
&\quad -\frac{1}{2}\left(\frac{a_1+(2+a_1K)\left(\widehat{\nu_0^*}+a_0\right)}{1+\left(\widehat{\nu_0^*}+a_0\right)K}\right)\left(\frac{\left(\widehat{\nu_0^*}+a_0\right)(G(b_1)+D)-(b_1-b_0)}{a_1+(2+a_1K)\left(\widehat{\nu_0^*}+a_0\right)}\right)^2 \\
&= \frac{1}{2}K\,(2+a_1K)\left(\frac{\left(\frac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)}{(1+a_1K)+(1+a_0K)\,(2+a_1K)}\right)^2 \\
&\quad -\frac{1}{2}\frac{1}{1+\left(\widehat{\nu_0^*}+a_0\right)K}\frac{\left[\left(\widehat{\nu_0^*}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2}{a_1+(2+a_1K)\left(\widehat{\nu_0^*}+a_0\right)} \\
&= \frac{1}{2}\frac{V_2}{W_2},
\end{aligned}
\tag{A.200}
$$

where

$$
\begin{aligned}
V_2 =& K\,(2+a_1K)\left[\left(\frac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2 \times \\
&\left[1+\left(\widehat{\nu_0^*}+a_0\right)K\right]\left[a_1+(2+a_1K)\left(\widehat{\nu_0^*}+a_0\right)\right] \\
&-[(1+a_1K)+(1+a_0K)\,(2+a_1K)]^2 \times \\
&\left[\left(\widehat{\nu_0^*}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2
\end{aligned}
\tag{A.201}
$$

and

$$
\begin{aligned}
W_2 =& [(1+a_1K)+(1+a_0K)\,(2+a_1K)]^2 \times \\
&\left[1+\left(\widehat{\nu_0^*}+a_0\right)K\right]\left[a_1+(2+a_1K)\left(\widehat{\nu_0^*}+a_0\right)\right].
\end{aligned}
\tag{A.202}
$$

For arbitrary fixed values of a_0, a_1, $\widehat{\nu_0^*}$ and K, it is evident that $W_2 > 0$. Hence, to prove (A.192) it lacks only to show that $V_2 > 0$. Indeed,

$$
\begin{aligned}
V_2 =& \left(\left[(1+a_1K)+(2+a_1K)\left(\widehat{\nu_0^*}+a_0\right)K\right]^2-1\right)\left[\left(\tfrac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2 \\
&-[(1+a_1K)+(1+a_0K)\,(2+a_1K)]^2\left[\left(\widehat{\nu_0^*}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2 \\
=& \left[(1+a_1K)+\left(\widehat{\nu_0^*}+a_0\right)(2+a_1K)\,K\right]^2\left[\left(\tfrac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2
\end{aligned}
$$

$$- \left[(1 + a_1 K) + \left(\tfrac{1}{K} + a_0\right)(2 + a_1 K)\, K\right]^2 \left[\left(\widehat{\nu_0^*} + a_0\right)(G(b_1) + D) - (b_1 - b_0)\right]^2 \\ - \left[\left(\tfrac{1}{K} + a_0\right)(G(b_1) + D) - (b_1 - b_0)\right]^2. \tag{A.203}$$

Now introduce the following notation:

$$\eta = 1 + a_1 K > 0, \tag{A.204}$$

$$\xi = K\,(1 + \eta) = K\,(2 + a_1 K) > 0, \tag{A.205}$$

$$\mathscr{Z} = \eta + a_0 \xi = (1 + a_1 K) + a_0 K\,(2 + a_1 K) > 0, \tag{A.206}$$

$$G_1 = G(b_1) + D > 0 \tag{A.207}$$

and

$$G_3 = a_0 G_1 - (b_1 - b_0) = a_0\,(G(b_1) + D) - (b_1 - b_0) > 0. \tag{A.208}$$

Based on that, we can rewrite (A.203) as follows:

$$\begin{aligned}
V_2 =& \left[(1 + a_1 K) + \left(\widehat{\nu_0^*} + a_0\right)(2 + a_1 K)\, K\right]^2 \left[\left(\tfrac{1}{K} + a_0\right)(G(b_1) + D) - (b_1 - b_0)\right]^2 \\
& - \left[(1 + a_1 K) + \left(\tfrac{1}{K} + a_0\right)(2 + a_1 K)\, K\right]^2 \left[\left(\widehat{\nu_0^*} + a_0\right)(G(b_1) + D) - (b_1 - b_0)\right]^2 \\
& - \left[\left(\tfrac{1}{K} + a_0\right)(G(b_1) + D) - (b_1 - b_0)\right]^2 \\
=& \left(\widehat{\nu_0^*}\xi + \mathscr{Z}\right)^2 \left(\tfrac{1}{K} G_1 + G_3\right)^2 - \left(\tfrac{1}{K}\xi + \mathscr{Z}\right)^2 \left(\widehat{\nu_0^*} G_1 + G_3\right)^2 - \left(\tfrac{1}{K} G_1 + G_3\right)^2 \\
=& \left(\tfrac{1}{K}\widehat{\nu_0^*}\xi G_1 + \tfrac{1}{K}\mathscr{Z} G_1 + \widehat{\nu_0^*}\xi G_3 + \mathscr{Z} G_3\right)^2 - \left(\tfrac{1}{K}\widehat{\nu_0^*}\xi G_1 + \widehat{\nu_0^*}\mathscr{Z} G_1 + \tfrac{1}{K}\xi G_3 + \mathscr{Z} G_3\right)^2 \\
& - \left(\frac{1}{K^2} G_1^2 + 2\tfrac{1}{K} G_1 G_3 + G_3^2\right) \\
=& \left[2\left(\tfrac{1}{K}\widehat{\nu_0^*}\xi G_1 + \mathscr{Z} G_3\right) + \left(\tfrac{1}{K} + \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 + \xi G_3)\right]\left[\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3)\right] \\
& - \left(\frac{1}{K^2} G_1^2 + 2\tfrac{1}{K} G_1 G_3\right) - G_3^2 \\
=& 2\left(\tfrac{1}{K}\widehat{\nu_0^*}\xi G_1 + \mathscr{Z} G_3\right)\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) + \left(\tfrac{1}{K^2} - \widehat{\nu_0^*}^2\right)\left(\mathscr{Z}^2 G_1^2 - \xi^2 G_3^2\right) \\
& - \tfrac{1}{K} G_1 \left(\tfrac{1}{K} G_1 + 2 G_3\right) - G_3^2 \\
=& \left(\tfrac{1}{K^2} - \widehat{\nu_0^*}^2\right)\left(\mathscr{Z}^2 G_1^2 - \xi^2 G_3^2\right) - \tfrac{1}{K} G_1 \left(\tfrac{1}{K} G_1 + 2 G_3\right) \\
& + 2\mathscr{Z} G_3 \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) - G_3^2 \\
& + 2\tfrac{1}{K}\widehat{\nu_0^*}\xi G_1 \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) \\
=& \mathscr{P}_1 + \mathscr{Q}_1 + \mathscr{R}_1,
\end{aligned} \tag{A.209}$$

where

$$\mathscr{P}_1 = \left(\frac{1}{K^2} - \widehat{\nu_0^*}^2\right)\left(\mathscr{Z}^2 G_1^2 - \xi^2 G_3^2\right) - \frac{1}{K} G_1 \left(\frac{1}{K} G_1 + 2G_3\right), \tag{A.210}$$

$$\mathscr{Q}_1 = 2\mathscr{Z} G_3 \left(\frac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) - G_3^2 \tag{A.211}$$

and

$$\mathscr{R}_1 = 2\frac{1}{K}\widehat{\nu_0^*}\xi G_1 \left(\frac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3). \tag{A.212}$$

Now, we are going to show that

$$\mathscr{Z} G_1 - \xi G_3 > 0. \tag{A.213}$$

Using (A.206) and (A.208) we have that

$$\begin{aligned} \mathscr{Z} G_1 - \xi G_3 &= (\eta + a_0\xi) G_1 - \xi (a_0 G_1 - (b_1 - b_0)) \\ &= \eta G_1 + \xi(b_1 - b_0) \geq \eta G_1 > 0, \end{aligned} \tag{A.214}$$

which proves (A.213).

Thus, given the values of $\widehat{\nu_0^*}$, K, Eqs. (A.205), (A.207), (A.150) and (A.213), we can conclude that $\mathscr{R}_1 > 0$.

Now,

$$\begin{aligned} \mathscr{Q}_1 &= 2\mathscr{Z} G_3 \left(\frac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) - G_3^2 \\ &= \left[2\mathscr{Z}\left(\frac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) - G_3\right] G_3, \end{aligned} \tag{A.215}$$

and using (A.208) we can rewrite (A.215)as follows:

$$\begin{aligned} \mathscr{Q}_1 &= \left[2\mathscr{Z}\left(\frac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) - G_3\right] G_3 \\ &= \left\{2\mathscr{Z}\left(\frac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi\left[a_0 G_1 - (b_1 - b_0)\right]) - \left[a_0 G_1 - (b_1 - b_0)\right]\right\} G_3 \\ &= \left\{2\mathscr{Z}\left(\frac{1}{K} - \widehat{\nu_0^*}\right)\left[\mathscr{Z} G_1 - a_0\xi G_1 + \xi(b_1 - b_0)\right] - a_0 G_1 + (b_1 - b_0)\right\} G_3 \\ &= \left\{2\mathscr{Z}\left(\frac{1}{K} - \widehat{\nu_0^*}\right)\left[(\mathscr{Z} - a_0\xi) G_1 + \xi(b_1 - b_0)\right] - a_0 G_1 + (b_1 - b_0)\right\} G_3 \\ &= \left\{2\mathscr{Z}(\mathscr{Z} - a_0\xi)\left(\frac{1}{K} - \widehat{\nu_0^*}\right) G_1 + 2\xi\mathscr{Z}\left(\frac{1}{K} - \widehat{\nu_0^*}\right)(b_1 - b_0) - a_0 G_1 + (b_1 - b_0)\right\} G_3 \\ &= \left\{\left[2\mathscr{Z}(\mathscr{Z} - a_0\xi)\left(\frac{1}{K} - \widehat{\nu_0^*}\right) - a_0\right] G_1 + \left[2\xi\mathscr{Z}\left(\frac{1}{K} - \widehat{\nu_0^*}\right) + 1\right](b_1 - b_0)\right\} G_3. \end{aligned} \tag{A.216}$$

Moreover, from (A.206), we have that

$$\eta = \mathscr{Z} - a_0\xi = 1 + a_1 K. \tag{A.217}$$

Substituting (A.217) in (A.216) we have that

$$\begin{aligned}\mathcal{Q}_1 &= \left\{\left[2\mathcal{Z}\left(\mathcal{Z}-a_0\xi\right)\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-a_0\right]G_1+\left[2\xi\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)+1\right](b_1-b_0)\right\}G_3\\ &= \left\{\left[2\mathcal{Z}\left(1+a_1K\right)\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-a_0\right]G_1+\left[2\xi\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)+1\right](b_1-b_0)\right\}G_3\\ &= \left\{\left[2a_1K\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)+2\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-a_0\right]G_1+\left[2\xi\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)+1\right](b_1-b_0)\right\}G_3\\ &= \left[\left(V_3+W_3\right)G_1+U_3(b_1-b_0)\right]G_3,\end{aligned} \tag{A.218}$$

where

$$V_3 = 2a_1K\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right), \tag{A.219}$$

$$W_3 = 2\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-a_0 \tag{A.220}$$

and

$$U_3 = 2\xi\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)+1. \tag{A.221}$$

For any given values of a_1, K, ξ and $\mathcal{Z}$, one easily deduces that $V_3 > 0$ and $U_3 > 0$. Now, we are going to show that $W_3 > 0$. In order to do that, we first substitute (A.206) in (A.220) to get:

$$\begin{aligned}W_3 =& 2\mathcal{Z}\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-a_0\\ =& 2\left[(1+a_1K)+a_0K\left(2+a_1K\right)\right]\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-a_0\\ >& a_0K\left(2+a_1K\right)\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-a_0\\ =& a_0\left[K\left(2+a_1K\right)\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-1\right].\end{aligned} \tag{A.222}$$

Now, making use of the expression of $\widehat{\nu_0^*}$ given by (A.141) we have that

$$\begin{aligned}\tfrac{1}{K}-\widehat{\nu_0^*} =& \tfrac{1}{K}-\frac{2(a_0+a_1+a_0a_1K)}{(2a_0K+a_0a_1K^2)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}}\\ =& \frac{(2a_0K+a_0a_1K^2)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}-2K(a_0+a_1+a_0a_1K)}{K\left[(2a_0K+a_0a_1K^2)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}\right]}\\ =& \frac{\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}-\left(2a_1K+a_0a_1K^2\right)}{K\left[(2a_0K+a_0a_1K^2)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}\right]}\\ =& \frac{\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}-a_1K(2+a_0K)}{K\left[(2a_0K+a_0a_1K^2)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}\right]}.\end{aligned} \tag{A.223}$$

Now plugging (A.223) in (A.222) we get:

$$
\begin{aligned}
W_3 &> a_0\left[K\left(2+a_1K\right)\left(\tfrac{1}{K}-\widehat{\nu_0^*}\right)-1\right]\\
&=a_0\left[K\left(2+a_1K\right)\frac{\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}-a_1K\left(2+a_0K\right)}{K\left[\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}\right]}-1\right]\\
&=a_0\left[\left(2+a_1K\right)\frac{\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}-a_1K\left(2+a_0K\right)}{\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}}-1\right]\\
&=\frac{a_0}{\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}}\Bigg[\\
&\quad\left(2+a_1K\right)\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}\\
&\quad-a_1K\left(2+a_0K\right)\left(2+a_1K\right)-\left(2a_0K+a_0a_1K^2\right)\\
&\quad-\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}\Bigg]\\
&=\frac{a_0}{\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}}\Bigg[\\
&\quad\left[\left(2+a_1K\right)-1\right]\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}\\
&\quad-a_1K\left(2+a_0K\right)\left(2+a_1K\right)-a_0K\left(2+a_1K\right)\Bigg]\\
&=\frac{a_0}{\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}}\Bigg\{\\
&\quad\left(1+a_1K\right)\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}\\
&\quad-K\left(2+a_1K\right)\left[a_1\left(2+a_0K\right)+a_0\right]\Bigg\}\\
&=\frac{a_0}{\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}}\Bigg\{\\
&\quad\left(1+a_1K\right)\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}\\
&\quad-K\left(2+a_1K\right)\left[a_0\left(1+a_1K\right)+2a_1\right]\Bigg\}\\
&=a_0\frac{V_4}{W_4},
\end{aligned}
\tag{A.224}
$$

where

$$V_4 = (1 + a_1 K) \sqrt{\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)(a_0 + a_1 + a_0 a_1 K)} \\ - K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1] \tag{A.225}$$

and

$$W_4 = \left(2a_0 K + a_0 a_1 K^2\right) + \sqrt{\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)(a_0 + a_1 + a_0 a_1 K)}. \tag{A.226}$$

For any values of a_0, a_1 and K, we see that $W_4 > 0$, thus, to compute the value of (A.224) we need only to estimate the value of V_4. Suppose that

$$V_4 = (1 + a_1 K) \sqrt{\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)(a_0 + a_1 + a_0 a_1 K)} \\ - K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1] \leq 0. \tag{A.227}$$

Then, we would have

$$(1 + a_1 K) \sqrt{\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)(a_0 + a_1 + a_0 a_1 K)} \\ \leq K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1]. \tag{A.228}$$

Both sides of (A.228) are positive, then, by squaring them we have that

$$(1 + a_1 K)^2 \left[\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)(a_0 + a_1 + a_0 a_1 K)\right] \\ \leq K^2 (2 + a_1 K)^2 [a_0(1 + a_1 K) + 2a_1]^2. \tag{A.229}$$

Thus,

$$(1 + a_1 K)^2 \left[a_0^2 K^2 (2 + a_1 K)^2 + 4K(2 + a_1 K)(a_0 + a_1 + a_0 a_1 K)\right] \\ \leq K^2 (2 + a_1 K)^2 [a_0(1 + a_1 K) + 2a_1]^2, \tag{A.230}$$

which leads to

$$K(2 + a_1 K)(1 + a_1 K)^2 \left[a_0^2 K(2 + a_1 K) + 4(a_0 + a_1 + a_0 a_1 K)\right] \\ \leq K^2 (2 + a_1 K)^2 [a_0(1 + a_1 K) + 2a_1]^2, \tag{A.231}$$

where, $K(2 + a_1 K) > 0$. Therefore,

$$(1 + a_1 K)^2 \left[a_0^2 K(2 + a_1 K) + 4(a_0 + a_1 + a_0 a_1 K)\right] \\ \leq K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1]^2. \tag{A.232}$$

Now, expanding the squares we find that:

$$(1 + a_1 K)^2 \left[a_0^2 K (2 + a_1 K) + 4 (a_0 + a_1 + a_0 a_1 K)\right] \\ \leq K (2 + a_1 K) \left[a_0^2 (1 + a_1 K)^2 + 4a_0 a_1 (1 + a_1 K) + 4a_1^2\right], \tag{A.233}$$

and by distributing some terms:

$$a_0^2 K (2 + a_1 K) (1 + a_1 K)^2 + 4 (a_0 + a_1 + a_0 a_1 K) (1 + a_1 K)^2 \\ \leq a_0^2 K (2+a_1 K) (1 + a_1 K)^2 + 4a_0 a_1 K (2 + a_1 K) (1 + a_1 K) + 4a_1^2 K (2 + a_1 K). \tag{A.234}$$

Thus,

$$4 (a_0 + a_1 + a_0 a_1 K) (1 + a_1 K)^2 \\ \leq 4a_0 a_1 K (2 + a_1 K) (1 + a_1 K) + 4a_1^2 K (2 + a_1 K), \tag{A.235}$$

and distributing again:

$$4 (a_0 + a_1) (1 + a_1 K)^2 + 4a_0 a_1 K (1 + a_1 K)^2 \\ \leq 4a_0 a_1 K (1 + a_1 K) + 4a_0 a_1 K (1 + a_1 K)^2 + 4a_1^2 K (2 + a_1 K), \tag{A.236}$$

which leads to

$$4 (a_0 + a_1) (1 + a_1 K)^2 \\ \leq 4a_0 a_1 K (1 + a_1 K) + 4a_1^2 K (2 + a_1 K). \tag{A.237}$$

Finally,

$$(a_0 + a_1) (1 + a_1 K)^2 \\ \leq a_0 a_1 K (1 + a_1 K) + a_1^2 K (2 + a_1 K), \tag{A.238}$$

and by distributing some terms and expanding the squares again we deduce that

$$(a_0 + a_1) \left(1 + 2a_1 K + a_1^2 K^2\right) \\ \leq a_0 a_1 K (1 + a_1 K) + a_1^2 K (1 + a_1 K) + a_1^2 K, \tag{A.239}$$

thus,

$$(a_0 + a_1) \left(1 + 2a_1 K + a_1^2 K^2\right) \\ \leq (a_0 + a_1) a_1 K (1 + a_1 K) + a_1^2 K, \tag{A.240}$$

which leads to

$$(a_0 + a_1) \left(1 + 2a_1 K + a_1^2 K^2\right) \\ \leq (a_0 + a_1) \left(a_1 K + a_1^2 K^2\right) + a_1^2 K. \tag{A.241}$$

Hence,

$$(a_0 + a_1)(1 + a_1 K) \le a_1^2 K, \tag{A.242}$$

which finally leads to

$$a_0 + a_1 + a_0 a_1 K + a_1^2 K \le a_1^2 K, \tag{A.243}$$

which is impossible since $a_0 + a_1 + a_0 a_1 K > 0$.

On a base of that, we conclude that

$$V_4 > 0, \tag{A.244}$$

so,

$$W_3 > a_0 \frac{V_4}{W_4} > 0. \tag{A.245}$$

Now, since $V_3 > 0$, $W_3 > 0$ and $U_3 > 0$, and given the values of b_0, b_1, G_1 and G_3, we have that

$$\mathscr{Q}_1 = [(V_3 + W_3) G_1 + U_3(b_1 - b_0)] G_3 > 0. \tag{A.246}$$

So we need only to estimate the value of $\mathscr{P}_1$.

$$\begin{aligned}
\mathscr{P}_1 &= \left(\tfrac{1}{K^2} - \widehat{\nu_0^*}^2\right)\left(\mathscr{Z}^2 G_1^2 - \xi^2 G_3^2\right) - \tfrac{1}{K} G_1 \left(\tfrac{1}{K} G_1 + 2G_3\right) \\
&= \left(\tfrac{1}{K} + \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 + \xi G_3)\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) - \tfrac{1}{K} G_1 \left(\tfrac{1}{K} G_1 + 2G_3\right) \\
&= \left(\tfrac{1}{K}\mathscr{Z} G_1 + \tfrac{1}{K}\xi G_3 + \widehat{\nu_0^*}\mathscr{Z} G_1 + \widehat{\nu_0^*}\xi G_3\right)\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z} G_1 - \xi G_3) \\
&\quad - \tfrac{1}{K} G_1 \left(\tfrac{1}{K} G_1 + 2G_3\right).
\end{aligned} \tag{A.247}$$

By Eqs. (A.205), (A.208) and (A.217), we have that

$$\xi = K(2 + a_1 K) > 2K, \tag{A.248}$$

$$\mathscr{Z} G_1 - \xi G_3 = \mathscr{Z} G_1 - \xi [a_0 G_1 - (b_1 - b_0)] \ge \mathscr{Z} G_1 - a_0 \xi G_1 = \eta G_1, \tag{A.249}$$

$$\mathscr{Z} = \eta + a_0 \xi > \eta \tag{A.250}$$

and

$$\widehat{\nu_0^*} \xi G_3 > 0. \tag{A.251}$$

Using inequalities (A.248)–(A.251) in (A.247) we get:

$$
\begin{aligned}
\mathscr{P}_1 &= \left(\tfrac{1}{K}\mathscr{Z}G_1 + \tfrac{1}{K}\xi G_3 + \widehat{\nu_0^*}\mathscr{Z}G_1 + \widehat{\nu_0^*}\xi G_3\right)\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)(\mathscr{Z}G_1 - \xi G_3) \\
&\quad - \tfrac{1}{K}G_1\left(\tfrac{1}{K}G_1 + 2G_3\right) \\
&> \left(\tfrac{1}{K}\mathscr{Z}G_1 + 2G_3 + \widehat{\nu_0^*}\eta G_1\right)\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\eta G_1 \\
&\quad - \tfrac{1}{K}G_1\left(\tfrac{1}{K}G_1 + 2G_3\right) \\
&= \left[\eta\left(\tfrac{1}{K}\mathscr{Z}G_1 + 2G_3 + \widehat{\nu_0^*}\eta G_1\right)\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\right. \\
&\quad \left. - \tfrac{1}{K}\left(\tfrac{1}{K}G_1 + 2G_3\right)\right]G_1 \\
&= \left\{\left[\left(\tfrac{1}{K}\mathscr{Z} + \widehat{\nu_0^*}\eta\right)\eta G_1 + 2\eta G_3\right]\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\right. \\
&\quad \left. - \tfrac{1}{K}\left(\tfrac{1}{K}G_1 + 2G_3\right)\right\}G_1.
\end{aligned} \tag{A.252}
$$

Now, making use of (A.204), (A.206) and (A.248), we come to

$$\eta = 1 + a_1 K > 1 \tag{A.253}$$

and

$$\mathscr{Z} = \eta + a_0\xi > \eta + 2a_0 K. \tag{A.254}$$

Then, applying inequalities (A.253) and (A.254) to (A.252) one gets:

$$
\begin{aligned}
\mathscr{P}_1 &= \left\{\left[\left(\tfrac{1}{K}\mathscr{Z} + \widehat{\nu_0^*}\eta\right)\eta G_1 + 2\eta G_3\right]\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\right. \\
&\quad \left. - \tfrac{1}{K}\left(\tfrac{1}{K}G_1 + 2G_3\right)\right\}G_1 \\
&> \left\{\left[\left(\tfrac{1}{K}(\eta + 2a_0 K) + \widehat{\nu_0^*}\right)G_1 + 2\eta G_3\right]\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\right. \\
&\quad \left. - \tfrac{1}{K}\left(\tfrac{1}{K}G_1 + 2G_3\right)\right\}G_1 \\
&= \left\{\left[\left(\tfrac{1}{K}\eta + 2a_0 + \widehat{\nu_0^*}\right)G_1 + 2\eta G_3\right]\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\right. \\
&\quad \left. - \tfrac{1}{K}\left(\tfrac{1}{K}G_1 + 2G_3\right)\right\}G_1.
\end{aligned} \tag{A.255}
$$

Substituting the value of G_3 given by (A.208) in (A.255), we have:

$$
\begin{aligned}
\mathscr{P}_1 &> \left\{\left[\left(\tfrac{1}{K}\eta + 2a_0 + \widehat{\nu_0^*}\right)G_1 + 2\eta G_3\right]\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\right. \\
&\quad \left. - \tfrac{1}{K}\left(\tfrac{1}{K}G_1 + 2G_3\right)\right\}G_1. \\
&= \left\{\left[\left(\tfrac{1}{K}\eta + 2a_0 + \widehat{\nu_0^*}\right)G_1 + 2\eta\left[a_0 G_1 - (b_1 - b_0)\right]\right]\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\right. \\
&\quad \left. - \tfrac{1}{K}\left\{\tfrac{1}{K}G_1 + 2\left[a_0 G_1 - (b_1 - b_0)\right]\right\}\right\}G_1. \\
&= \left\{\left[\left(\tfrac{1}{K}\eta + 2a_0 + \widehat{\nu_0^*}\right)G_1 + 2a_0\eta G_1 - 2\eta(b_1 - b_0)\right]\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right)\right. \\
&\quad \left. - \tfrac{1}{K}\left[\tfrac{1}{K}G_1 + 2a_0 G_1 - 2(b_1 - b_0)\right]\right\}G_1.
\end{aligned} \tag{A.256}
$$

$$
\begin{aligned}
&= \big\{\big[\big(\tfrac{1}{K}\eta + 2a_0 + 2a_0\eta + \widehat{\nu_0^*}\big)\, G_1 - 2\eta(b_1 - b_0)\big]\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) \\
&\quad - \tfrac{1}{K}\,\big[\big(\tfrac{1}{K} + 2a_0\big)\, G_1 - 2(b_1 - b_0)\big]\,\big\}G_1. \\
&= \big\{\big(\tfrac{1}{K}\eta + 2a_0 + 2a_0\eta + \widehat{\nu_0^*}\big)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big)\, G_1 - 2\eta\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big)\,(b_1 - b_0) \\
&\quad - \tfrac{1}{K}\,\big(\tfrac{1}{K} + 2a_0\big)\, G_1 + 2\tfrac{1}{K}(b_1 - b_0)\,\big\}G_1. \\
&= \big\{\big[\big(\tfrac{1}{K}\eta + 2a_0 + 2a_0\eta + \widehat{\nu_0^*}\big)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) - \tfrac{1}{K}\,\big(\tfrac{1}{K} + 2a_0\big)\big]\, G_1 \\
&\quad + 2\,\big[\tfrac{1}{K} - \eta\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big)\big]\,(b_1 - b_0)\,\big\}G_1.
\end{aligned}
$$

Next, we substitute the value of η given by (A.204) in (A.256) to obtain:

$$
\begin{aligned}
\mathscr{P}_1 &> \big\{\big[\big(\tfrac{1}{K}\eta + 2a_0 + 2a_0\eta + \widehat{\nu_0^*}\big)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) - \tfrac{1}{K}\,\big(\tfrac{1}{K} + 2a_0\big)\big]\, G_1 \\
&\quad + 2\,\big[\tfrac{1}{K} - \eta\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big)\big]\,(b_1 - b_0)\,\big\}G_1. \\
&= \big\{\big[\big(\tfrac{1}{K}\,(1 + a_1 K) + 2a_0 + 2a_0\eta + \widehat{\nu_0^*}\big)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) - \tfrac{1}{K}\,\big(\tfrac{1}{K} + 2a_0\big)\big]\, G_1 \\
&\quad + 2\,\big[\tfrac{1}{K} - (1 + a_1 K)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big)\big]\,(b_1 - b_0)\,\big\}G_1. \\
&= \big\{\big[\big(\tfrac{1}{K} + a_1 + 2a_0 + 2a_0\eta + \widehat{\nu_0^*}\big)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) - \tfrac{1}{K}\,\big(\tfrac{1}{K} + 2a_0\big)\big]\, G_1 \\
&\quad + 2\,\big[\tfrac{1}{K} - \big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) - a_1 K\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big)\big]\,(b_1 - b_0)\,\big\}G_1. \\
&= \big\{\big[-\widehat{\nu_0^*}\,\big(\tfrac{1}{K} + 2a_0\big) + (a_1 + 2a_0\eta)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) + \widehat{\nu_0^*}\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big)\big]\, G_1 \\
&\quad + 2\,\big[(1 + a_1 K)\,\widehat{\nu_0^*} - a_1\big]\,(b_1 - b_0)\,\big\}G_1. \\
&= \big\{\big[(a_1 + 2a_0\eta)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) - \widehat{\nu_0^*}\,\big(2a_0 + \widehat{\nu_0^*}\big)\big]\, G_1 \\
&\quad + 2\,\big[(1 + a_1 K)\,\widehat{\nu_0^*} - a_1\big]\,(b_1 - b_0)\,\big\}G_1. \\
&= \left[V_5 G_1 + 2W_5(b_1 - b_0)\right] G_1,
\end{aligned}
\tag{A.257}
$$

where

$$V_5 = (a_1 + 2a_0\eta)\,\big(\tfrac{1}{K} - \widehat{\nu_0^*}\big) - \widehat{\nu_0^*}\,\big(2a_0 + \widehat{\nu_0^*}\big) \tag{A.258}$$

and

$$W_5 = (1 + a_1 K)\,\widehat{\nu_0^*} - a_1. \tag{A.259}$$

Now, we only need to show that $V_5 > 0$ and $W_5 > 0$.

First, since $\nu_0^*(\beta)$ is strictly decreasing, we have $\widehat{\nu_0^*} = \lim_{\beta \to 0} \nu_0^*(\beta) > \nu_0^*(1) = \overline{\nu_0^*}$, thus,

$$W_5 = (1 + a_1 K)\,\widehat{\nu_0^*} - a_1 > (1 + a_1 K)\,\overline{\nu_0^*} - a_1. \tag{A.260}$$

Substituting the value of $\overline{\nu_0^*}$ given by (A.193) in (A.260) we have that:

$$\begin{aligned}
W_5 &> (1 + a_1 K)\, \overline{\nu_0^*} - a_1 \\
&= (1 + a_1 K) \left(\frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \right) - a_1 \\
&= \frac{(1 + a_1 K)(a_0 + a_1 + a_0 a_1 K) - a_1 \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right)}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\
&= \frac{V_6}{W_6},
\end{aligned} \tag{A.261}$$

where

$$V_6 = (1 + a_1 K)(a_0 + a_1 + a_0 a_1 K) - a_1 \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) \tag{A.262}$$

and

$$W_6 = 1 + 2a_0 K + a_1 K + a_0 a_1 K^2. \tag{A.263}$$

Given the values of a_0, a_1 and K, it's easy to see that $W_6 > 0$. Now, we calculate the value of V_6:

$$\begin{aligned}
V_6 &= (1 + a_1 K)(a_0 + a_1 + a_0 a_1 K) - a_1 \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) \\
&= (a_0 + a_1 + a_0 a_1 K) + a_1 K (a_0 + a_1 + a_0 a_1 K) - a_1 \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) \\
&= a_0 + a_1 (1 + a_0 K) + a_1 \left(a_0 K + a_1 K + a_0 a_1 K^2\right) - a_1 \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) \\
&= a_0 + a_1 \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) - a_1 \left(1 + 2a_0 K + a_1 K + a_0 a_1 K^2\right) \\
&= a_0 > 0.
\end{aligned} \tag{A.264}$$

Therefore $V_6 > 0$ and

$$W_5 > \frac{V_6}{W_6} > 0. \tag{A.265}$$

So we only lack showing that $V_5 > 0$:

$$\begin{aligned}
V_5 &= (a_1 + 2a_0 \eta) \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*} \left(2a_0 + \widehat{\nu_0^*}\right) \\
&= (a_1 + a_0 \eta) \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) + a_0 \eta \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*} \left(a_0 + \widehat{\nu_0^*}\right) - a_0 \widehat{\nu_0^*} \\
&= \left[(a_1 + a_0 \eta) \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*} \left(a_0 + \widehat{\nu_0^*}\right)\right] + a_0 \left[\eta \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*}\right] \\
&= V_7 + a_0 W_7,
\end{aligned} \tag{A.266}$$

where

$$V_7 = (a_1 + a_0 \eta) \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*} \left(a_0 + \widehat{\nu_0^*}\right) \tag{A.267}$$

and

$$W_7 = \eta \left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*}. \tag{A.268}$$

Finally, we will demonstrate that $V_7 > 0$ and $W_7 > 0$. Substituting the value of η given by (A.204) in V_7 we get:

$$\begin{aligned} V_7 &= (a_1 + a_0\eta)\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*}\left(a_0 + \widehat{\nu_0^*}\right) \\ &= [a_1 + a_0(1 + a_1K)]\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*}\left(a_0 + \widehat{\nu_0^*}\right) \\ &= (a_1 + a_0 + a_0a_1K)\left(\tfrac{1}{K} - \widehat{\nu_0^*}\right) - \widehat{\nu_0^*}\left(a_0 + \widehat{\nu_0^*}\right) \\ &= \tfrac{1}{K}(a_1 + a_0 + a_0a_1K) - \widehat{\nu_0^*}(a_1 + a_0 + a_0a_1K) - a_0\widehat{\nu_0^*} - \widehat{\nu_0^*}^2 \\ &= \tfrac{1}{K}(a_1 + a_0 + a_0a_1K) - \widehat{\nu_0^*}(a_1 + 2a_0 + a_0a_1K) - \widehat{\nu_0^*}^2. \end{aligned} \tag{A.269}$$

we get:

$$(1-\beta)\left(-2\tau + a_1\tau^2\right)\nu_0^2 + \left(\beta - 2a_0\tau - \beta a_1\tau + a_0a_1\tau^2\right)\nu_0 - (a_0 + a_1 - a_0a_1\tau) = 0,$$

given by (A.16), for $\tau = -K$. Now by applying the limit when $\beta \to 0$, one obtains the following equality:

$$\left(2K + a_1K^2\right)\widehat{\nu_0^*}^2 + \left(2a_0K + a_0a_1K^2\right)\widehat{\nu_0^*} - (a_0 + a_1 + a_0a_1K) = 0. \tag{A.270}$$

Therefore,

$$\begin{aligned} &\left(2K + a_1K^2\right)\widehat{\nu_0^*}^2 + \left(2a_0K + a_0a_1K^2\right)\widehat{\nu_0^*} - (a_0 + a_1 + a_0a_1K) \\ &= (2 + a_1K)K\widehat{\nu_0^*}^2 + (2a_0 + a_0a_1K)K\widehat{\nu_0^*} - \tfrac{1}{K}(a_0 + a_1 + a_0a_1K)K \\ &= \left[(2 + a_1K)\widehat{\nu_0^*}^2 + (2a_0 + a_0a_1K)\widehat{\nu_0^*} - \tfrac{1}{K}(a_0 + a_1 + a_0a_1K)\right]K \\ &= \left[(1 + 1 + a_1K)\widehat{\nu_0^*}^2 + (a_1 + 2a_0 + a_0a_1K - a_1)\widehat{\nu_0^*} - \tfrac{1}{K}(a_0 + a_1 + a_0a_1K)\right]K \\ &= \left[\widehat{\nu_0^*}^2 + (a_1 + 2a_0 + a_0a_1K)\widehat{\nu_0^*} - \tfrac{1}{K}(a_0 + a_1 + a_0a_1K) + (1 + a_1K)\widehat{\nu_0^*}^2 - a_1\widehat{\nu_0^*}\right]K \\ &= 0. \end{aligned} \tag{A.271}$$

Since $K > 0$, we have that

$$\widehat{\nu_0^*}^2 + (a_1 + 2a_0 + a_0a_1K)\widehat{\nu_0^*} - \tfrac{1}{K}(a_0 + a_1 + a_0a_1K) + (1 + a_1K)\widehat{\nu_0^*}^2 - a_1\widehat{\nu_0^*} = 0, \tag{A.272}$$

thus,

$$(1 + a_1K)\widehat{\nu_0^*}^2 - a_1\widehat{\nu_0^*} = \tfrac{1}{K}(a_0 + a_1 + a_0a_1K) - (a_1 + 2a_0 + a_0a_1K)\widehat{\nu_0^*} - \widehat{\nu_0^*}^2. \tag{A.273}$$

Applying equality (A.273) to (A.269) we have that:

$$\begin{aligned} V_7 =& \frac{1}{K}\left(a_0+a_1+a_0a_1K\right)-\left(a_1+2a_0+a_0a_1K\right)\widehat{v_0^*}-\widehat{v_0^*}^2 \\ =& \left(1+a_1K\right)\widehat{v_0^*}^2-a_1\widehat{v_0^*} \\ =& \left[\left(1+a_1K\right)\widehat{v_0^*}-a_1\right]\widehat{v_0^*} \\ =& W_5\widehat{v_0^*}. \end{aligned} \tag{A.274}$$

Now recalling that $W_5>0$ and $\widehat{v_0^*}>0$, we have:

$$V_7 = W_5\widehat{v_0^*} > 0. \tag{A.275}$$

So we only need to show that $W_7>0$. To do this, we plug the value of η given by (A.204) in W_7 to get:

$$\begin{aligned} W_7 =& \eta\left(\frac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*} \\ =& \left(1+a_1K\right)\left(\frac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*} \\ =& \frac{1}{K}\left(1+a_1K\right)-\left(1+a_1K\right)\widehat{v_0^*}-\widehat{v_0^*} \\ =& \frac{1}{K}\left(1+a_1K\right)-\left(2+a_1K\right)\widehat{v_0^*}. \end{aligned} \tag{A.276}$$

Using relationship (A.270) we have that:

$$\begin{aligned} &\left(2K+a_1K^2\right)\widehat{v_0^*}^2+\left(2a_0K+a_0a_1K^2\right)\widehat{v_0^*}-\left(a_0+a_1+a_0a_1K\right) \\ &= \left[\left(2+a_1K\right)\widehat{v_0^*}^2+\left(2a_0+a_0a_1K\right)\widehat{v_0^*}-\frac{1}{K}\left(a_0+a_1+a_0a_1K\right)\right]K \\ &= \left\{\left(2+a_1K\right)\widehat{v_0^*}^2+a_0\left(2+a_1K\right)\widehat{v_0^*}-\frac{1}{K}\left[a_1+a_0\left(1+a_1K\right)\right]\right\}K \\ &= \left\{\left(2+a_1K\right)\widehat{v_0^*}^2+a_0\left(2+a_1K\right)\widehat{v_0^*}-\frac{1}{K}a_1-\frac{1}{K}a_0\left(1+a_1K\right)\right\}K \\ &= \left\{\left(2+a_1K\right)\widehat{v_0^*}^2-\frac{1}{K}a_1+a_0\left[\left(2+a_1K\right)\widehat{v_0^*}-\frac{1}{K}\left(1+a_1K\right)\right]\right\}K. \end{aligned} \tag{A.277}$$

Since $K>0$, we have that:

$$\left(2+a_1K\right)\widehat{v_0^*}^2-\frac{1}{K}a_1+a_0\left[\left(2+a_1K\right)\widehat{v_0^*}-\frac{1}{K}\left(1+a_1K\right)\right]=0, \tag{A.278}$$

which implies

$$\left(2+a_1K\right)\widehat{v_0^*}^2-\frac{1}{K}a_1=-a_0\left[\left(2+a_1K\right)\widehat{v_0^*}-\frac{1}{K}\left(1+a_1K\right)\right]. \tag{A.279}$$

As $a_0>0$, one gets:

$$\frac{1}{a_0}\left[\left(2+a_1K\right)\widehat{v_0^*}^2-\frac{1}{K}a_1\right]=\frac{1}{K}\left(1+a_1K\right)-\left(2+a_1K\right)\widehat{v_0^*}. \tag{A.280}$$

Now, applying equality (A.280) to (A.276), we deduce:

$$\begin{aligned} W_7 =& \tfrac{1}{K}\left(1 + a_1 K\right) - \left(2 + a_1 K\right)\widehat{\nu_0^*} \\ =& \tfrac{1}{a_0}\left[\left(2 + a_1 K\right)\widehat{\nu_0^*}^2 - \tfrac{1}{K}a_1\right] \\ =& \frac{U_8}{a_0}, \end{aligned} \tag{A.281}$$

where

$$U_8 = \left(2 + a_1 K\right)\widehat{\nu_0^*}^2 - \tfrac{1}{K}a_1. \tag{A.282}$$

Finally, let's suppose that

$$U_8 \leq 0. \tag{A.283}$$

Substituting the value of $\widehat{\nu_0^*}$, given by (A.141), in U_8 we have:

$$\begin{aligned} U_8 &= \left(2 + a_1 K\right)\widehat{\nu_0^*}^2 - \tfrac{1}{K}a_1 \\ &= \left(2 + a_1 K\right)\left[\frac{2(a_0+a_1+a_0a_1K)}{(2a_0K+a_0a_1K^2)+\sqrt{(2a_0K+a_0a_1K^2)^2+4(2K+a_1K^2)(a_0+a_1+a_0a_1K)}}\right]^2 - \tfrac{1}{K}a_1 \\ &= \Bigg\{\left(2 + a_1 K\right)\left[2\left(a_0 + a_1 + a_0 a_1 K\right)\right]^2 - \tfrac{1}{K}a_1\Bigg[\left(2a_0 K + a_0 a_1 K^2\right) \\ &\quad + \sqrt{\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)\left(a_0 + a_1 + a_0 a_1 K\right)}\Bigg]^2\Bigg\} \\ &\quad / \left[\left(2a_0 K + a_0 a_1 K^2\right) + \sqrt{\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)\left(a_0 + a_1 + a_0 a_1 K\right)}\right]^2 \\ &= \frac{V_9}{W_9} \leq 0, \end{aligned} \tag{A.284}$$

where

$$\begin{aligned} V_9 = &\left(2 + a_1 K\right)\left[2\left(a_0 + a_1 + a_0 a_1 K\right)\right]^2 - \tfrac{1}{K}a_1\Bigg[\left(2a_0 K + a_0 a_1 K^2\right) \\ &+ \sqrt{\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)\left(a_0 + a_1 + a_0 a_1 K\right)}\Bigg]^2 \end{aligned} \tag{A.285}$$

and

$$W_9 = \left[\left(2a_0 K + a_0 a_1 K^2\right) + \sqrt{\left(2a_0 K + a_0 a_1 K^2\right)^2 + 4\left(2K + a_1 K^2\right)\left(a_0 + a_1 + a_0 a_1 K\right)}\right]^2. \tag{A.286}$$

For any given values of a_0, a_1 and K, one has $W_9 > 0$. Therefore, Eq. (A.284) implies

$$V_9 \le 0. \tag{A.287}$$

Hence

$$\begin{aligned}
V_9 =& (2+a_1K)\left[2\left(a_0+a_1+a_0a_1K\right)\right]^2 - \tfrac{1}{K}a_1\Bigg[\left(2a_0K+a_0a_1K^2\right) \\
&+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)\left(a_0+a_1+a_0a_1K\right)}\Bigg]^2 \\
=&4\left(2+a_1K\right)\left[a_1+a_0\left(1+a_1K\right)\right]^2 - \tfrac{1}{K}a_1\Bigg[a_0K\left(2+a_1K\right) \\
&+\sqrt{K\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\}}\Bigg]^2 \\
=&4a_1^2\left(2+a_1K\right)+8a_0a_1\left(2+a_1K\right)\left(1+a_1K\right)+4a_0^2\left(2+a_1K\right)\left(1+a_1K\right)^2 \\
&-a_1a_0^2K\left(2+a_1K\right)^2-a_1\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\} \\
&-2a_0a_1\left(2+a_1K\right)\sqrt{K\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\}} \\
=&\left(2+a_1K\right)\Big[4a_1^2+8a_0a_1\left(1+a_1K\right)+4a_0^2\left(1+a_1K\right)^2 \\
&-a_1a_0^2K\left(2+a_1K\right)-a_1a_0^2K\left(2+a_1K\right)-4a_1\left[a_1+a_0\left(1+a_1K\right)\right]\Big] \\
&-2a_0a_1\left(2+a_1K\right)\sqrt{K\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\}} \\
=&\left(2+a_1K\right)\left[4a_0a_1\left(1+a_1K\right)+4a_0^2\left(1+a_1K\right)^2-2a_1a_0^2K\left(2+a_1K\right)\right] \\
&-2a_0a_1\left(2+a_1K\right)\sqrt{K\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\}} \\
=&a_0\left(2+a_1K\right)\left[4a_1\left(1+a_1K\right)+4a_0\left(1+2a_1K+a_1^2K^2\right)-2a_0a_1K\left(2+a_1K\right)\right] \\
&-2a_0a_1\left(2+a_1K\right)\sqrt{K\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\}} \\
=&a_0\left(2+a_1K\right)\left(4a_0+4a_1+4a_0a_1K+4a_1^2K+2a_0a_1^2K^2\right) \\
&-2a_0a_1\left(2+a_1K\right)\sqrt{K\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\}} \\
=&2a_0\left(2+a_1K\right)\Bigg[\left(2a_0+2a_1+2a_0a_1K+2a_1^2K+a_0a_1^2K^2\right) \\
&-a_1\sqrt{K\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\}}\Bigg] \\
=&2a_0\left(2+a_1K\right)\Bigg[2a_0+a_1K\left(2\tfrac{1}{K}+2a_0+2a_1+a_0a_1K\right) \\
&-a_1\sqrt{K\left(2+a_1K\right)\left\{a_0^2K\left(2+a_1K\right)+4\left[a_1+a_0\left(1+a_1K\right)\right]\right\}}\Bigg] \le 0.
\end{aligned} \tag{A.288}$$

Since $2a_0\left(2+a_1K\right)>0$, then, from (A.288) the following condition must be met:

$$\begin{aligned}&2a_0 + a_1K\left(2\tfrac{1}{K} + 2a_0 + 2a_1 + a_0a_1K\right)\\&- a_1\sqrt{K\left(2 + a_1K\right)\left\{a_0^2K\left(2 + a_1K\right) + 4\left[a_1 + a_0\left(1 + a_1K\right)\right]\right\}} \leq 0.\end{aligned} \tag{A.289}$$

Therefore,

$$\begin{aligned}&2a_0 + a_1K\left(2\tfrac{1}{K} + 2a_0 + 2a_1 + a_0a_1K\right)\\&\leq a_1\sqrt{K\left(2 + a_1K\right)\left\{a_0^2K\left(2 + a_1K\right) + 4\left[a_1 + a_0\left(1 + a_1K\right)\right]\right\}}.\end{aligned} \tag{A.290}$$

Since both sides of inequality (A.290) are positive, then,

$$\begin{aligned}&\left[2a_0 + a_1K\left(2\tfrac{1}{K} + 2a_0 + 2a_1 + a_0a_1K\right)\right]^2\\&\leq a_1^2\left[K\left(2 + a_1K\right)\left\{a_0^2K\left(2 + a_1K\right) + 4\left[a_1 + a_0\left(1 + a_1K\right)\right]\right\}\right],\end{aligned} \tag{A.291}$$

which leads to

$$\begin{aligned}0 \leq& a_1^2\left[K\left(2 + a_1K\right)\left\{a_0^2K\left(2 + a_1K\right) + 4\left[a_1 + a_0\left(1 + a_1K\right)\right]\right\}\right]\\&- \left[2a_0 + a_1K\left(2\tfrac{1}{K} + 2a_0 + 2a_1 + a_0a_1K\right)\right]^2\\=& a_1^2K\left(2 + a_1K\right)\left(2a_0^2K + a_0^2a_1K^2 + 4a_1 + 4a_0 + 4a_0a_1K\right)\\&- \left(2a_0 + 2a_1 + 2a_0a_1K + 2a_1^2K + a_0a_1^2K^2\right)^2\\=& a_1^2K\left(2 + a_1K\right)\left[4\left(a_0 + a_1 + a_0a_1K\right) + a_0^2K\left(2 + a_1K\right)\right]\\&- \left[2\left(a_0 + a_1 + a_0a_1K\right) + a_1^2K\left(2 + a_0K\right)\right]^2\\=& 4a_1^2K\left(a_0 + a_1 + a_0a_1K\right)\left[\left(2 + a_1K\right) - \left(2 + a_0K\right)\right]\\&+ a_1^2K^2\left[a_0^2\left(2 + a_1K\right)^2 - a_1^2\left(2 + a_0K\right)^2\right]\\&- 4\left(a_0 + a_1 + a_0a_1K\right)^2\\=& 4a_1^2K^2\left(a_0 + a_1 + a_0a_1K\right)\left(a_1 - a_0\right)\\&+ a_1^2K^2\left(2a_0 + a_0a_1K + 2a_1 + a_0a_1K\right)\left(2a_0 + a_0a_1K - 2a_1 - a_0a_1K\right)\\&- 4\left(a_0 + a_1 + a_0a_1K\right)^2\\=& 4a_1^2K^2\left(a_0 + a_1 + a_0a_1K\right)\left(a_1 - a_0\right)\\&- 4a_1^2K^2\left(a_0 + a_1 + a_0a_1K\right)\left(a_1 - a_0\right)\\&- 4\left(a_0 + a_1 + a_0a_1K\right)^2\\=& - 4\left(a_0 + a_1 + a_0a_1K\right)^2.\end{aligned} \tag{A.292}$$

Thus,

$$-4\left(a_0 + a_1 + a_0a_1K\right)^2 \geq 0,$$

which cannot happen due to $a_0 + a_1 + a_0a_1K > 0$.

Therefore, the assumption was false, so $U_8 > 0$. Thus,

$$W_7 = \frac{U_8}{a_0} > 0, \quad (A.293)$$

then,

$$V_5 = V_7 + a_0 W_7 > 0, \quad (A.294)$$

which proves that

$$\mathscr{P}_1 > [V_5 G_1 + 2W_5(b_1 - b_0)]\, G_1 \geq V_5 {G_1}^2 > 0. \quad (A.295)$$

So we have that $\mathscr{P}_1 > 0$, $\mathscr{Q}_1 > 0$ and $\mathscr{R}_1 > 0$, then,

$$V_2 = \mathscr{P}_1 + \mathscr{Q}_1 + \mathscr{R}_1 > 0, \quad (A.296)$$

and therefore,

$$\lim_{\beta \to 0} (\pi_1^c - \pi_1^*)(\beta) = \frac{1}{2} \frac{V_2}{W_2} > 0, \quad (A.297)$$

which proves (A.192). The proof of the theorem is complete ■

Appendix B: Proofs of Results from Chapter 3

B.1 Proofs of Results from Sect. 3.3

Theorem 3.2 *Let the number of oligopoly producers be at least three, i.e., $n \geq 3$, then, under assumptions **A**4–**A**6, there exists an interior equilibrium. Moreover, if the number of producers is two, i.e., $n = 2$, in addition to assumptions **A**4–**A**6, suppose that there exists an $\varepsilon > 0$ such that $G'(p) \leq -\varepsilon$ for all $p > 0$, then, there exists interior equilibrium.*

Proof For any given set of influence coefficients $\nu = (\nu_1, \ldots, \nu_n) \geq 0$, by Theorem 3.1, there exists the unique exterior equilibrium $(p(\nu), q_1(\nu), \ldots, q_n(\nu))$.

Now, we define the following functions:

$$F_i(\nu) = \frac{1}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \dfrac{1}{\nu_j + f_j''(q_j(\nu))} - G'(p(\nu))}, \quad i = 1, \ldots, n. \tag{B.1}$$

These functions are well-defined and continuous with respect to $\nu = (\nu_1, \ldots, \nu_n) \geq 0$, due to assumptions **A**4 and **A**5.

Therefore, the function $H = (F_1, \ldots, F_n) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is also continuous.

Next, we define the value $s = \max\{f_i''(q_i) \mid 0 \leq q_i \leq G(p_0),\ i = 1, \ldots, n\} > 0$.

For $n \geq 3$, if $0 \leq \nu_i \leq \dfrac{s}{n-2}$ for all $i = 1, \ldots, n$, we have:

J. G. Flores Muñiz et al., *Public Interest and Private Enterprize: New Developments*, Lecture Notes in Networks and Systems 138,
https://doi.org/10.1007/978-3-030-58349-1

$$
\begin{aligned}
0 \le F_i(\nu) &= \frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n} \dfrac{1}{\nu_j + f_j''(q_j(\nu))} - G'(p(\nu))} \le \frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n} \dfrac{1}{\nu_j + f_j''(q_j(\nu))}} \\
&\le \frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n} \dfrac{1}{\dfrac{s}{n-2} + s}} = \frac{1}{\dfrac{n-1}{\dfrac{s}{n-2} + s}} = \frac{s}{n-2}, \quad i = 1, \dots, n.
\end{aligned} \tag{B.2}
$$

Thus, the function $H = (F_1, \dots, F_n)$ maps the convex compact subset $\left[0, \dfrac{s}{n-2}\right]^n$ onto itself. Therefore, by Brouwer's fixed-point theorem, H has a fixed point, i.e., there exists $\nu^* = (\nu_1^*, \dots, \nu_n^*) \ge 0$ such that $F_i(\nu^*) = \nu_i^*$ for all $i = 1, \dots, n$.

On the other hand, for $n = 2$ and $G'(p) \le -\varepsilon$ for some $\varepsilon > 0$, if $0 \le \nu_i \le \dfrac{1}{\varepsilon}$ for all $i = 1, \dots, n$, we have:

$$
\begin{aligned}
0 \le F_1(\nu) &= \frac{1}{\dfrac{1}{\nu_2 + f_2''(q_2(\nu))} - G'(p(\nu))} \le \frac{1}{-G'(p(\nu))} \le \frac{1}{\varepsilon}, \\
0 \le F_2(\nu) &= \frac{1}{\dfrac{1}{\nu_1 + f_1''(q_1(\nu))} - G'(p(\nu))} \le \frac{1}{-G'(p(\nu))} \le \frac{1}{\varepsilon}.
\end{aligned} \tag{B.3}
$$

Thus, the function $H = (F_1, F_2)$ maps the convex compact subset $\left[0, \dfrac{1}{\varepsilon}\right]^2$ onto itself, then, again by Brouwer's fixed-point theorem, H has a fixed point $F_i(\nu^*) = \nu_i^*$, $i = 1, 2$.

By the definition of the functions (B.1), the influence coefficients $\nu^* = (\nu_1^*, \dots, \nu_n^*) \ge 0$, given by Brouwer's fixed-point theorem, satisfy the Consistency Criterion 3.2 and, therefore, the vector $(p(\nu^*), q_1(\nu^*), \dots, q_n(\nu^*), \nu_1^*, \dots, \nu_n^*)$ is the interior equilibrium. The proof is complete ■

B.2 Proofs of Results from Sect. 3.4

Theorem 3.6 *Suppose that the stronger assumption* ***A****7 is true, together with* ***A****4 and* ***A****6, and suppose that the function G is concave. Then, the Consistency Criterion for the original oligopoly is a necessary and sufficient condition for the collection of influence conjectures* $\nu = (\nu_1, \dots, \nu_n)$ *to produce Nash equilibrium in the meta-game.*

Proof Note that the necessity is a particular case of Theorem 3.5, thus, to prove Theorem 3.6, we just need to establish the sufficiency.

We assume **A**7, that is, for all i, the cost functions f_i are quadratic (and strictly convex) with $f_i(0) = 0$, $f_i'(0) > 0$, and $f_i''(0) > 0$, i.e.,

$$f_i(q_i) = \frac{1}{2}a_i {q_i}^2 + b_i q_i,$$

where $a_i > 0, b_i > 0, i = 1, \ldots, n$. Now we are in a position to demonstrate that in this specific case, each interior equilibrium $(p^*; q_1^*, \ldots, q_n^*; \nu_1^*, \ldots, \nu_n^*)$ of the original oligopoly provides the Nash equilibrium in the meta-game $\Gamma = (N, V, \Pi, D)$. Namely, the consistent conjectures (influence coefficients) $\nu^* = (\nu_1^*, \ldots, \nu_n^*)$ satisfying (3.14) form the Nash equilibrium in the meta-game.

Indeed, first of all, Eq. (3.14) in this particular case are reduced to the system

$$\nu_i^* = \frac{1}{\sum\limits_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} - G'(p^*)}, \quad i = 1, \ldots, n, \tag{B.4}$$

which clearly implies that all components of the vector ν^* are positive: $\nu_i^* > 0$, $i = 1, \ldots, n$.

Next, equations

$$\begin{aligned}\frac{\partial \pi_i}{\partial \nu_i} &= \frac{q_i^2}{\nu_i + f_i''(q_i)} \left[\frac{1}{\sum\limits_{k=1}^{n} \frac{1}{\nu_k + f_k''(q_k)} - G'(p)} - \frac{\sum\limits_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k + f_k''(q_k)} - G'(p)}{\sum\limits_{k=1}^{n} \frac{1}{\nu_k + f_k''(q_k)} - G'(p)} \nu_i \right] \\ &= \frac{q_i^2}{\nu_i + f_i''(q_i)} \frac{\sum\limits_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k + f_k''(q_k)} - G'(p)}{\sum\limits_{k=1}^{n} \frac{1}{\nu_k + f_k''(q_k)} - G'(p)} \left[\frac{1}{\sum\limits_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k + f_k''(q_k)} - G'(p)} - \nu_i \right] \\ &= 0, \quad i = 1, \ldots, n.\end{aligned} \tag{B.5}$$

guarantee that the first-order optimality conditions for the meta-game payoff functions hold:

$$\frac{\partial \pi_i}{\partial \nu_i}(\nu^*) = 0, \quad i = 1, \ldots, n. \tag{B.6}$$

Therefore, the value ν_i^* may be the maximum point of the i-th producer's payoff function

$$\tilde{\pi}_i(\nu_i) \equiv \pi(\nu_i, \nu_{-i}^*), \quad i = 1, \dots, n, \tag{B.7}$$

where $\nu_{-i}^* = (\nu_1^*, \dots, \nu_{i-1}^*, \nu_{i+1}^*, \dots, \nu_n^*)$. In order to establish the maximum point property, we are going to fix an arbitrary i and to show that the function $\tilde{\pi}_i = \tilde{\pi}_i(\nu_i)$:

(a) doesn't increase along the ray $(\nu_i^*, +\infty)$,
(b) doesn't decrease in the interval $(0, \nu_i^*)$.

In order to prove (a), taking into account (B.5), it suffices to show that

$$\frac{1}{\sum\limits_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} - G'(p(\nu_i^* + \delta, \nu_{-i}^*))} - (\nu_i^* + \delta) \le 0, \quad \forall \delta > 0. \tag{B.8}$$

By inverting both sides of the consistency equation (B.4) one gets

$$\frac{1}{\nu_i^*} = \sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} - G'(p^*), \tag{B.9}$$

which clearly implies the relationships

$$\sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} = \frac{1}{\nu_i^*} + G'(p^*) = \frac{1 + \nu_i^* G'(p^*)}{\nu_i^*} > 0. \tag{B.10}$$

Making use of (B.10), rewrite the left-hand side of (B.8) in the form

$$\begin{aligned}
&\frac{1}{\sum\limits_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} - G'(p(\nu_i^* + \delta, \nu_{-i}^*))} - (\nu_i^* + \delta) \\
&\quad = \frac{1}{\frac{1}{\nu_i^*} + G'(p^*) - G'(p(\nu_i^* + \delta, \nu_{-i}^*))} - (\nu_i^* + \delta) \\
&\quad = \frac{\nu_i^*}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* + \delta, \nu_{-i}^*))} - (\nu_i^* + \delta) \\
&\quad = \frac{(\nu_i^*)^2 \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right] - \delta + \nu_i^* \delta \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right]}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* + \delta, \nu_{-i}^*))} \\
&\quad = \frac{\nu_i^* \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* + \delta) - \delta}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* + \delta, \nu_{-i}^*))}.
\end{aligned} \tag{B.11}$$

Since $1 + \nu_i^* G'(p^*) > 0$ from (B.10), and $-\nu_i^* G'(p(\nu_i^* + \delta, \nu_{-i}^*)) \geq 0$ by assumption **A**4, then the denominator of (B.11) is strictly positive, thus the sign of ratio (B.11) is determined by that of its numerator. Now since the derivative $G'(p)$ is non-increasing by hypothesis, and $\frac{\partial p}{\partial \nu_i} > 0$ by (3.10), it isn't difficult to show that $\left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right] \leq 0$, hence the numerator of (B.11) is strictly negative for any $\delta > 0$:

$$\nu_i^* \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right] (\nu_i^* + \delta) - \delta < 0, \quad \forall \delta > 0. \tag{B.12}$$

The latter brings about the desired inequality

$$\frac{d\tilde{\pi}_i}{d\nu_i}(\nu_i, \nu_{-i}^*) < 0, \quad \forall \nu_i > \nu_i^*, \tag{B.13}$$

which finishes the proof of (a).

Now to demonstrate that (b) is also true, again taking into account (B.5), it is enough to check that

$$\frac{1}{\displaystyle\sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} - G'(p(\nu_i^* - \delta, \nu_{-i}^*))} - (\nu_i^* - \delta) \geq 0, \quad \forall \delta \text{ such that } 0 < \delta < \nu_i^*. \tag{B.14}$$

Again employing (B.10) yields the following transformation of the left-hand side of (B.14):

$$\begin{aligned}
&\frac{1}{\displaystyle\sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} - G'(p(\nu_i^* - \delta, \nu_{-i}^*))} - (\nu_i^* - \delta) \\
&\quad = \frac{1}{\frac{1}{\nu_i^*} + G'(p^*) - G'(p(\nu_i^* - \delta, \nu_{-i}^*))} - (\nu_i^* - \delta) \\
&\quad = \frac{\nu_i^*}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* - \delta, \nu_{-i}^*))} - (\nu_i^* - \delta) \\
&\quad = \frac{(\nu_i^*)^2 \left[G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)\right] + \delta - \nu_i^* \delta \left[G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)\right]}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* - \delta, \nu_{-i}^*))} \\
&\quad = \frac{\nu_i^* \left[G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)\right] (\nu_i^* - \delta) + \delta}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* - \delta, \nu_{-i}^*))}.
\end{aligned} \tag{B.15}$$

Similar to the proof of case (a), the denominator of the fraction (B.15) is strictly positive, hence, the fraction's sign coincides with that of its numerator. Again, since the derivative $G'(p)$ is non-increasing by hypothesis, and $\frac{\partial p}{\partial \nu_i} > 0$ by (3.10), it is evident that $\left[G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)\right] \geq 0$, hence, the numerator of (B.15) is strictly positive for any $0 < \delta < \nu_i^*$:

$$\nu_i^* \left[G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)\right] (\nu_i^* - \delta) + \delta > 0, \quad \forall \delta \text{ that } 0 < \delta < \nu_i^*, \tag{B.16}$$

which deduces the desired inequality:

$$\frac{d\tilde{\pi}_i}{d\nu_i}(\nu_i, \nu_{-i}^*) > 0, \quad \forall \nu_i < \nu_i^*. \tag{B.17}$$

Therefore, the proof of (b) is also completed.

Now we can conclude that the Nash equilibrium condition has been established:

$$\pi_i(\nu^*) = \max_{\nu_i > 0} \pi_i(\nu_i, \nu_{-i}^*), \text{ for any } i \in \{1, \ldots, n\}, \tag{B.18}$$

which finishes the proof of Theorem 3.6 ■

Theorem 3.8 *Suppose that apart from assumptions* ***A****4,* ***A****6 and* ***A****7, the regular demand function's derivative is Lipschitz continuous. In more detail, for $n \geq 3$ assume that for any $p_1 > 0$ and $p_2 > 0$ the following inequality holds:*

$$|G'(p_1) - G'(p_2)| \leq \frac{1}{2s^2 G(p_0)} |p_1 - p_2|, \tag{B.19}$$

where $s = \max\{a_1, \ldots, a_n\}$, and the price p_0 is the one defined in the assumption ***A****6. Next, if $n = 2$ (duopoly), we again suppose that there exists $\varepsilon > 0$ such that $G'(p) \leq -\varepsilon$ for all $p > 0$, and the Lipschitz continuity of the demand function is described in the form:*

$$|G'(p_1) - G'(p_2)| \leq \frac{2}{\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3\max\{a_1, a_2\}\right)^2 G(p_0)} |p_1 - p_2|, \ \forall p_1, p_2 > 0. \tag{B.20}$$

Then, the Consistency Criterion for the original oligopoly is a necessary and sufficient condition for the collection of influence conjectures $\nu = (\nu_1, \ldots, \nu_n)$ to be the Nash equilibrium in the meta-game.

Proof Again, the necessity is just a particular case of Theorem 3.5, then, we proceed to show the sufficiency.

Just like in the proof of Theorem 3.6, we need to establish that the i-th producer's payoff function

$$\tilde{\pi}_i(\nu_i) \equiv \pi(\nu_i, \nu_{-i}^*), \quad i = 1, \ldots, n, \tag{B.7}$$

has a maximum point at $\nu_i = \nu_i^*$ for a fixed value of i, for which, again, it will suffice to show that:

(a)

$$\nu_i^* \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* + \delta) - \delta < 0, \quad \forall 0 < \delta < s, \quad \text{(B.21)}$$

(b)

$$\nu_i^* \left[G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* - \delta) + \delta > 0, \quad \forall 0 < \delta < \nu_i^*, \quad \text{(B.22)}$$

where $s = \max\{a_1, \ldots, a_n\} > 0$. From the proof of Theorem 3.2 [1] we have:

$$0 \le \nu_i^* \le \frac{s}{n-2}, \quad i = 1, \ldots, n. \quad \text{(B.23)}$$

Now, from assumption (B.19) and the fact that $\frac{\partial p}{\partial \nu_i} > 0$, it follows that

$$\begin{aligned}
&\nu_i^* \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* + \delta) - \delta \\
&\le \nu_i^* \left|G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right|(\nu_i^* + \delta) - \delta \\
&\le \nu_i^*(\nu_i^* + \delta)\frac{1}{2s^2 G(p_0)}|p(\nu_i^* + \delta, \nu_{-i}^*) - p^*| - \delta \\
&\le \nu_i^*(\nu_i^* + \delta)\frac{1}{2s^2 G(p_0)}(p(\nu_i^* + \delta, \nu_{-i}^*) - p^*) - \delta.
\end{aligned} \quad \text{(B.24)}$$

By the mean value theorem here exists a value $\hat{\nu}_i$ such that $\nu_i < \hat{\nu}_i < \nu_i + \delta$ and

$$p(\nu_i^* + \delta, \nu_{-i}^*) - p^* = \delta \frac{\partial p}{\partial \nu_i}(\hat{\nu}_i, \nu_{-i}^*). \quad \text{(B.25)}$$

Using (3.10) we get

$$\begin{aligned}
\frac{\partial p}{\partial \nu_i}(\hat{\nu}_i, \nu_{-i}^*) &= \frac{\dfrac{q_i(p(\hat{\nu}_i, \nu_{-i}^*), (\hat{\nu}_i, \nu_{-i}^*))}{\hat{\nu}_i + a_i}}{\displaystyle\sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} + \frac{1}{\hat{\nu}_i + a_i} - G'(p(\hat{\nu}_i, \nu_{-i}^*))} \le \frac{\dfrac{q_i(p(\hat{\nu}_i, \nu_{-i}^*), (\hat{\nu}_i, \nu_{-i}^*))}{\hat{\nu}_i + a_i}}{\displaystyle\sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{\nu_k^* + a_k} + \frac{1}{\hat{\nu}_i + a_i}} \\
&\le \frac{\dfrac{q_i(p(\hat{\nu}_i, \nu_{-i}^*), (\hat{\nu}_i, \nu_{-i}^*))}{\hat{\nu}_i + a_i}}{\dfrac{1}{\hat{\nu}_i + a_i}} = q_i(p(\hat{\nu}_i, \nu_{-i}^*), (\hat{\nu}_i, \nu_{-i}^*)) \le G(p_0).
\end{aligned} \quad \text{(B.26)}$$

Applying (B.25) and (B.26) to (B.24) we find:

$$\begin{aligned}\nu_i^* \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* + \delta) - \delta \leq& \nu_i^*(\nu_i^* + \delta)\frac{1}{2s^2}\delta - \delta \\ =& \left[\nu_i^*(\nu_i^* + \delta)\frac{1}{2s^2} - 1\right]\delta,\end{aligned} \tag{B.27}$$

moreover, since $0 < \nu_i^* \leq s$ and $0 < \delta < s$, it follows that

$$\nu_i^* \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* + \delta) - \delta \leq \left[\nu_i^*(\nu_i^* + \delta)\frac{1}{2s^2} - 1\right]\delta < 0, \tag{B.28}$$

which proves (a).

Analogous to the previous case, we can find that

$$\nu_i^* \left[G'(p^*) - G'(p(\nu_i^* - \delta, \nu_{-i}^*))\right](\nu_i^* - \delta) - \delta \leq \left[\nu_i^*(\nu_i^* - \delta)\frac{1}{2s^2} - 1\right]\delta, \tag{B.29}$$

and, since $0 < \delta < \nu_i^* \leq s$, we have

$$\nu_i^* \left[G'(p^*) - G'(p(\nu_i^* - \delta, \nu_{-i}^*))\right](\nu_i^* - \delta) - \delta \leq \left[\nu_i^*(\nu_i^* - \delta)\frac{1}{2s^2} - 1\right]\delta < 0, \tag{B.30}$$

then

$$\nu_i^* \left[G'(p(\nu_i^* - \delta, \nu_{-i}^*) - G'(p^*))\right](\nu_i^* - \delta) + \delta > 0. \tag{B.31}$$

Therefore, the vector ν^* is Nash equilibrium for $n \geq 3$.

Finally, let $n = 2$. We can repeat the steps for the case $n \geq 3$ to get the following inequality:

$$\begin{aligned}&\nu_i^* \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* + \delta) - \delta \\ &\leq \left[\nu_i^*(\nu_i^* + \delta)\frac{2}{\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3\max\{a_1, a_2\}\right)^2} - 1\right]\delta.\end{aligned} \tag{B.32}$$

From

$$\begin{aligned}\nu_i \leq& \frac{1}{2}\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + \max\{a_1, a_2\}\right) + \max\{a_1, a_2\} \\ =& \frac{1}{2}\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3\max\{a_1, a_2\}\right), \quad i = 1, 2.\end{aligned} \tag{B.33}$$

we have

$$0 < \nu_i^* \leq \frac{1}{2}\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3\max\{a_1, a_2\}\right) \tag{B.34}$$

and

$$0 < \delta < \max\{a_1, a_2\} < \frac{1}{2}\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\}\right), \tag{B.35}$$

thus

$$\begin{aligned} & \nu_i^* \left[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* + \delta) - \delta \\ & \le \left[\nu_i^*(\nu_i^* + \delta) \frac{2}{\left(\dfrac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\}\right)^2} - 1\right]\delta < 0. \end{aligned} \tag{B.36}$$

which finally proves (a).

Analogously, to prove (b), it is easy to show that

$$\begin{aligned} & \nu_i^* \left[G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)\right](\nu_i^* - \delta) + \delta \\ & \ge \left[1 - \nu_i^*(\nu_i^* - \delta) \frac{2}{\left(\dfrac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\}\right)^2}\right]\delta > 0. \end{aligned} \tag{B.37}$$

The theorem has been proved ■

Appendix C: Proofs of Results from Chapter 4

Theorem 4.1 *The quadratic programming problem (4.11)–(4.15) is convex and any of its solutions provides the Nash equilibrium for the non-cooperative game (4.6)–(4.10).*

Proof First, in order to prove that the quadratic programming problem (4.11)–(4.15) is convex, we only need to prove that the symmetric matrix associated with the quadratic objective function (4.12) is positive semidefinite, i.e., we need to prove that

$$\sum_{k\in K}\sum_{\ell\in K\setminus\{k\}}\sum_{a\in A}\frac{1}{2}d_a x_a^k x_a^\ell + \sum_{k\in K}\sum_{a\in A} d_a (x_a^k)^2 \geq 0, \ \forall x \in \mathbb{R}^{M\kappa}. \tag{C.1}$$

Indeed, let $x \in \mathbb{R}^{M\kappa}$, then, we have that

$$\begin{aligned}
&\sum_{k\in K}\sum_{\ell\in K\setminus\{k\}}\sum_{a\in A}\frac{1}{2}d_a x_a^k x_a^\ell + \sum_{k\in K}\sum_{a\in A} d_a (x_a^k)^2 \\
&= \sum_{k\in K}\sum_{a\in A}\frac{1}{2}d_a (x_a^k)^2 + \sum_{k\in K}\sum_{\ell\in K}\sum_{a\in A}\frac{1}{2}d_a x_a^k x_a^\ell \\
&= \sum_{k\in K}\sum_{a\in A}\frac{1}{2}d_a (x_a^k)^2 + \sum_{a\in A}\frac{1}{2}d_a\left(\sum_{k\in K}\sum_{\ell\in K} x_a^k x_a^\ell\right) \\
&= \sum_{k\in K}\sum_{a\in A}\frac{1}{2}d_a (x_a^k)^2 + \sum_{a\in A}\frac{1}{2}d_a\left(\sum_{k\in K} x_a^k\right)^2.
\end{aligned} \tag{C.2}$$

Since all of the congestion coefficients d_a are nonnegative, we can easily see that (C.2) is also nonnegative. Moreover, if all congestion coefficients d_a are strictly positive, then, (C.2) is also strictly positive and the quadratic programming problem (4.11)–(4.15) is strictly convex so it has a unique solution.

J. G. Flores Muñiz et al., *Public Interest and Private Enterprize: New Developments*, Lecture Notes in Networks and Systems 138,
https://doi.org/10.1007/978-3-030-58349-1

Now, we will prove that any solution of (4.11)–(4.15) provides the Nash equilibrium for (4.6)–(4.10). In order to do this, we rewrite both problems in their matrix forms.

Let $\{t_a \mid a \in A_1\}$ satisfy (4.4) and (4.5), then, we can consider the vector $z \in R^M$ whose a-th component is given by c_a if $a \in A_2$ and by $t_a + c_a$ if $a \in A_1$. Thus, the Nash equilibrium problem (4.6)–(4.10) is given as follows:

$$x^k \in \Psi_k(t, x^{-k}),\ \forall k \in K, \tag{C.3}$$

where

$$\Psi_k(t, x^{-k}) = \underset{x^k}{\text{Argmin}}\ f_k(x^k) = z^T x^k + \sum_{\ell \in K \setminus \{k\}} {x^k}^T H x^\ell + {x^k}^T H x^k, \tag{C.4}$$

$$\text{subject to} \qquad B x^k = b^k, \tag{C.5}$$

$$x^k \le q - \sum_{\ell \in K \setminus \{k\}} x^\ell, \tag{C.6}$$

$$x^k \ge 0. \tag{C.7}$$

Here, the matrix H is the diagonal matrix $M \times M$ matrix corresponding to the congestion factors, i.e., the a-th diagonal element of H is d_a. The matrix $B \in \mathbb{R}^{\eta \times M}$ and the vector $b^k \in \mathbb{R}^\eta$ corresponds to the equality constraints (4.8) (the matrix B depends solely upon the network so it is the same for any commodity k), and the vector $q \in \mathbb{R}^M$ has the capacity upper bounds q_a, $a \in A$, as its components. Using the above notation, the quadratic programming problem (4.11)–(4.15) is given by:

$$x \in \Psi(t), \tag{C.8}$$

where

$$\Psi(t) = \underset{x}{\text{Argmin}}\ f(x) = \sum_{k \in K} z^T x^k + \sum_{k \in K} \sum_{\ell \in K \setminus \{k\}} \frac{1}{2} {x^k}^T H x^\ell + \sum_{k \in K} {x^k}^T H x^k, \tag{C.9}$$

$$\text{subject to} \qquad B x^k = b^k,\ \forall k \in K, \tag{C.10}$$

$$\sum_{\ell \in K} x^\ell \le q, \tag{C.11}$$

$$x \ge 0. \tag{C.12}$$

The matrix D is a $M\kappa \times M\kappa$ symmetric block matrix whose $\kappa \times \kappa$ block components are the matrix H in its diagonal blocks and the matrix $\frac{1}{2}H$ in its non-diagonal blocks. Since all the values d_a are nonnegative, both, the matrix H and the matrix D are symmetric and positive semi-definite, and if all the values d_a are strictly positive, the matrices H and D will be positive definite.

The programs appearing in (C.3)–(C.7) and program (C.8)–(C.12) are differentiable and convex (with linear constraints) quadratic programming problems, so these problems can be equivalently transformed into a nonlinear system of equations and inequalities using the Lagrange multipliers and Karush-Kuhn-Tucker conditions. Therefore, in order to show that a solution of (C.8)–(C.12) generates Nash-equilibrium for (C.3)–(C.7), it will suffice to demonstrate that a solution for the Lagrange multipliers and Karush-Kuhn-Tucker conditions of (C.3)–(C.7) lead to a solution for the Lagrange multipliers and Karush-Kuhn-Tucker conditions of (C.3)–(C.7). The Lagrange multipliers and Karush-Kuhn-Tucker condition for problem (C.3)–(C.7) are as follows:

$$\frac{df_k}{dx^k} + \mu^k + B^T\lambda^k = z + \sum_{\ell \in K\setminus\{k\}} Hx^\ell + 2Hx^k + \mu^k + B^T\lambda^k \geq 0, \quad \text{(C.13)}$$

$$x^k\left(z + \sum_{\ell \in K\setminus\{k\}} Hx^\ell + 2Hx^k + \mu^k + B^T\lambda^k\right) = 0, \quad \text{(C.14)}$$

$$Bx^k = b^k, \quad \text{(C.15)}$$

$$x^k \leq q - \sum_{\ell \in K\setminus\{k\}} x^\ell, \quad \text{(C.16)}$$

$$\mu^k\left(\sum_{\ell \in K} x^\ell - q\right) = 0, \quad \text{(C.17)}$$

$$x^k, \mu^k \geq 0, \quad \text{(C.18)}$$

where $\mu^k \in \mathbb{R}^M$ and $\lambda^k \in \mathbb{R}^\eta$, for all $k \in K$. And the Lagrange multipliers and Karush-Kuhn-Tucker conditions for problem (C.8)–(C.12) are:

$$\frac{\partial f}{\partial x^k} + \mu + B^T\lambda^k = z + \sum_{\ell \in K\setminus\{k\}} Hx^\ell + 2Hx^k + \mu + B^T\lambda^k \geq 0, \ \forall k \in K, \quad \text{(C.19)}$$

$$x^k\left(z + \sum_{\ell \in K\setminus\{k\}} Hx^\ell + 2Hx^k + \mu + B^T\lambda^k\right) = 0, \ \forall k \in K, \quad \text{(C.20)}$$

$$Bx^k = b^k, \ \forall k \in K, \quad \text{(C.21)}$$

$$\sum_{\ell \in K} x^\ell \leq q, \quad \text{(C.22)}$$

$$\mu\left(\sum_{\ell \in K} x^\ell - q\right) = 0, \quad \text{(C.23)}$$

$$x, \mu \geq 0, \quad \text{(C.24)}$$

where $\mu \in \mathbb{R}^M$ and $\lambda^k \in \mathbb{R}^\eta, k \in K$. Now let x^k, $\mu \in \mathbb{R}^M$ and $\lambda^k \in \mathbb{R}^\eta, k \in K$, satisfy (C.19)–(C.24). Then, for a fixed $k \in K$, we have that:

$$z + \sum_{\ell \in K \setminus \{k\}} Hx^{\ell} + 2Hx^k + \mu + B^T \lambda^k \geq 0, \tag{C.25}$$

$$x^k \left(z + \sum_{\ell \in K \setminus \{k\}} Hx^{\ell} + 2Hx^k + \mu + B^T \lambda^k \right) = 0, \tag{C.26}$$

$$Bx^k = b^k, \tag{C.27}$$

$$x^k \leq q - \sum_{\ell \in K \setminus \{k\}} x^{\ell}, \tag{C.28}$$

$$\mu \left(\sum_{\ell \in K} x^{\ell} - q \right) = 0, \tag{C.29}$$

$$x^k, \mu \geq 0. \tag{C.30}$$

Therefore, the vectors $x^k, \mu \in \mathbb{R}^M$ and $\lambda^k \in \mathbb{R}^\eta$ satisfy (C.13)–(C.18), for all $k \in K$, which proves the theorem.

Finally, if we remove the capacity constraints, we can easily see that the KKT conditions (C.13)–(C.18) imply the KKT conditions (C.19)–(C.24) taking $\mu := \max\{\mu^k | k \in K\}$, which proves Corollary 4.1 ■

Reference

1. Bulavsky, V.A.: Structure of demand and equilibrium in a model of oligopoly. Econ. Math. Methods (Ekonomika i Matematicheskie Metody) **33**, 112–134 (1997). In Russian