Appendix A
Background Material

A.1 Measure and Integration

As noted in the text, a *measure* on a set X consists of a σ-algebra $\mathcal{M}$ of measurable subsets of X and a countably additive function $\mu : \mathcal{M} \to [0, \infty]$. To recall the basic definitions, a σ-algebra is a collection of subsets that is closed under countable unions and complements (and hence countable intersections as well). The countable additivity property means that for any countable disjoint sequence of sets $A_1, A_2, \ldots,$

$$\mu\big(\cup_{j=1}^{\infty} A_j\big) = \sum_{j=1}^{\infty} \mu(A_j). \tag{A.1}$$

In the text we considered only σ-finite measure spaces, for which X can be decomposed into a countable union of sets of finite measure.

Two important special cases can be defined with $\mathcal{M}$ equal to the collection of all subsets of X:

Example A.1. The *counting measure* ν is defined by

$$\nu(A) := \begin{cases} \#A, & A \text{ is finite}, \\ \infty, & A \text{ is infinite}. \end{cases}$$

Counting measure is the default choice for a discrete set, such as $\mathbb{N}$ or $\mathbb{Z}$. (Clearly, ν is σ-finite if and only if X is countable.)

D. Borthwick, *Spectral Theory*, Graduate Texts in Mathematics 284,
https://doi.org/10.1007/978-3-030-38002-1

Example A.2. For $p \in X$, the *point measure* δ_p is defined by

$$\delta_p(A) := \begin{cases} 1, & p \in A, \\ 0, & p \notin A. \end{cases}$$

For a topological space X, the collection $\mathcal{B}$ of *Borel sets* is defined as the σ-algebra generated by the open subsets of X. A *Borel measure* is simply a measure defined on $\mathcal{B}$.

In this section we will review some of the basics of measure and integration theory. This material is standard and covered in many texts, so we will omit most of the proofs. For additional background, see, e.g., Folland [31], Royden [76], Rudin [78], or Stein and Shakarchi [87].

A.1.1 Lebesgue Measure

To define Lebesgue measure on $\mathbb{R}^n$, we start from the standard definition of the volume of a closed rectangle,

$$\mathrm{vol}\big(I_1 \times I_2 \times \cdots \times I_n\big) := \prod_{j=1}^{n} \ell(I_j),$$

where $\ell[a, b] := b - a$. Let $\mathcal{R}$ denote the collection of closed rectangles in $\mathbb{R}^n$. The *outer measure* of a set $A \subset \mathbb{R}^n$ is then defined by taking the infimum over coverings by countable unions of rectangles,

$$m^*(A) := \inf\left\{\sum_{j=1}^{\infty} \mathrm{vol}(R_j) \, : \, A \subset \bigcup_{j=1}^{\infty} R_j\right\}.$$

In general, an outer measure is defined as a countably subadditive set function, meaning that

$$m^*(\cup A_j) \le \sum_{j=1}^{\infty} m^*(A_j),$$

for a countable sequence of sets $\{A_j\}$.

To obtain a measure, we need to restrict m^* to an appropriate class of measurable sets. Constantin Carathéodory established a criterion for this: a set E is Lebesgue measurable if, for each $A \subset \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c). \tag{A.2}$$

This condition defines the collection $\mathcal{M}$ of measurable sets, which forms a σ-algebra. Lebesgue measure is defined as the restriction,

$$m := m^*|_{\mathcal{M}}.$$

The criterion (A.2) can be used to produce a measure μ from any outer measure μ^*.

The Lebesgue class $\mathcal{M}$ includes all open sets. This is not immediately clear from the condition (A.2), but in fact one can show that $E \in \mathcal{M}$ if and only if for each $\varepsilon > 0$ there exists an open set $U \supset E$ such that

$$m^*(U \backslash E) \leq \varepsilon. \tag{A.3}$$

A similar characterization in terms of closed sets contained in E is also possible. Since $\mathcal{M}$ is a σ-algebra, the fact that it contains all open sets implies that $\mathcal{B} \subset \mathcal{M}$.

Since Lebesgue measure generalizes the classical notion of volume, it is common to denote $m(A)$ by $\mathrm{vol}(A)$, especially in geometric contexts.

A.1.2 Integration

On a measure space $(X, \mathcal{M}, \mu)$, a *simple* function φ is a finite linear combination of characteristic functions,

$$\varphi = \sum_{j=1}^{m} c_j \chi_{E_j},$$

where $c_j \in \mathbb{C}$, $E_j \in \mathcal{M}$, and $\mu(E_j) < \infty$ for each j. The integral of a simple function is defined by the obvious sum,

$$\int_X \varphi \, d\mu := \sum_{j=1}^{m} c_j \mu(E_j).$$

A function $f : X \to \mathbb{R}^n$ is *measurable* if the preimage of each Borel set is contained in $\mathcal{M}$. The measurability of f implies that there exists a sequence of simple functions $\{\varphi_j\}$ such that $\varphi_j \to f$ pointwise and $|\varphi_j| \to |f|$ monotonically. With such an approximation we can define

$$\int_X |f| \, d\mu := \lim_{j\to\infty} \int_X |\varphi_j| \, d\mu.$$

We say that f is *integrable* if the integral of $|f|$ is finite, in which case we can define

$$\int_X f \, d\mu := \lim_{j\to\infty} \int_X \varphi_j \, d\mu.$$

The integral is well defined independently of the approximating sequence, and linear in the sense that for two integrable functions,

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu.$$

It is also monotonic, meaning that

$$f \le g \Longrightarrow \int_X f\,d\mu \le \int_X g\,d\mu.$$

In the case of Lebesgue measure on $\mathbb{R}^n$, this integral construction generalizes the ordinary Riemann integral. In the main text we write the Lebesgue integral on $\mathbb{R}^n$ in the traditional notation, replacing dm by $d^n x$.

The integral construction sketched here yields a trio of very useful convergence theorems.

Theorem A.3 (Monotone Convergence Theorem). *Suppose that* $\{f_j\}$ *is a sequence of measurable functions with*

$$0 \le f_1 \le f_2 \le \dots .$$

Then,

$$\lim_{j\to\infty} \int_X f_j\,d\mu = \int_X \left(\lim_{j\to\infty} f_j\right) d\mu$$

(where both sides could be infinite).

Theorem A.4 (Fatou's Lemma). *If* $\{f_j\}$ *is a sequence of measurable functions with* $f_j \ge 0$*, then*

$$\int_X \left(\liminf_{j\to\infty} f_j\right) d\mu \le \liminf_{j\to\infty} \int_X f_j\,d\mu$$

(where both sides could be infinite).

Theorem A.5 (Dominated Convergence Theorem). *Suppose that* $\{f_j\}$ *is a sequence of measurable functions such that* $f_j \to f$ *pointwise. If there exists an integrable function* g *such that* $|f_j| \le g$ *for all* j*, then*

$$\lim_{j\to\infty} \int_X f_j\,d\mu = \int_X f\,d\mu.$$

A.1.3 Product Measure

Given two measure spaces $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$, we can construct a measure on $X_1 \times X_2$ using a generalization of the approach outlined in Appendix A.1.1. We start by defining the measure of a "rectangular set" in the obvious way. For $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$,

$$\pi_0(A \times B) := \mu_1(A)\mu_2(B). \tag{A.4}$$

We then use coverings by rectangles to construct an outer measure, for $E \subset X_1 \times X_2$,

$$\pi^*(E) := \inf\left\{\sum_{j=1}^{\infty} \pi_0(A_j \times B_j) : A \subset \bigcup_{j=1}^{\infty}(A_j \times B_j)\right\}.$$

Carathéodory's condition defines a class $\mathcal{M}$ of subsets of $X_1 \times X_2$ measurable with respect to π^*. The restriction of π^* to $\mathcal{M}$ then defines the product measure π, which is commonly written as $\pi = \mu_1 \times \mu_2$.

In the Lebesgue case, we could apply the product construction to obtain the measure on $\mathbb{R}^n$ as a product of measures on $\mathbb{R}$.

In principle, integrating with respect to a product measure on $X_1 \times X_2$ could give different results from an iterated integral defined by integrating separately over the original X_1 and X_2. However, we can avoid this issue under fairly general conditions.

Theorem A.6 (Fubini). *Suppose that $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ are σ-finite measure spaces, with the product space $(X_1 \times X_2, \mathcal{M}, \mu_1 \times \mu_2)$. If f is an integrable function on $X_1 \times X_2$, then the iterated integrals make sense in either order and*

$$\begin{aligned}\int_{X_1 \times X_2} f \, d(\mu_1 \times \mu_2) &= \int_{X_1}\left(\int_{X_2} f(x, y)\, d\mu_2(y)\right) d\mu_1(x) \\ &= \int_{X_2}\left(\int_{X_1} f(x, y)\, d\mu_1(x)\right) d\mu_2(y).\end{aligned}$$

The same conclusion holds without the integrability assumption if $f \geq 0$.

A.1.4 Differentiation

A function $f : [a, b] \to \mathbb{C}$ is *absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite collection of disjoint subintervals (a_j, b_j) satisfying

$$\sum_{j=1}^{k}(b_j - a_j) < \delta,$$

we have

$$\sum_{j=1}^{k} \left| f(b_j) - f(a_j) \right| < \varepsilon.$$

One way to obtain an absolutely continuous function is by integration. If $h \in L^1[a, b]$, then the function

$$f(x) := \int_a^x h(t)\, dt \tag{A.5}$$

is absolutely continuous. Indeed, by a general measure theory argument, if $g \in L^1(X, d\mu)$, then for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) < \delta \quad \Longrightarrow \quad \int_E |g|\, d\mu < \varepsilon. \tag{A.6}$$

Applying this in the case of Lebesgue measure on $\mathbb{R}$, with E a finite union of intervals, shows that (A.5) is absolutely continuous.

It turns that all absolutely continuous functions can be expressed as definite integrals.

Theorem A.7 (Lebesgue Differentiation Theorem). *If f is absolutely continuous on $[a, b]$, then f' exists almost everywhere, $f' \in L^1[a, b]$, and*

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

Conversely, for $g \in L^1(a, b)$ the function defined by

$$f(x) := \int_a^x g\, dt$$

is absolutely continuous, with $f' = g$ a.e.

The property (A.6) suggests a related definition for measures. On a measure space $(X, \mathcal{M}, \mu)$, a measure ν on $\mathcal{M}$ is absolutely continuous with respect to μ if for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) < \delta \quad \Longrightarrow \quad \nu(E) < \varepsilon.$$

By standard measure theory arguments, absolute continuity holds if and only if every set of measure zero with respect to μ also has measure zero with respect to ν. (This latter condition is frequently taken as the definition.)

A.1.5 Decomposition of Measures

In this section we will review the Lebesgue decomposition theorem on $\mathbb{R}$, which was applied in Section 5.4.2 to establish a classification of the spectrum.

A Borel measure on $\mathbb{R}$ is *regular* if $\mu(K) < \infty$ when K is compact. Let m denote the Lebesgue measure on $\mathbb{R}$. For the decomposition theorem, we distinguish the following types of Borel measure:

(i) A *pure point* measure is a linear combination of point measures.
(ii) A measure μ is *absolutely continuous* if $\mu(E) = 0$ whenever $m(E) = 0$.
(iii) A measure μ is *singular continuous* if μ is supported on a set of Lebesgue measure zero, but $\mu\{x\} = 0$ for each $x \in \mathbb{R}$.

The Cantor measure (a probability measure supported on the Cantor set) is the classic example of a singular continuous measure.

Theorem A.8 (Lebesgue Decomposition Theorem). *A regular Borel measure μ on $\mathbb{R}$ admits a unique decomposition,*

$$\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}},$$

where μ_{pp} is pure point, μ_{ac} is absolutely continuous, and μ_{sc} is singular continuous.

Proof Consider the subset

$$Z := \{x \in \mathbb{R} : \ \mu\{x\} > 0\}.$$

Since $\mu[-n, n] < \infty$ for all n by the regularity assumption, $Z \cap [-n, n]$ is finite. Hence Z is at most countable. Let $Z = \{z_1, \ldots\}$, and define

$$\mu_{\mathrm{pp}} := \sum_j \mu\{z_j\}\delta_{z_j}.$$

If we then define $\mu_{\mathrm{c}} := \mu - \mu_{\mathrm{pp}}$, then μ_{c} is a continuous Borel measure, meaning that single point has measure zero. Let α be the associated cumulative distribution function, centered at 0,

$$\alpha(x) := \begin{cases} -\mu_{\mathrm{c}}(x, 0], & x < 0, \\ \mu_{\mathrm{c}}[0, x], & x \geq 0. \end{cases}$$

Since μ_{c} is a continuous measure, α is a continuous increasing function. It follows that α' exists almost everywhere (with respect to Lebesgue measure) and is locally integrable. We can thus define

$$\mu_{\mathrm{ac}}(E) := \int_E \alpha' \, dm,$$

and then set $\mu_{\mathrm{sc}} = \mu_{\mathrm{c}} - \mu_{\mathrm{ac}}$. □

For additional details on this construction, we refer the reader to Folland [31, §3.5].

A.1.6 Riesz Representation

Let X be a compact topological space. In this section, we will develop the version of the Riesz representation theorem stated as Theorem 5.3, which relates certain continuous linear functionals on $C(X)$ with Borel measures on X. Here $C(X)$ denotes the space of continuous functions $X \to \mathbb{C}$.

A linear functional $\beta : C(X) \to \mathbb{C}$ is *positive* if

$$\beta(f) \geq 0 \text{ for } f \geq 0.$$

Applying the positivity condition to $(\sup |f| - |f|) \geq 0$ shows that

$$\beta(|f|) \leq \beta(1) \sup |f|.$$

It follows that a positive functional is bounded with respect to the sup norm, because

$$\begin{aligned} |\beta(f)| &= \sqrt{\beta(\mathrm{Re}\, f)^2 + \beta(\mathrm{Im}\, f)^2} \\ &\leq \sqrt{2}\beta(1) \sup |f|. \end{aligned}$$

The existence of a partition of unity will play an important role in the proof. This is easy to establish for a compact metric space.

Lemma A.9 (Partition of Unity). *Let X be a compact metric space, and $\{U_j\}_{j=1}^m$ a finite open cover. There exists a set of functions $\psi_j \in C(X)$ such that $0 \leq \psi_j \leq 1$, $\operatorname{supp} \psi_j \subset U_j$ for each $j = 1, \ldots, n$, and*

$$\sum_{j=1}^m \psi_j = 1.$$

Proof For each $x \in X$, there exists an open metric ball B_x for which $\overline{B_x} \subset U_j$ for some j. Since X is compact, a finite number of these balls, say B_{x_i} for $i = 1, \ldots m$, is sufficient to cover X. Set $g_i(x) := \mathrm{dist}(x, B_{x_i}^c)$ so that g_i is continuous and $g_i(x) > 0$ if and only if $x \in B_{x_i}$.

For each j let h_j be the sum of the g_i for which $\overline{B_{x_i}} \subset U_j$. Since the B_{x_i} form a cover, for every $x \in X$ we have $h_j(x) > 0$ for at least one j. The desired functions ψ_j are then obtained by normalizing

$$\psi_j(x) := \frac{h_j(x)}{\sum_k h_k(x)}.$$

□

For $f \in C(X)$ and $U \subset X$ an open set, we use the notation

$$f \prec U$$

to mean that $0 \leq f \leq 1$ and $\operatorname{supp} f \subset U$.

Theorem A.10 (Riesz Representation Theorem). *Let X be a compact metric space. Given a positive linear functional $\beta : C(X) \to \mathbb{C}$, there exists a unique Borel measure μ on X such that*

$$\beta(f) = \int_X f \, d\mu \tag{A.7}$$

for $f \in C(X)$.

Proof For an open set $U \subset X$, let

$$\mu(U) := \sup\big\{\beta(f) : f \in C(X), f \prec U\big\}.$$

For an arbitrary subset $A \subset X$, we then set

$$\mu^*(A) := \inf\big\{\mu(U) : U \text{ open and } A \subset U\big\}. \tag{A.8}$$

To see that μ^* is an outer measure, we need to establish countable subadditivity. Suppose that $\{A_j\}$ is a countable sequence of subsets of X, and set $A := \cup A_j$. Our goal is to prove that

$$\mu^*(A) \leq \sum \mu^*(A_j). \tag{A.9}$$

Given $\varepsilon > 0$, for each j we can find an open set $U_j \supset A_j$ such that

$$\mu(U_j) \leq \mu^*(A_j) + 2^{-j}\varepsilon.$$

Adding these together thus gives

$$\sum_{j=1}^{\infty} \mu(U_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon. \tag{A.10}$$

Now let $U := \cup_j U_j$. By the definition (A.8),

$$\mu^*(A) \le \mu^*(U). \tag{A.11}$$

Consider $f \in C(X)$ with $f \prec U$. Since $\mathrm{supp}(f)$ is compact, we have $\mathrm{supp}(f) \subset \cup_{j=1}^n U_j$ for some finite n. If we set $U_0 := \mathrm{supp}(f)^c$, then $\{U_j\}_{j=0}^\infty$ is a cover for X. By Lemma A.9, there exists a partition of unity $\{\psi_j\}_{j=0}^n \subset C(X)$, with $0 \le \psi_j \le 1$, $\mathrm{supp}\,\psi_j \in U_j$ and

$$\sum_{j=0}^n \psi_j = 1.$$

This construction yields $f = \sum_{j=1}^n f\psi_j$ and $\psi_j f \prec U_j$ for $j = 1, \dots, n$. Hence,

$$\begin{aligned}\beta(f) &= \sum_{j=1}^n \beta(f\psi_j)\\ &\le \sum_{j=1}^n \mu(U_j).\end{aligned}$$

By (A.10), this gives

$$\beta(f) \le \sum_{j=1}^\infty \mu^*(A_j) + \varepsilon.$$

Since this holds for all $f \prec U$, it implies that

$$\mu^*(U) \le \sum_{j=1}^\infty \mu^*(A_j) + \varepsilon.$$

Applying (A.11) and taking $\varepsilon \to 0$ thus prove (A.9).

With μ^* established as an outer measure, we can now apply the standard Carathéodory construction to obtain a measure μ on the σ-algebra $\mathcal{M}$ defined by the condition that $E \subset X$ is measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E) \tag{A.12}$$

for all $A \subset X$.

To show that μ is a Borel measure, we must check that each open set $U \subset X$ satisfies (A.12). Given $A \subset X$ and $\varepsilon > 0$, choose an open set $V \supset A$ such that

$$\mu^*(V) \le \mu^*(A) + \varepsilon. \tag{A.13}$$

Since $V \cap U$ is open, there exists $f \in C(X)$ with $f \prec V \cap U$ and

$$\beta(f) \geq \mu^*(V \cap U) - \varepsilon.$$

Similarly, there exists $g \in C(X)$ with $g \prec V - \text{supp}(f)$ and

$$\beta(g) \geq \mu^*(V - \text{supp } f) - \varepsilon.$$

Then $f + g \prec V$, and so

$$\begin{aligned}\mu^*(V) &\geq \beta(f) + \beta(g)\\ &\geq \mu^*(V \cap U) + \mu^*(V - \text{supp } f) - 2\varepsilon\\ &\geq \mu^*(A \cap U) + \mu^*(A - U) - 2\varepsilon.\end{aligned}$$

Using (A.13) and then taking $\varepsilon \to 0$ give

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A - U).$$

Since the opposite inequality is automatic by subadditivity, this proves that U is measurable. Therefore, μ is a Borel measure on X.

The final step is to prove the integral formula (A.7), which will also establish the uniqueness of μ. Consider $f \in C(X)$ with $f \geq 0$. We first claim that

$$\mu\{f \geq 1\} \leq \beta(f) \leq \mu(\text{supp } f). \tag{A.14}$$

The upper bound follows from the definition (A.8), because $f \prec U$ for any $U \supset \text{supp}(f)$. For the lower bound, let $U_\varepsilon := \{f > 1 - \varepsilon\}$ for $\varepsilon > 0$. For $g \prec U_\varepsilon$ we have $f \geq (1 - \varepsilon)g$, which by the positivity of β implies that

$$\beta(g) \leq (1 - \varepsilon)^{-1}\beta(f).$$

Taking the supremum over all $g \prec U_\varepsilon$ gives

$$\mu(U_\varepsilon) \leq (1 - \varepsilon)^{-1}\beta(f).$$

Therefore, by (A.8),

$$\mu\{f \geq 1\} \leq (1 - \varepsilon)^{-1}\beta(f).$$

Taking $\varepsilon \to 0$ yields the lower bound in (A.14).

To refine the estimate (A.14), we fix some $n \in \mathbb{N}$ and decompose f into layers of height $1/n$ by setting

$$f_j(x) := \begin{cases} 0, & f(x) < j/n, \\ f(x) - j/n, & j/n \le f(x) \le (j+1)/n, \\ 1/n, & f(x) > (j+1)/n. \end{cases}$$

Note that $0 \le f_j \le 1/n$, and

$$f = \sum_{j=1}^{m} f_j \tag{A.15}$$

for m large enough that $\sup f \le m/n$.

If we set $K_j := \operatorname{supp}(f_j)$, then the fact that $\chi_{K_{j+1}} \le nf_j \le \chi_{K_j}$ implies

$$\mu(K_{j+1}) \le n \int_X f_j \, d\mu \le \mu(K_j),$$

by the monotonicity of the integral. By (A.15), we can sum over j to obtain

$$\frac{1}{n} \sum_{j=1}^{m} \mu(K_j) \le \int_X f \, d\mu \le \frac{1}{n} \sum_{j=0}^{m} \mu(K_j). \tag{A.16}$$

(Note that $K_m = \emptyset$.)

On the other hand, we can apply (A.14) to the function nf_j to conclude that

$$\mu(K_{j+1}) \le n\beta(f_j) \le \mu(K_j).$$

Summing over j gives

$$\frac{1}{n} \sum_{j=1}^{m} \mu(K_j) \le \beta(f) \le \frac{1}{n} \sum_{j=0}^{m} \mu(K_j).$$

In conjunction with (A.16), this shows that

$$\left| \int_X f \, d\mu - \beta(f) \right| \le \frac{\mu(K_0)}{n}.$$

Taking $n \to \infty$ then completes the proof of (A.7) for $f \ge 0$. The general case follows by linearity. □

For the application of the Riesz representation theorem to spectral measures in Section 5.1.2, we need to know that $C(X)$ is dense in $L^2(X, d\mu)$. This holds for general Borel measures under certain regularity conditions. For measures obtained via the Riesz theorem, we can give a simple direct proof.

Lemma A.11. *Let X be a compact metric space and μ a Borel measure constructed from a positive functional on $C(X)$ as in Theorem A.10. Then $C(X)$ is dense in $L^p(X, d\mu)$ for $1 \le p < \infty$.*

Proof By the definition of the integral on a measure space, simple functions are dense in L^p. It therefore suffices to show that the characteristic function χ_E for a Borel subset $E \subset X$ can be approximated in the L^p sense by continuous functions. Given $\varepsilon > 0$, the definition of μ^* gives an open set $U \supset E$ such that $\mu(U - E) < \varepsilon$. Then, by the definition of μ in terms of the functional β, there exists $f \in C(X)$ such that $f \prec U$ and

$$\mu(U) - \int_X f \, d\mu < \varepsilon.$$

We can then estimate

$$\begin{aligned} \|\chi_E - f\|_p &\le \|\chi_E - \chi_U\|_p + \|\chi_U - f\|_p \\ &\le 2\varepsilon^{\frac{1}{p}}. \end{aligned}$$

□

The Riesz representation theorem can be extended to more general topological spaces. The limiting factor is essentially the existence of the partition of unity. For a locally compact Hausdorf space, Urysohn's lemma implies the existence of locally finite partitions of unity, and the Riesz theorem can be extended to this case by a very similar argument. For further details, see, e.g., Rudin [78, Thm. 6.19] or Folland [31, §7.1].

A.2 L^p Spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $p \in [1, \infty)$, the L^p norm of a measurable function $f : X \to \mathbb{C}$ is defined by

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}. \tag{A.17}$$

For $p = \infty$, the integral is replaced by the essential supremum,

$$\|f\|_\infty := \inf\left\{ m \in \mathbb{R} : \ |f| \le m \text{ a.e.} \right\}.$$

The L^p spaces are defined as

$$L^p(X, d\mu) := \left\{ f \text{ measurable } X \to \mathbb{C} : \ \|f\|_p < \infty \right\}, \tag{A.18}$$

subject to the standard equivalence of functions that agree almost everywhere with respect to μ.

The function $\|\cdot\|_p$ is homogeneous because of the power $1/p$ included in (A.17), and positive definiteness is a consequence of the equivalence relation imposed on $L^p(X, d\mu)$. The L^p version of the triangle inequality is known as the Minkowski inequality. It is obvious for $p = 1$ or ∞, and follows from Cauchy–Schwarz for $p = 2$. Its proof in the general case relies on the following:

Lemma A.12 (Hölder Inequality). *Let f, g be measurable functions on X. For $p, q \geq 1$ with $1/p + 1/q = 1$,*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{A.19}$$

Proof Assume that p, q are as in the statement. For $x > 0$, calculus shows that the function

$$h(x) := x - \frac{x^p}{p}$$

is maximized when $x = 1$. Therefore,

$$x - \frac{x^p}{p} \leq 1 - \frac{1}{p}$$

for all $x > 0$. Setting $x = ab^{-q/p}$ for $a, b > 0$ gives, after some simplification, the inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{A.20}$$

which clearly extends to the case where a or $b = 0$,

Now suppose f, g are measurable functions on X. The inequality (A.19) is trivial if either $f \equiv 0$ or $g \equiv 0$, so we can assume that these functions have nonzero norms. Setting $a = |f(x)|/\|f\|_p$ and $b = |g(x)|/\|g\|_q$ in (A.20) gives

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \leq \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q}.$$

Integration over x gives

$$\frac{\|fg\|_1}{\|f\|_p\|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

yielding (A.19). □

Corollary A.13 (Minkowski Inequality). *Let f, g be measurable functions on X. For $p \geq 1$,*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof As in the proof of Hölder, we can assume f and g are not $\equiv 0$, since this case is trivial. From (A.19), we have

$$\|f\|_p \geq \|fh\|_1 \tag{A.21}$$

under the assumption that $\|h\|_q = 1$ where $q = p/(p-1)$. Since equality holds in (A.21) for $h = |f|^{p-1}/\|f^{p-1}\|_q$, we can conclude that

$$\|f\|_p = \sup_{\|h\|_q=1} \|fh\|_1,$$

The triangle inequality for L^p now follows,

$$\begin{aligned}
\|f+g\|_p &= \sup_{\|h\|_q=1} \|(f+g)h\|_1 \\
&\leq \sup_{\|h\|_q=1} \|fh\|_1 + \sup_{\|h\|_q=1} \|gh\|_1 \\
&= \|f\|_p + \|g\|_p.
\end{aligned}$$

□

On $\mathbb{R}^n$, a *step function* is defined as a linear combination of characteristic functions of rectangles. From the construction of Lebesgue measure described in Appendix A.1.1, we can deduce that the step functions are dense in $L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$. (See, e.g., Royden [76, §6.4].) By smoothing the edges of the characteristic functions, we can thus conclude that $C_0^\infty(\mathbb{R}^n)$ is also dense as a subset of $L^p(\mathbb{R}^n)$.

A.2.1 Completeness

This section is devoted to the proof of the completeness of L^p spaces (Theorem 2.5). The result is a straightforward consequence of the convergence theorems from the Lebesgue integration theory.

Theorem A.14 (Riesz–Fischer). *For $p \in [1, \infty]$, $L^p(X, d\mu)$ is complete as a metric space.*

Proof By Theorem 2.4, it suffices to show that an absolutely convergent series is convergent. Consider first the case $p = \infty$. For $\{u_k\}_{k=1}^\infty \subset L^\infty(X, d\mu)$, set $m_k := \|u_k\|_\infty$. We assume that the series is absolutely convergent, which means that

$$\sum m_k < \infty. \tag{A.22}$$

Since each $|u_k|$ is bounded by m_k almost everywhere, we can define an exceptional set

$$E := \bigcup_{k=1}^{\infty} \{x : |u_k(x)| > m_k\},$$

of measure zero, such that $|u_k| \leq m_k$ for all k on $X \backslash E$. For $x \notin E$, the series $\sum u_k(x)$ converges absolutely by (A.22). We can thus define a function $f \in L^{\infty}(X, d\mu)$ by

$$f(x) = \sum_{k=1}^{\infty} u_k(x), \quad \text{for } x \notin E,$$

with the values on E being irrelevant because the set has measure zero. The function f lies in $L^{\infty}(X, d\mu)$, with

$$\|f\|_{\infty} \leq \sum_{k=1}^{\infty} m_k.$$

It follows from (A.22) that $\sum u_k$ converges to f in the L^{∞} sense, because

$$\left\| f - \sum_{k=1}^{n} u_k \right\|_{\infty} \leq \sum_{k=n+1}^{\infty} m_k.$$

Now let $p \in [1, \infty)$ and assume that $\sum u_k$ is an absolutely convergent series in $L^p(X, d\mu)$. Define

$$g(x) := \sum_{k=1}^{\infty} |u_k(x)|.$$

The triangle inequality implies that

$$\left\| \sum_{k=1}^{N} |u_k| \right\|_p \leq \sum_{k=1}^{N} \|u_k\|_p. \tag{A.23}$$

By the monotone convergence theorem, the left-hand side of (A.23) converges to $\|g\|_p$ as $N \to \infty$, implying that

$$\|g\|_p \leq \sum_{k=1}^{\infty} \|u_k\|_p.$$

Since $\sum u_k$ converges absolutely, this shows $g \in L^p(X, d\mu)$.

In particular, g is finite a.e., so the series $\sum u_k(x)$ converges absolutely for a.e. $x \in X$. Hence the series $\sum u_k$ converges pointwise a.e. to some function f. Moreover, $|f| \leq g$, so $f \in L^p(X, d\mu)$ also. Since

$$\left|\sum_{k=0}^{m} u_k - f\right|^p \leq (2g)^p,$$

and g^p is integrable, the dominated convergence theorem implies that

$$\lim_{m\to\infty} \int_X \left|\sum_{k=0}^{m} u_k - f\right|^p d\mu = 0.$$

Hence the series $\sum u_k$ converges to f in L^p. □

A.2.2 Convolution

The convolution of two measurable functions on $\mathbb{R}^n$ is defined by

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\, d^n y, \tag{A.24}$$

assuming the integral is well defined. For $f, g \in L^1(\mathbb{R}^n)$, we can deduce from Fubini's theorem that the integral (A.24) exists for almost every x, defining a function $f * g \in L^1(\mathbb{R}^n)$ which satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

This basic result can be extended to combinations of L^p spaces, as follows.

Theorem A.15 (Young's Convolution Inequality). *Suppose $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then, if r satisfies*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \tag{A.25}$$

then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof It suffices to consider the case $f, g \in C_0^\infty(\mathbb{R}^n)$. Applying Hölder's inequality twice yields a triple product version,

$$\|fgh\|_1 \leq \|f\|_s \|g\|_t \|h\|_r, \tag{A.26}$$

where

$$\frac{1}{s} + \frac{1}{t} + \frac{1}{r} = 1.$$

To prove Young's inequality, we first divide up the convolution integrand into three terms,

$$\begin{aligned} |f * g(x)| &\leq \int_{\mathbb{R}^n} |f(x-y)g(y)| \, d^n y \\ &\leq \int_{\mathbb{R}^n} |f(x-y)|^{1-\frac{p}{r}} \cdot |g(y)|^{1-\frac{q}{r}} \cdot \left| f(x-y)^p g(y)^q \right|^{\frac{1}{r}} \, d^n y. \end{aligned}$$

Applying the Hölder inequality (A.26) to this expression gives

$$|f * g(x)| \leq \left\| f^{1-\frac{p}{r}} \right\|_s \left\| g^{1-\frac{q}{r}} \right\|_t \left\| f(x-\cdot)^{\frac{p}{r}} g^{\frac{q}{r}} \right\|_r. \tag{A.27}$$

Assuming that (A.25) holds, we can choose s and t so that

$$\frac{1}{s} = \frac{1}{p} - \frac{1}{r}, \qquad \frac{1}{t} = \frac{1}{p} - \frac{1}{r}.$$

With these choices, (A.27) gives

$$|f * g(x)|^r \leq \|f\|_p^{1-\frac{p}{r}} \, \|g\|_q^{1-\frac{q}{r}} \int_{\mathbb{R}^n} |f(x-y)|^p \, |g(y)|^q \, d^n y. \tag{A.28}$$

For the remaining integral over x, note that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)|^p \, |g(y)|^q \, d^n y \, d^n x \leq \|f\|_p^p \, \|g\|_q^q \, ,$$

by Fubini's theorem. Integrating (A.28) over x thus yields

$$\begin{aligned} \|f * g\|_r^r &\leq \|f\|_p^{r-p} \, \|g\|_q^{r-q} \, \|f\|_p^p \, \|g\|_q^q \\ &= \|f\|_p^r \, \|g\|_q^r. \end{aligned}$$

□

A.3 Fourier Transform

In this section we review some standard background material on the Fourier transform, which is used extensively in the text.

The *Fourier transform* of a function in $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, d^n x. \tag{A.29}$$

As a map the transform is denoted by

$$\mathcal{F} : f \mapsto \hat{f}.$$

The properties of the Fourier transform of an integrable function are characterized by the following:

Lemma A.16 (Riemann–Lebesgue). *For $f \in L^1(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ is continuous and bounded, with*

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

Proof From (A.29) we see immediately that

$$|\hat{f}(\xi)| \le (2\pi)^{-\frac{n}{2}} \|f\|_1 \tag{A.30}$$

for all $\xi \in \mathbb{R}^n$. This also shows that $\mathcal{F}$ maps L^1 convergent sequences to uniformly convergence sequences.

For $f \in L^1(\mathbb{R}^n)$, let $\{\psi_k\} \subset C_0^\infty(\mathbb{R}^n)$ be an approximating sequence such that $\psi_k \to f$ in L^1. Using integration by parts, we can check that $\hat{\psi}_k$ is smooth and approaches zero as $|\xi| \to \infty$. Since $\hat{\psi}_k \to \hat{f}$ uniformly on $\mathbb{R}^n$, it follows that $\hat{f}$ is continuous and decays to zero at infinity. □

The primary goal of this section is to explain how the definition of $\mathcal{F}$ is extended to L^2 functions, for which the integral (A.29) may not exist.

Theorem A.17 (Plancherel). *The Fourier transform defined by* (A.29) *extends to a unitary map $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

To prove Plancherel's theorem, we first study the restriction of $\mathcal{F}$ to the space of *Schwartz functions*,

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n);\ \left\| x^\alpha D^\beta f \right\|_\infty < \infty \text{ for all } \alpha, \beta \in (\mathbb{N}_0)^n \right\}. \tag{A.31}$$

For a Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^n)$, integration by parts implies that

$$\mathcal{F}[D_x^\alpha \psi](\xi) = (i\xi)^\alpha \hat{\psi}(\xi), \tag{A.32}$$

and

$$\mathcal{F}[x^\alpha \psi](\xi) = (iD_\xi)^\alpha \hat{\psi}(\xi). \tag{A.33}$$

This means that, under the Fourier transform, smoothness translates to rapid decay and vice versa.

Lemma A.18. *The Fourier transform $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.*

Proof Suppose that $f \in \mathcal{S}$. In order to show that $\hat{f}$ is Schwartz, we need to produce a bound on the function $\xi^\beta D^\alpha \hat{f}$ for each α, β. By (A.32) and (A.33),

$$\xi^\beta D_\xi^\alpha \hat{f}(\xi) = i^{|\alpha|+|\beta|} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} x^\alpha D_x^\beta f(x)\, d^n x. \tag{A.34}$$

Because $(1+|x|^2)^{-N}$ is integrable for N sufficiently large, we can estimate (A.34) by

$$\left|\xi^\beta D_\xi^\alpha \hat{f}(\xi)\right| \le C_N \sup \left|(1+|x|^2)^N x^\alpha D_x^\beta f\right|.$$

The right side is finite for all N by the definition (A.31). □

The final ingredient for the proof of Theorem A.17 is the formula for the Fourier transform of a Gaussian function. Let

$$g(x) := e^{-a|x|^2}, \tag{A.35}$$

for $a > 0$. By completing the square, and then using contour integration to make a complex change of variables, we can compute

$$\hat{g}(\xi) = (2a)^{-\frac{n}{2}} e^{-|\xi|^2/4a}. \tag{A.36}$$

Theorem A.19. *The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ defined by (A.29) has an inverse $\mathcal{F}^{-1}$ given by*

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} \hat{f}(\xi)\, d^n\xi. \tag{A.37}$$

Proof For $f, g \in \mathcal{S}(\mathbb{R}^n)$, consider the integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x) e^{-ix\cdot y} g(y)\, d^n x\, d^n y. \tag{A.38}$$

The integrals over x and y can be taken in either order, by Fubini, yielding the identity

$$\int_{\mathbb{R}^n} f\hat{g}\, d^n x = \int_{\mathbb{R}^n} \hat{f} g\, d^n y. \tag{A.39}$$

Now let g be the Gaussian function (A.35). By (A.36),

$$(2a)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-|x|^2/4a} \, d^n x = \int_{\mathbb{R}^n} \hat{f}(y) e^{-a|y|^2} \, d^n y. \tag{A.40}$$

Rescaling the variable on the left-hand side of (A.40) yields

$$2^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\sqrt{a} x) e^{-|x|^2/4} \, d^n x = \int_{\mathbb{R}^n} \hat{f}(y) e^{-a|y|^2} \, d^n y.$$

By the dominated convergence theorem, taking $a \to 0$ then gives

$$(2\pi)^{\frac{n}{2}} f(0) = \int_{\mathbb{R}^n} \hat{f}(y) \, d^n y. \tag{A.41}$$

This verifies (A.37) for $x = 0$.

The general inverse formula can be deduced from (A.41) by a simple translation argument. For $w \in \mathbb{R}^n$, define the translation operator T_w on $\mathcal{S}(\mathbb{R}^n)$ by

$$T_w f(x) := f(x + w).$$

A change of variables shows that

$$\begin{aligned}
\widehat{T_w f}(y) &= \int_{\mathbb{R}^n} e^{-ix\cdot y} f(x+w) \, d^n x \\
&= \int_{\mathbb{R}^n} e^{-i(x-w)\cdot y} f(x) \, d^n x \\
&= e^{iw\cdot y} \hat{f}(y).
\end{aligned}$$

Plugging $T_w f$ into (A.41) yields

$$(2\pi)^{n/2} f(w) = \int_{\mathbb{R}^n} e^{iw\cdot y} \hat{f}(y) \, d^n y.$$

□

From the pairing formula (A.39) and the invertibility of $\mathcal{F}$, we can immediately deduce that

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle \tag{A.42}$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$. It is straightforward to extend a unitary map from a dense subspace to the full Hilbert space, so Theorem A.17 follows from (A.42), Theorem A.19, and the fact that $\mathcal{S}(\mathbb{R}^n)$ is dense as a subspace of $L^2(\mathbb{R}^n)$.

A.4 Elliptic Regularity

In Section 6.3 we gave a simple argument for the interior regularity of eigenfunctions. This approach, based on the Fourier characterization of Sobolev spaces, is easily generalized to elliptic PDE with constant coefficients. To handle elliptic operators with variable coefficients, or to include regularity up to the boundary, a different strategy is required.

In this section we will prove the more general elliptic regularity result stated in the text as Theorem 9.27. For more general versions of this result, see, e.g., Evans [29, §6.3.2], Gilbarg and Trudinger [36, §8.4], or Taylor [89, §5.1]

Theorem A.20 (Elliptic Regularity). *Let $\overline{\Omega}$ be a compact Riemannian manifold with boundary, with $-\Delta$ be the Dirichlet Laplacian defined in Section 9.3. If $u \in \mathcal{D}(\Delta)$ and $\Delta u \in H^m(\Omega)$ for some $m \in \mathbb{N}_0$, then $u \in H^{m+2}(\Omega)$, with*

$$\|u\|_{H^{m+2}} \leq C\big(\|\Delta u\|_{H^m} + \|u\|\big),$$

where C depends only on Ω and m.

The strategy for the proof of Theorem A.20 is to use difference quotients to estimate weak derivatives. For $h \in \mathbb{R}$ and f a function on $\mathbb{R}^n$, define

$$\partial_j^h f(x) := \frac{f(x + he_j) - f(x)}{h}$$

for $j = 1, \dots n$, where $\{e_j\}$ denotes the standard basis for $\mathbb{R}^n$. We first need to prove a basic estimate for difference quotients in terms of derivatives.

Lemma A.21. *Suppose $u \in H_0^1(U)$ where $U \subset \mathbb{R}^n$ is a bounded open set. For $\varepsilon > 0$, let $U_\varepsilon := \{x \in U : d(x, \partial U) > \varepsilon\}$. If $\operatorname{supp}(u) \subset U_\varepsilon$ for $\varepsilon > 0$, then for $|h| < \varepsilon$,*

$$\|\partial_j^h u\|_{L^2} \leq \|\partial_k u\|_{L^2}.$$

Proof For $\psi \in C_0^\infty(\Omega)$, the difference quotient can be expressed as an integral,

$$\partial_j^h \psi(x) = \frac{1}{h}\int_0^h \partial_j \psi(x + te_j)\, dt,$$

for h sufficiently small. Applying Cauchy–Schwarz then gives the estimate,

$$|\partial_j^h \psi(x)|^2 \leq \frac{1}{h}\int_0^h \big|\partial_j \psi(x + te_j)\big|^2\, dt.$$

Integrating over x then gives

$$\begin{aligned}\|\partial_j^h \psi\|^2 &\le \frac{1}{h}\int_U \int_0^h \left|\partial_j \psi(x+te_j)\right|^2 dt\, dx \\ &= \frac{1}{h}\int_0^h \int_U \left|\partial_j \psi(x+te_j)\right|^2 dx\, dt \\ &= \|\partial_j \psi\|^2.\end{aligned}$$

The estimate can now be extended to a function $u \in H_0^1(U)$ with support in U_ε, by choosing an approximating sequence $\{\phi_k\} \subset C_0^\infty(\Omega)$ such that $\phi_k \to u$ in H^1.
□

Proof of Theorem A.20 The first observation is that the proof can be localized to coordinate neighborhoods. Let $\{U_j\}_{j=1}^q$ and $\{\chi_j\}$ be the coordinate atlas and corresponding partition of unity used to define the Sobolev norms as in (9.39). A simple concavity argument shows that

$$\|u\|_{H^m} \asymp \sum_{j=1}^q \|\chi_j u\|_{H^m},$$

for each m, with constants that depend only on q. Furthermore,

$$\begin{aligned}\|\Delta(\chi_j u)\|_{H^m} &\le \|\chi_j \Delta u\|_{H^m} + \|[\Delta, \chi_j]u\|_{H^m} \\ &\le \|\chi_j \Delta u\|_{H^m} + C\|u\|_{H^{m+1}}.\end{aligned}$$

We will first show that

$$\|\chi_j u\|_{H^2} \le C\big(\|\Delta(\chi_j u)\|_{L^2} + \|\chi_j u\|_{L^2}\big), \tag{A.43}$$

for each j. We only need to consider the case of a boundary neighborhood U_j, as the interior estimate can be considered as a special case where the cutoff vanishes near the boundary.

To prove (A.43), we can specialize to the case of a bounded domain $U := \{|x| < R, x^n > 0\}$ in $\mathbb{R}^n$, with the metric g represented in coordinates as matrix g_{ij}. On $H_0^1(U)$ we define the sesquilinear form

$$\begin{aligned}Q[u, v] &= \int_U g(\nabla \bar{u}, \nabla v)\, dV \\ &:= \int_U g^{ij}(\partial_i \bar{u})(\partial_j v)\sqrt{g}\, d^n x.\end{aligned}$$

For simplicity, we will write this as

$$Q[u, v] = \int_U a^{ij}(\partial_i \bar{u})(\partial_j v)\, d^n x,$$

where $a^{ij} := g^{ij}\sqrt{g}$.

Recall the local formula (9.33) for the Laplacian,

$$\Delta = \frac{1}{\sqrt{g}} \partial_i \big(\sqrt{g} g^{ij} \partial_j\big). \tag{A.44}$$

By the definition of $\mathcal{D}(-\Delta)$ from Section 9.4.1, we have

$$Q[u, v] = \langle -\Delta u, v \rangle, \tag{A.45}$$

for $u \in \mathcal{D}(-\Delta)$ and $v \in H_0^1(U)$.

For the application to (A.43), we may assume that u is supported away from the boundary of U, except possibly at $x^n = 0$. Assuming the $v \in H_0^1(U)$ shares this property, we can apply ∂_k^h to v for $k = 1, \dots, n-1$ and h sufficiently small. We then deduce from (A.45) that

$$Q[u, \partial_k^h v] = \langle -\Delta u, \partial_k^h v \rangle. \tag{A.46}$$

Writing out the left-hand side gives

$$Q[u, \partial_k^h v] = \int_U a^{ij}(\partial_i \bar{u})(\partial_j \partial_k^h v)\, d^n x.$$

By a linear change of variables, the difference quotient can be transferred from one term to the other (as if integrating by parts) to obtain

$$Q[u, \partial_k^h v] = -\int_U \partial_k^{-h}(a^{ij}\partial_i \bar{u})\, \partial_j v\, dx.$$

A simple product-rule computation shows that

$$\partial_k^{-h}(a^{ij}\partial_i \bar{u}) = a^{ij}(x - he_k)\partial_i(\partial_k^{-h}\bar{u}) + (\partial_k^{-h} a^{ij})\partial_i \bar{u}.$$

(As above, $\{e_k\}$ denotes the standard basis for $\mathbb{R}^n$.) Applying these computations to (A.46) gives

$$\int_U a^{ij}(x - he_k)\partial_i(\partial_k^{-h}\bar{u})\, \partial_j v\, dx = \int_U \Big[(L\bar{u})\partial_k^h v - (\partial_k^{-h} a^{ij})\partial_i \bar{u}\, \partial_j v\Big]\, dx.$$

By Lemma A.21, the L^2 norm of $\partial_k^h v$ can be estimated by that of $\partial_k v$, for h sufficiently small. Thus Cauchy–Schwarz gives the estimate

$$\begin{aligned}\int_U a^{ij}(x - he_k)\partial_i(\partial_k^{-h}\bar{u})\,\partial_j v\,dx &\leq C\big(\|\Delta u\| + \|\nabla u\|\big)\|\nabla v\| \\ &\leq C\big(\|\Delta u\| + \|u\|_{H^1}\big)\|\nabla w\|.\end{aligned} \tag{A.47}$$

Here C is independent of h, because the coefficients $(\partial_k^{-h}a^{ij})$ are bounded uniformly for small h, by the mean value theorem.

Now let us set $v = \partial_k^{-h}u$. The fact that g^{ij} is smooth and positive definite on $\overline{U}$ implies

$$\int_U a^{ij}(x - he_k)\partial_i(\partial_k^{-h}\bar{u})\,\partial_j(\partial_k^{-h}u)\,dx \geq c\left\|\partial_k^{-h}\nabla u\right\|^2.$$

Applying this in (A.47) gives

$$\left\|\partial_k^{-h}\nabla u\right\|^2 \leq C\big(\|\Delta u\| + \|u\|_{H^1}\big)\left\|\partial_k^{-h}\nabla u\right\|,$$

which yields

$$\left\|\partial_k^{-h}\nabla u\right\| \leq C\big(\|\Delta u\| + \|u\|_{H^1}\big). \tag{A.48}$$

By Cauchy–Schwarz, we can estimate

$$\begin{aligned}\|u\|_{H^1}^2 &= \langle -\Delta u, u\rangle \\ &\leq \|\Delta u\|\|u\| \\ &\leq \frac{1}{2}(\|\Delta u\| + \|u\|)^2.\end{aligned}$$

Therefore, (A.48) can be reduced to

$$\left\|\partial_k^{-h}\nabla u\right\| \leq C\big(\|\Delta u\| + \|u\|\big), \tag{A.49}$$

with C independent of h.

By Alaoglu's theorem (Theorem 2.37), the uniform estimate (A.49) implies that there exists functions $f_j \in L^2(U)$ satisfying

$$\|f_j\| \leq C\big(\|\Delta u\| + \|u\|\big),$$

and a sequence $h_l \to 0$ such that $\partial_k^{-h_l}\partial_j u \to f_j$ weakly as $l \to 0$. For $\psi \in C_0^\infty(U)$ we have

$$\begin{aligned}\int_U f_j\psi\, d^n x &= \lim_{l\to 0}\int_U (\partial_k^{-h_l}\partial_j u)\psi\, d^n x\\ &= -\lim_{l\to 0}\int_U \partial_j u\, \partial_k^{h_l}\psi\, d^n x\\ &= -\int_U \partial_j u\, \partial_k\psi\, d^n x.\end{aligned}$$

This shows that $f_j = \partial_k\partial_j u$ as a weak derivative, and (A.49) gives the estimate

$$\|\partial_k \nabla u\| \le C\big(\|\Delta u\| + \|u\|\big). \tag{A.50}$$

This argument required $k \le n-1$, so the estimate (A.50) covers all of the second partial derivatives of u except for $\partial_n^2 u$. To handle this case, note that (A.44) implies that

$$\Delta u = \frac{1}{\sqrt{g}}\Bigg[\sum_{(i,j)\neq(n,n)} \partial_i[a^{ij}\partial_j u] + a^{nn}\partial_n^2 u + (\partial_n a^{nn})\partial_n u\Bigg].$$

All of the terms here except $a^{nn}\partial_n^2 u$ are in $L^2(U)$, either by assumption or by (A.50). Since a^{nn} is bounded below by a positive constant, by the positive definiteness of the metric on $\overline{U}$, it follows that $\partial_n^2 u \in L^2(U)$, with

$$\big\|\partial_n^2 u\big\| \le C\big(\|\Delta u\| + \|u\|\big).$$

Together with (A.50), this shows that $u \in H^2(U)$, with

$$\|u\|_{H^2} \le C\big(\|\Delta u\| + \|u\|\big). \tag{A.51}$$

This completes the proof of (A.43), which settles the case $m = 0$.

To prove the estimate for higher m we proceed by induction. Assume that $u \in H^{m+1}(U)$ (with support as described above) and satisfies

$$\|u\|_{H^{m+1}} \le C\big(\|\Delta u\|_{H^{m-1}} + \|u\|_{H^m}\big).$$

For the inductive step we assume that $\Delta u \in H^m(U)$, and need to show that this implies $u \in H^{m+2}(U)$, with the corresponding estimate.

Since $u \in H^{m+1}$, the weak derivative $D^\alpha u$ exists for $|\alpha| = m$ and lies in $H_0^1(\Omega)$. It satisfies the equation

$$\Delta(D^\alpha u) = D^\alpha(\Delta u) - [\Delta, D^\alpha]u. \tag{A.52}$$

By the assumptions on u and Δu, and the fact that $[\Delta, D^\alpha]$ is a differential operator of order $m+1$, the right-hand side of (A.52) lies in $L^2(\Omega)$. We can therefore apply the bound (A.51) to $D^\alpha u$ to conclude that $D^\alpha u \in H^2(\Omega)$ with

$$\|D^\alpha u\|_{H^2} \leq C\big(\|\Delta(D^\alpha u)\| + \|D^\alpha u\|_{H^1}\big).$$

Furthermore, by (A.52) we can estimate

$$\|\Delta(D^\alpha f)\| \leq \|\Delta u\|_{H^m} + C\|u\|_{H^{m+1}}.$$

It thus follows that

$$\|u\|_{H^{m+2}} \leq C\big(\|\Delta u\|_{H^m} + \|u\|_{H^{m+1}}\big).$$

□

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