Appendix A
Mellin-Barnes Integrals

In analogy to integer-order calculus, fractional calculus also presents classes of functions that can be called special. In solving a fractional differential equation with constant coefficients, the classical Mittag-Leffler function and its particular cases emerge naturally. The most general special function appearing in problems of fractional calculus is the Fox's H-function, which is defined in terms of the Mellin-Barnes integral. Here we present Mellin-Barnes integrals associated with Fox's H-function and some particulares cases, among them, the classical hypergeometric function.

In this appendix we will present the called Mellin-Barnes integrals [1, 2], because of their importance in the inversion of the Laplace and Mellin transforms. The particularity of these integrals lies in the fact that they are integral in the complex plane and, therefore, the integration contour plays a fundamental rule [3].

First, before presenting the definition of the Mellin-Barnes integral, it should be pointed out that Pincherle [4], in 1888, therefore, before Mellin [2] and Barnes [1], obtained the following formula

$$\Psi(t) = \int_\gamma \frac{\displaystyle\prod_{i=1}^{m} \Gamma(x-\rho_i)}{\displaystyle\prod_{i=1}^{m-1} \Gamma(x-\sigma_i)} e^{xt}\,\mathrm{d}x$$

whose convergence was proved using an asymptotic formula for the gamma function.

We note the similarity of this expression with the inverse Laplace transform, since by defining the quotient involving the gamma functions, by $f(x, \rho_i, \sigma_i)$ for example, we have exactly the integral, in the complex plane, that recovers the function through the inverse Laplace transform. Thus, this integral can be considered as the first example in the literature of what is now known as the Mellin-Barnes integral [5].

In order to make explicit the calculations, we will discuss, through examples, involving the Mittag-Leffler function with three parameters and the classical hyper-

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geometric function, how to present the function as an integral in the complex plane and vice versa, that is, from the integral in complex plane, to recover the particular function.

Definition A.1 (*Mellin-Barnes integral*) Let γ be a contour in the complex plane starting at $c - i\infty$ and ending at $c + i\infty$, with $\text{Re}(s) = c > 0$. It is called Mellin-Barnes integral to any integral in the complex plane whose integrand contemplates at least one gamma function, given by

$$I(z) = \frac{1}{2\pi i} \int_{\gamma} f(s)\, z^{-s}\, \mathrm{d}s \tag{A.1}$$

where the density function, $f(s)$, in general, solution of a differential equation with polynomial coefficients, is given by a quotient of products of gamma functions depending on parameters. The choice of the contour γ, as we have already mentioned, plays a crucial rule, and will be explained through examples, involving a Mittag-Leffler function with three parameters, the hypergeometric function as well as, the confluent hypergeometric function, because with these functions we can discuss the case of the called ‘empty product’, expressing these functions as a Meijer’s G-function or a Fox’s H-function.

Moreover, we will direct this appendix to be able to use it, when appropriate, in the inversion of integrals, in particular, involving the Meijer and Fox functions. Therefore, in this sense, we will consider the density function in the following form

$$f(s) = \frac{A(s)B(s)}{C(s)D(s)}$$

with $A(s)$, $B(s)$, $C(s)$ and $D(s)$ are products of gamma functions, as in Sect. 2.4.3.

Example A.1 (Mittag-Leffler function with three parameters) Let $\alpha \in \mathbb{R}_+$; $\beta, \rho \in \mathbb{C}$, $\beta \neq 0$ and $\text{Re}(\rho) > 0$. Show that the integral

$$\Lambda \equiv \frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{\Gamma(s)\Gamma(\rho - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s}\, \mathrm{d}s$$

for a convenient integration contour, $\mathfrak{L}$, in the complex plane, is a less than adequate gamma function, a Mittag-Leffler function with three parameters.

We consider $|\arg(z)| < \pi$ and the contour $\mathfrak{L}$, as in Fig. A.1, starting at $c - i\infty$ going to $c + i\infty$, being $0 < c < \text{Re}(\rho)$, separating all the poles $s = -k$, with $k = 0, 1, 2, \ldots$ to the left and all the poles $s = \rho + n$, with $n = 0, 1, 2, \ldots$ to the right (do not contribute to the residue theorem) relatively to the line $\text{Re}(s) = c$.

To compute the integral in the complex plane, we close the integration contour so that only the poles at $s = -k$, with $k = 0, 1, 2, \ldots$ contribute, and we use the residue theorem, that is, the sum of the residues at the poles $s = 0, -1, -2, \ldots$, from where follows

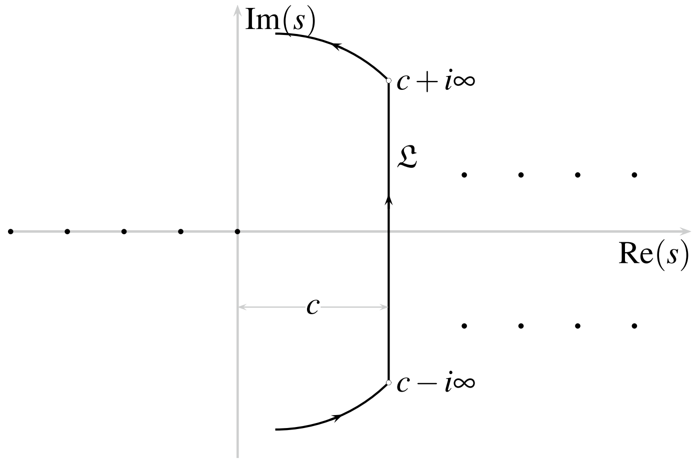

Fig. A.1 Contour $\mathfrak{L}$ for Mittag-Leffler with three parameters

$$\Lambda = \sum_{k=0}^{\infty} \lim_{s \to -k} \left\{ (s+k) \left[\frac{\Gamma(s)\Gamma(\rho - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \right] \right\}$$

that, using the fact that the limit of the product is equal to the product of the limits,

$$\Lambda = \sum_{k=0}^{\infty} \frac{\Gamma(k+\rho)}{\Gamma(\alpha k + \beta)} (-z)^k \lim_{s \to -k} (s+k)\Gamma(s)\cdot$$

Note that we have an indetermination of type $0 \cdot \infty$. In order to raise this indetermination, we will calculate the remaining limit separately,

$$\lim_{s \to -k} (s+k)\Gamma(s) = \lim_{s \to -k} (s+k) \frac{(s+k-1)\cdots s\Gamma(s)}{(s+k-1)\cdots s} = \lim_{s \to -k} \frac{\Gamma(s+k+1)}{(s+k-1)\cdots s}$$

$$= \frac{\Gamma(1)}{(-1)(-2)\cdots(-k)} = \frac{1}{(-1)^k(1 \cdot 2 \cdots k)} = \frac{(-1)^k}{k!}\cdot$$

This result is the residue of the gamma function at the poles $s = -k$ with $k = 0, 1, 2, \ldots$ Another way to obtain this result is through the called Mittag-Leffler expansion, according to Exercises (2) and (3) from Chap. 2.

Thus, returning with this result in the expression for Λ, we obtain

$$\Lambda = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\rho + k)}{\Gamma(\beta + \alpha k)} (-z)^k = \sum_{k=0}^{\infty} \frac{\Gamma(\rho + k)}{\Gamma(\beta + \alpha k)} \frac{z^k}{k!}.$$

Using the definition of the Pochhammer symbol, we can write

$$\Lambda = \Gamma(\rho) \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}$$

which can be identified with a Mittag-Leffler function with three parameters, that is,

$$\Lambda = \Gamma(\rho) E^{\rho}_{\alpha,\beta}(z)$$

from which it follows that the integral is the product of a gamma function by the Mittag-Leffler function with three parameters

$$\frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{\Gamma(s)\Gamma(\rho - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \, \mathrm{d}s = \Gamma(\rho) E^{\rho}_{\alpha,\beta}(z) \tag{A.2}$$

which is the desired result ◇

Example A.2 (Particular cases) Since the Mittag-Leffler function with three parameters contains as particular cases Mittag-Leffler functions with two and one parameter, we will express such functions in terms of an integral in the complex plane. Using the previous result, taking $\rho = 1$ in Eq. (A.2) we obtain

$$\frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{\Gamma(s)\Gamma(1 - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \, \mathrm{d}s = E_{\alpha,\beta}(z)$$

an integral representation for the Mittag-Leffler function with two parameters. Also, taking $\rho = 1 = \beta$ in Eq. (A.2) we obtain

$$\frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{\Gamma(s)\Gamma(1 - s)}{\Gamma(1 - \alpha s)} (-z)^{-s} \, \mathrm{d}s = E_{\alpha}(z)$$

an integral representation for the classical Mittag-Leffler function. Finally, for $\rho = 1 = \beta = \alpha$ in Eq. (A.2) we obtain

$$\frac{1}{2\pi i} \int_{\mathfrak{L}} \Gamma(s)(-z)^{-s} \, \mathrm{d}s = e^{z}$$

an integral representation for the exponential function. The last three expressions can be interpreted as inverse Mellin transforms ◇

Example A.3 (Mittag-Leffler function as a Fox's H-function) Here we will address the inverse problem, that is, we have the integral in the complex plane and we want to determine the function, that is, to calculate the integral explicitly.

Let us show that this function can not be written in terms of a Meijer function, which, as particular cases, contains all classical hypergeometric functions. Just to mention, the particular Mittag-Leffler function that can be written in terms of a confluent hypergeometric function, yes, can be written as a Meijer function.

In order to write the Mittag-Leffler function with three parameters, let us first identify its representation in terms of a Mellin-Barnes integral, Eq. (A.2), with the Mellin-Barnes integral defining Fox's H-function, that is,

$$\frac{1}{2\pi i}\int_{\mathfrak{L}} \frac{\displaystyle\prod_{i=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\displaystyle\prod_{i=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}(-z)^s\,\mathrm{d}s\,\cdot$$

In order to identify, we start with the numerator, that is, the integrand numerator in Eq. (A.2), has two gamma functions, where we consider $m = 1 = n$ while in the denominator we have only one gamma function, then we consider $q = 2$ and $p = 1$, the smallest that does not contribute, that is, it will generate an 'empty product', since the initial index begins with $j = n + 1$ and, like $n = 1$, the beginning will be $j = 2$, so it is equal to unity.

Let's explain these products individually, identifying them as follows:

$$\begin{aligned}
\prod_{j=1}^{1}\Gamma(b_j+\beta_j s) &= \Gamma(b_1+\beta_1 s) && \text{numerator,}\\
\prod_{j=1}^{1}\Gamma(1-a_j-\alpha_j s) &= \Gamma(1-a_1-\alpha_1 s) && \text{numerator,}\\
\prod_{j=2}^{2}\Gamma(1-b_j-\beta_j s) &= \Gamma(1-b_2-\beta_2 s) && \text{denominator,}\\
\prod_{j=2}^{1}\Gamma(a_j+\alpha_j s) &= 1(\text{not contributes}) && \text{denominator.}
\end{aligned}$$

Using the integrand in Eq. (A.2) and identifying with the products,

$$\frac{\Gamma(s)\Gamma(\rho-s)}{\Gamma(\beta-\alpha s)} = \frac{\Gamma(b_1+\beta_1 s)\Gamma(1-a_1-\alpha_1 s)}{\Gamma(1-b_2-\beta_2 s)}$$

we have $a_1 = 1 - \rho, \alpha_1 = 1, b_1 = 0, \beta_1 = 1, b_2 = 1 - \beta, \beta_2 = \alpha$ and $z \to -z$. Identifying the symbols for the Fox's H-function, we have

$$H^{1,1}_{1,2}\left[-z \left| \begin{matrix} (a_1, \alpha_1) \\ (b_1, \beta_1), (b_2, \beta_2) \end{matrix} \right.\right] = H^{1,1}_{1,2}\left[-z \left| \begin{matrix} (1-\rho, 1) \\ (0, 1), (1-\beta, \alpha) \end{matrix} \right.\right]$$

which, finally, allows us to write

$$\Gamma(\rho) E^{\rho}_{\alpha,\beta}(z) = H^{1,1}_{1,2}\left[-z \left| \begin{matrix} (1-\rho, 1) \\ (0, 1), (1-\beta, \alpha) \end{matrix} \right.\right]$$

which is the desired result ◇

Example A.4 (A particular case. Meijer's G-function) From the above expression it is clear that Fox's H-function can not be reduced to a Meijer's function, except if $\alpha = 1$, a particular Mittag-Leffler function $E^{\rho}_{1,\beta}(\cdot)$, that is,

$$\Gamma(\rho) E^{\rho}_{1,\beta}(z) = G^{1,1}_{1,2}\left[-z \left| \begin{matrix} 1-\rho \\ 0, 1-\beta \end{matrix} \right.\right]$$

with $G^{1,1}_{1,2}\left[\cdot \left| \begin{matrix} \cdot \\ \cdot \end{matrix} \right.\right]$ is a Meijer's G-function.

Since this particular Mittag-Leffler function with three parameters is related to a confluent hypergeometric function, through the expression

$$\Gamma(\beta) E^{\rho}_{1,\beta}(z) = {}_1F_1(\rho; \beta; z)$$

we can write this confluent hypergeometric function in terms of a Meijer's G-function,

$${}_1F_1(\rho; \beta; x) = G^{1,1}_{1,2}\left[-z \left| \begin{matrix} 1-\rho \\ 0, 1-\beta \end{matrix} \right.\right]$$

which is the desired result ◇

Example A.5 (Hypergeometric function) Here, we star with the calculation of the Mellin transform, denoted by $\mathscr{M}$, of the classical hypergeometric function ${}_2F_1(a, b; c; -x)$, that is

$$\Omega \equiv \mathscr{M}[{}_2F_1(a, b; c; -x)] = \int_0^\infty x^{s-1}\, {}_2F_1(a, b; c; -x)\, \mathrm{d}x \cdot$$

By introducing the integral representation for the hypergeometric function, given by Eq. (2.10) and changing the integration orders, we can write

$$\Omega = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \int_0^\infty \frac{x^{s-1}}{(1+xt)^a}\, \mathrm{d}x\ \mathrm{d}t \cdot$$

To calculate the integral in the variable x, we introduce the change of variable $xt = \xi$ from where we obtain, only for the integral in x

$$\int_0^\infty \frac{x^{s-1}}{(1+xt)^a}\,\mathrm{d}x = t^{-s}\int_0^\infty \frac{\xi^{s-1}}{(1+\xi)^a}\,\mathrm{d}\xi\cdot$$

In order to calculate the integral in the variable ξ, we introduce another change of variable, η, given by $\xi = \eta + \xi\eta$, from where it follows to the integral, already simplifying

$$t^{-s}\int_0^\infty \frac{\xi^{s-1}}{(1+\xi)^a}\,\mathrm{d}\xi = t^{-s}\int_0^1 \eta^{s-1}(1-\eta)^{-s+a-1}\,\mathrm{d}\eta\cdot$$

The remaining integral is nothing more than a beta function, according to Eq. (2.2), then

$$\int_0^\infty \frac{x^{s-1}}{(1+xt)^a}\,\mathrm{d}x = t^{-s}B(s,a-s) = t^{-s}\frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}$$

where, in the last step, we use the relation between the gamma and beta functions.

Returning in the expression to Ω we can write

$$\Omega = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}\int_0^1 t^{b-s-1}(1-t)^{c-b-1}\,\mathrm{d}t$$

which, also, is another beta function. Proceeding as in the previous one and expressing the result in terms of gamma functions, we obtain, already simplifying

$$\Omega = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}\cdot$$

Returning with the inverse we obtain the Mellin-Barnes integral

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}{}_2F_1(a,b;c;z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}(-z)^{-s}\,\mathrm{d}s \qquad \text{(A.3)}$$

which is also an integral representation for the classical hypergeometric function.

In order to show, from the previous integral representation, the inverse process, that is, from the integral representation, to obtain the hypergeometric function, we proceed as in the case of the Mittag-Leffler function with three parameters.

We explain only the steps, and then write this function as a particular case of the Meijer's G-function. Moreover, just to remember, we can also write it in terms of a Fox's H-function, but, as we have already mentioned, Meijer's G-function is a particular case of Fox's H-function, so, we write only in terms of Meijer's G-function.

Then we return the integral given by Eq. (A.3) and outline the steps, in analogy to the case of the Mittag-Leffler function with three parameters. First the contour, where we admit $\arg(-z) < \pi$ and separating the poles from the $\Gamma(a-s)$ and $\Gamma(b-s)$ those of function $\Gamma(s)$, as well as considering $(-z)^{-s}$ with its principal value. Then we consider the poles of $\Gamma(s)$ such that $-s = k = 0, 1, 2, \ldots$, so we must calculate the limit

$$\sum_{k=0}^{\infty} \lim_{s \to -k} \left\{ (s+k) \left[\Gamma(s) \frac{\Gamma(-s+a)\Gamma(-s+b)}{\Gamma(-s+c)} (-z)^{-s} \right] \right\} .$$

Even more, analogous to the previous one, we have

$$\lim_{s \to -k} (k+s)\Gamma(s) = \frac{(-1)^k}{k!}$$

from where follows the result for the integral

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \, {}_2F_1(a, b; c; z) \, ,$$

which is the desired result ◇

Example A.6 (Hypergeometric function as a Meijer's G-function) Consider the following integral in the complex plane

$$\Lambda = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} (-z)^s \, \mathrm{d}s \cdot \tag{A.4}$$

We recall that, as in all gamma functions in the integrand of the previous one, the variable s is not multiplied by any coefficient, we can directly identify with the Meijer's G-function, being the contour as described in the previous example. Then we must compare the integrants, given in Λ with that of the Meijer's G-function, given by

$$\mathscr{G}^{m,n}_{p,q}(s) = \frac{\displaystyle\prod_{i=1}^{m} \Gamma(b_i - s) \cdot \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\displaystyle\prod_{i=m+1}^{q} \Gamma(1 - b_j + s) \cdot \prod_{j=n+1}^{p} \Gamma(a_j - s)} .$$

Then, since in the denominator we only have a gamma function, we take $m = 1$ and $q = 2$ and $p < n + 1$ so that the product is 'empty', identified with the unit. Let $n = 2$ (we still have two more gamma functions in the numerator) and $p = 2$.

Let's explain these products individually, identifying them as follows:

$$\prod_{i=1}^{1} \Gamma(b_j - s) = \Gamma(b_1 - s) \quad \text{numerator,}$$

$$\prod_{j=1}^{2}\Gamma(1-a_j+s)=\Gamma(1-a_1+s)\Gamma(1-a_2+s)\quad \text{numerator},$$
$$\prod_{i=2}^{2}\Gamma(1-b_j+s)=\Gamma(1-b_2+s)\quad \text{denominator},$$
$$\prod_{j=3}^{2}\Gamma(a_j-s)=1(\text{not contributes})\quad \text{denominator}.$$

Using the integrand in Eq. (A.4) and identifying with the products,

$$\frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}=\frac{\Gamma(b_1-s)\Gamma(1-a_1+s)\Gamma(1-a_2+s)}{\Gamma(1-b_2+s)}$$

we have $a_1=1-a$, $a_2=1-b$, $b_1=0$ and $b_2=1-c$. Identifying with the symbols for the Meijer's G-function, we have

$$G^{1,2}_{2,2}\left[-z\left|\begin{matrix}a_1, a_2\\ b_1, b_2\end{matrix}\right.\right]=G^{1,2}_{2,2}\left[-z\left|\begin{matrix}1-a, 1-b\\ 0, 1-c\end{matrix}\right.\right]$$

which finally allows us to write

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z)=G^{1,2}_{2,2}\left[-z\left|\begin{matrix}1-a, 1-b\\ 0, 1-c\end{matrix}\right.\right]$$

which is the desired result ◇

Example A.7 (Hypergeometric function. A particular case) Knowing that the relation is worth

$$(1+x^2)^{-1/2}={}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2};-x^2\right)$$

express this function in terms of the Meijer's G-function.

Using the previous result, we can write, already simplifying

$$(1+x^2)^{-1/2}=\frac{1}{\sqrt{\pi}}G^{1,2}_{2,2}\left[x^2\left|\begin{matrix}1/2, 1/2\\ 0, 1/2\end{matrix}\right.\right].$$

This expression can be simplified. To this end, we use Exercise (30) from Chap. 2, then

$$(1+x^2)^{-1/2}=\frac{1}{\sqrt{\pi}}G^{1,2}_{2,2}\left[x^2\left|\begin{matrix}1/2\\ 0\end{matrix}\right.\right]$$

which is the desired result. ◇

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