Appendix
Important Tools

In this appendix, we collect some tools that are necessary in the development of some algebraic properties.

A.1 Summation

The addition of elements $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n$ can be represented in a short form using the symbol Σ, that is:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

where i is the variable of summation (or index), m is the lower bound of summation, n the upper bound of summation and $i = m$ means that the index i starts out equal to m. This index is incremented by 1 for each successive term and stopping when $i = n$.

The summation is the addition of a sequence of numbers that can be integers, rational, real or complex numbers. Note that the sum does not depend of this index and can be replaced for another index $\left(\sum_{i=1}^{n} a_i = \sum_{l=1}^{n} a_l\right)$. As examples of sums we have:

$$\sum_{j=1}^{10} x_j = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

$$\sum_{j=1}^{4} 2^j \cdot j^2 = 2^1 \cdot 1^2 + 2^2 \cdot 2^2 + 2^3 \cdot 3^2 + 2^4 \cdot 4^2 = 2 + 16 + 72 + 256 = 346$$

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Summation $\sum_{i=1}^{n} a_i$ can be established by mathematical induction , considering that

$$\sum_{k=1}^{0} x_k = 0 \tag{A.1}$$

$$\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n} a_i + a_{n+1} \tag{A.2}$$

Some important properties of summation are the following:

1. **Partition of summation**. If $m \in \{1, \ldots, n\}$ then

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i \tag{A.3}$$

2. **Additive property of summation**.

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{A.4}$$

3. **Homogeneous property**.

$$\sum_{i=1}^{n} (\lambda a_i) = \lambda \sum_{i=1}^{n} a_i \quad \forall \lambda \in \mathbb{R} \tag{A.5}$$

4. **Exchange of finite summations**.

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{m} \sum_{i=1}^{n} a_{i,j} \tag{A.6}$$

5. **Finite summations of constants**.

$$\sum_{i=1}^{n} \lambda = \lambda n \quad \forall \lambda \in \mathbb{R} \tag{A.7}$$

6. **Linearity of the summation**.

$$\sum_{i=1}^{n} \left(\lambda a_i + \bar{\lambda} b_i\right) = \lambda \sum_{i=1}^{n} a_i + \bar{\lambda} \sum_{i=1}^{n} b_i \quad \forall \lambda, \bar{\lambda} \in \mathbb{R} \tag{A.8}$$

7. **Telescoping summation**.

$$\sum_{i=1}^{n}(a_{i+1}-a_i)=a_{n+1}-a_1,\quad \sum_{i=1}^{n}(a_i-a_{i+1})=a_1-a_{n+1} \tag{A.9}$$

Example A.1 The use of the properties mentioned above is shown in this exercise. Let consider:

$$a_n=\begin{cases}\frac{4^{2n+1}\cdot 2^{3n-2}}{8^{n-1}} & \text{if } 1\le n\le 27\\ \sqrt{2n+2}-\sqrt{2n} & \text{if } 28\le n\le 56\\ 7\left(n+n^2\right) & \text{if } 57\le n\end{cases}$$

and calculate the following sum:

$$\sum_{k=1}^{59}a_k$$

According to (A.3):

$$\sum_{k=1}^{59}a_k=\sum_{k=1}^{27}a_k+\sum_{k=28}^{56}a_k+\sum_{k=57}^{59}a_k$$

For the first addend, making algebraic reductions, taking account the homogeneous property (A.5) and the sum of the first n terms of a geometric series we have:

$$\sum_{k=1}^{27}a_k=\sum_{k=1}^{27}\frac{4^{2k+1}\cdot 2^{3k-2}}{8^{k-1}}=\sum_{k=1}^{27}\frac{4^{2k}\cdot 4\cdot 2^{3k}\cdot 2^{-2}}{8^k\cdot 8^{-1}}=\frac{4\cdot 2^{-2}}{8^{-1}}\sum_{k=1}^{27}\frac{4^{2k}\cdot 2^{3k}}{8^k}=8\sum_{k=1}^{27}16^k$$
$$=8\cdot\frac{16\left(1-16^{27}\right)}{1-16}=-\frac{128\left(1-16^{27}\right)}{15}$$

On the other hand, for the second addend using the telescoping summation (A.9):

$$\sum_{k=28}^{56}a_k=\sum_{k=28}^{56}\left(\sqrt{2k+2}-\sqrt{2k}\right)=\sum_{k=28}^{56}\left(\sqrt{2(k+1)}-\sqrt{2k}\right)=\sqrt{114}-2\sqrt{14}$$

and finally for the third addend:

$$\sum_{k=57}^{59}a_k=\sum_{k=57}^{59}7\left(k+k^2\right)=7\sum_{k=57}^{59}k\,(k+1)=7\,[57(58)+58(59)+59(60)]=71876$$

Therefore,

$$\sum_{k=1}^{59}a_k=-\frac{128\left(1-16^{27}\right)}{15}+\sqrt{114}-2\sqrt{14}+71876$$

Sometimes it is necessary the change of variable in a summation, for example, let consider the following summation:

$$\sum_{k=2}^{5} a_k = a_2 + a_3 + a_4 + a_5$$

In the above expression, let $l = k + 2$ then $k = l - 2$. If $k = 2$ and $k = 5$ then $l = 4$ and $l = 7$ respectively, then we have

$$\sum_{l=4}^{7} a_{l-2} = a_2 + a_3 + a_4 + a_5 = \sum_{k=2}^{5} a_k$$

In addition as example, note that

$$\sum_{j=3}^{6} p^{j+2} = p^5 + p^6 + p^7 + p^8 = \sum_{m=5}^{8} p^m$$

The first summation, can be expressed as the second summation taking account the change of variable $m = j + 2$, then $j = m - 2$ and the lower and upper bound of the summation are $m = 5$ and $m = 8$ respectively. To conclude this section, see what happens with the cancellation of addends as special case of partition of a summation (A.3). Let the summation

$$\sum_{j=m}^{m+n} a_j + \sum_{j=l}^{l+k} a_j = a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} - \left(a_l + a_{l+1} + a_{l+2} + \cdots + a_{l+k}\right)$$

where $l > m$, $l + k > m + n$. Let define the sets of index $\hat{A} := \{m, m+1, m+2, \ldots, m+n\}$, $\hat{B} = \{l, l+1, l+2, \ldots, l+k\}$. Suppose that from some i and even a $\hat{i}$:

$$a_{m+i} + a_{m+i+1} + a_{m+i+2} + \cdots + a_{m+\hat{i}} = a_{l+i} + a_{l+i+1} + a_{l+i+2} + \cdots + a_{l+\hat{i}}$$

then, the summation is reduced and these addends are eliminated (because each one is the additive inverse of the other). To determine which addends are in the final result, note that $\hat{C} := \hat{A} \cap \hat{B}$ represents a new set whose elements are the index of the addends to be canceled in the final result, that is,

$$\hat{A} \cap \hat{B} = \left\{m + i, m + i + 1, m + i + 2, \ldots, m + \hat{i}\right\} = \left\{l + i, l + i + 1, l + i + 2, \ldots, l + \hat{i}\right\}$$

Finally the index of addends of the final result are $\hat{A} \setminus \hat{C}$ for the summation $\sum_{j=m}^{m+n} a_j$ and $\hat{B} \setminus \hat{C}$ for the summation $\sum_{j=l}^{l+k} a_j$.

Example A.2 Consider the following summation

$$\mathscr{S} = \sum_{j=0}^{7} a_j - \sum_{j=1}^{11} a_j$$

According to theory discussed above

$$\hat{A} = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$\hat{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

then the set of indices of the addends to be canceled in the final result is $\hat{C} = \hat{A} \cap \hat{B} = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{A} \setminus \hat{C} = \{0\}$, $\hat{B} \setminus \hat{C} = \{8, 9, 10, 11\}$ represent the sets of indices of the addends in the final result for the first and second addends respectively. Finally:

$$\mathscr{S} = \sum_{j=0}^{7} a_j - \sum_{j=1}^{11} a_j = a_0 - a_8 - a_9 - a_{10} - a_{11} = a_0 - \sum_{j=8}^{11} a_j.$$

A.2 Kronecker Delta

The Kronecker delta is a function of two variables, usually just positive integers. This function $\delta : \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ is a piecewise function of variables i and j given by:

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases} \tag{A.10}$$

for example $\delta_{1,1} = 1$, $\delta_{10,5} = 0$, $\delta_{1,2} = 0$. In addition, for every $i, j \in \mathbb{Z}$, $\delta_{i,j} = \delta_{j,i}$. Some properties of Kronecker delta are listed below. These properties are useful in the simplification of summations with Kronecker delta.

1. $\sum_{i=1}^{n} \delta_{i,j} a_i = a_j$ for $j \in \{1, \ldots, n\}$.
2. $\sum_{i=1}^{n} \delta_{i,j} a_i = 0$ for $j \notin \{1, \ldots, n\}$.

3. $\sum_{k=1}^{n} \delta_{i,k}\delta_{k,j} = \delta_{i,j}$.

Example A.3 Considering the following summations

$$
\begin{aligned}
\sum_{i=1}^{5} \delta_{i,2}a_i &= \delta_{1,2}a_1 + \delta_{2,2}a_2 + \delta_{3,2}a_3 + \delta_{4,2}a_4 + \delta_{5,2}a_5 \\
&= \cancelto{0}{\delta_{1,2}}a_1 + \cancelto{1}{\delta_{2,2}}a_2 + \cancelto{0}{\delta_{3,2}}a_3 + \cancelto{0}{\delta_{4,2}}a_4 + \cancelto{0}{\delta_{5,2}}a_5 = a_2
\end{aligned}
$$

In this case, according to property 1, $j = 2$ and $j = 2 \in \{1, 2, 3, 4, 5\}$, then we can use this property and avoid to make all the summation. In the following summation, $j = 6$. This value not belongs to set of indices, therefore the property 2 is used.

$$
\begin{aligned}
\sum_{i=1}^{5} \delta_{i,6}a_i &= \delta_{1,6}a_1 + \delta_{2,6}a_2 + \delta_{3,6}a_3 + \delta_{4,6}a_4 + \delta_{5,6}a_5 \\
&= \cancelto{0}{\delta_{1,6}}a_1 + \cancelto{0}{\delta_{2,6}}a_2 + \cancelto{0}{\delta_{3,6}}a_3 + \cancelto{0}{\delta_{4,6}}a_4 + \cancelto{0}{\delta_{5,6}}a_5 = 0
\end{aligned}
$$

Finally,

$$
\sum_{k=1}^{5} \delta_{i,k}\delta_{k,j} = \delta_{i,1}\delta_{1,j} + \delta_{i,2}\delta_{2,j} + \delta_{i,3}\delta_{3,j} + \delta_{i,4}\delta_{4,j} + \delta_{i,5}\delta_{5,j}
$$

This summation is 1 if $i = j$ and belongs to set of indices or 0 if $i \neq j$ or $i = j$ but not belongs to the set of indices. In other words, this summation is the Kronecker delta $\delta_{i,j}$.

Kronecker delta is commonly used in matrix theory to define the identity matrix. This section conclude with other example where the entries of two matrices are written considering an explicit formula taking account the Kronecker delta.

Example A.4

$$
A = \left[6\delta_{i,3}\right]_{i,j=1}^{4} = \begin{bmatrix} 6\delta_{1,3} & 6\delta_{1,3} & 6\delta_{1,3} & 6\delta_{1,3} \\ 6\delta_{2,3} & 6\delta_{2,3} & 6\delta_{2,3} & 6\delta_{2,3} \\ 6\delta_{3,3} & 6\delta_{3,3} & 6\delta_{3,3} & 6\delta_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 \end{bmatrix}
$$

On the other hand, if $B \in \mathscr{M}_5(\mathbb{R})$ and $B_{i,j} = \delta_{i,4} + \delta_{j,2}$ then

$$
B = \begin{bmatrix}
\delta_{1,4} + \delta_{1,2} & \delta_{1,4} + \delta_{2,2} & \delta_{1,4} + \delta_{3,2} & \delta_{1,4} + \delta_{4,2} & \delta_{1,4} + \delta_{5,2} \\
\delta_{2,4} + \delta_{1,2} & \delta_{2,4} + \delta_{2,2} & \delta_{2,4} + \delta_{3,2} & \delta_{2,4} + \delta_{4,2} & \delta_{2,4} + \delta_{5,2} \\
\delta_{3,4} + \delta_{1,2} & \delta_{3,4} + \delta_{2,2} & \delta_{3,4} + \delta_{3,2} & \delta_{3,4} + \delta_{4,2} & \delta_{3,4} + \delta_{5,2} \\
\delta_{4,4} + \delta_{1,2} & \delta_{4,4} + \delta_{2,2} & \delta_{4,4} + \delta_{3,2} & \delta_{4,4} + \delta_{4,2} & \delta_{4,4} + \delta_{5,2} \\
\delta_{5,4} + \delta_{1,2} & \delta_{5,4} + \delta_{2,2} & \delta_{5,4} + \delta_{3,2} & \delta_{5,4} + \delta_{4,2} & \delta_{5,4} + \delta_{5,2}
\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}
$$

A.3 Matrices

In this part we consider important a resume of properties of matrices and determinants. Consider the matrices $A, B, C \in \mathscr{M}_{m\times n}(\mathbb{F})$ and $\alpha, \beta \in \mathbb{F}$. In addition $0 = 0_{m\times n}$ represents the zero matrix.

1. **Sum of matrices and multiplication of one scalar by a matrix**.
   a. $A + B = B + A$
   b. $A + B + (C) = (A + B) + C$
   c. $\alpha(A + B) = \alpha A + \alpha B$
   d. $(\alpha + \beta)A = \alpha A + \beta A$
   e. $\alpha(\beta A) = (\alpha\beta)A$
   f. $A + 0 = A$
   g. $A + (-A) = 0$
2. **Multiplication of matrices**.
   a. $A(B + C) = AB + AC$
   b. $(A + B)C = AC + BC$
   c. $A(BC) = (AB)C$
   d. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
   e. $A0 = 0A = 0$
   f. $BI = IB = B$
   g In general, $AB \neq AB$.
   h $AB = 0$ does not necessarily mean that $A = 0$ or $B = 0$.
   i $AB = AC$ does not necessarily mean that $B = C$.
3. **Trace properties**.
   The trace of a square matrix $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ is defined to be the sum of the elements on the main diagonal of A:

$$\mathrm{tr}(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii} \tag{A.11}$$

   a. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
   b. $\mathrm{tr}(\alpha A) = \alpha\,\mathrm{tr}(A)$.
   c. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
   d. $\mathrm{tr}(A^\intercal) = \mathrm{tr}(A)$.
   e. $\mathrm{tr}(A^\intercal B) = \mathrm{tr}(AB^\intercal) = \mathrm{tr}(B^\intercal A) = \mathrm{tr}(BA^\intercal) = \sum_{i,j} A_{ij}B_{ij}$.
   f. $\mathrm{tr}(ABCD) = \mathrm{tr}(BCDA) = \mathrm{tr}(CDAB) = \mathrm{tr}(DABC)$. Trace is invariant under cyclic permutations.
   g. In general $\mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$.
   h. If A, B, C are symmetric matrices then $\mathrm{tr}(ABC) = \mathrm{tr}(A^\intercal B^\intercal C^\intercal) = \mathrm{tr}(A^\intercal (CB)^\intercal) = \mathrm{tr}((CB)^\intercal A^\intercal) = \mathrm{tr}((ACB)^\intercal) = \mathrm{tr}(ACB)$.

i. The trace is similarity-invariant: $\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(P^{-1}(AP)) = \operatorname{tr}((AP)P^{-1}) = \operatorname{tr}(A(PP^{-1})) = \operatorname{tr}(AI) = \operatorname{tr}(A)$.
j If A is a symmetric matrix and B is antisymmetric, then $\operatorname{tr}(AB) = 0$.
k If A is an idempotent matrix then $\operatorname{tr}(A) = \operatorname{rank}(A)$.
l. If A is a nilpotent matrix then $\operatorname{tr}(A) = 0$.
m. $\operatorname{tr}(I_n) = n$.
n. If $f(x) = (x-\lambda_1)^{d_1}(x-\lambda_2)^{d_2}\cdots(x-\lambda_k)^{d_k}$ is the characteristic polynomial of A, then $\operatorname{tr}(A) = d_1\lambda_1 + d_2\lambda_2 + \cdots + d_k\lambda_k$.
o. If $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, $\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i$.
p. $\operatorname{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k$, where λ_i is an eigenvalue of A.
q. Let A^* the conjugate transpose of A, then $\operatorname{tr}(AA^*) \geq 0$. In addition $\operatorname{tr}(AA^*) = 0$ if and only if $A = 0$.
r. $\operatorname{tr}(B^*A) = \langle A, B\rangle$ defines an inner product on the space of matrices $\mathscr{M}_{m\times n}(\mathbb{F})$.
s. $0 \leq \operatorname{tr}(AB)^2 \leq \operatorname{tr}(A^2)\operatorname{tr}(B^2) \leq \operatorname{tr}(A)^2\operatorname{tr}(B)^2$.

4. **Diagonal matrices properties**.
If A, B are diagonal matrices then

a. $A + B = \operatorname{diag}(a_{11} + b_{11}, a_{22} + b_{22}, \ldots, a_{nn} + b_{nn})$.
b. $AB = \operatorname{diag}(a_{11}b_{11}, a_{22}b_{22}, \ldots, a_{nn}b_{nn})$.
c. $\alpha A = \operatorname{diag}(\alpha a_{11}, \alpha a_{22}, \ldots, \alpha a_{nn})$.
d. A is invertible if and only if the entries $a_1, \ldots, a_n$ are all non-zero, then $A^{-1} = \operatorname{diag}\left(a_1^{-1}, \ldots, a_n^{-1}\right)$.
e. The adjoint of a diagonal matrix is again diagonal.
f. A square matrix is diagonal if and only if it is triangular and normal.
g. $\det(A) = \prod_{i=1}^{n} a_{ii}$.

5. **Properties of the inverse matrix**.

a. A^{-1} is unique.
b. $\left(A^{-1}\right)^{-1} = A$.
c. $(AB)^{-1} = B^{-1}A^{-1}$
d. $(\alpha A)^{-1} = \alpha^{-1}A^{-1} \quad \forall\alpha \neq 0$.
e. $(A^n)^{-1} = \left(A^{-1}\right)^n$.
f. $(A^\mathsf{T})^{-1} = \left(A^{-1}\right)^\mathsf{T}$.
g. $A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A$.
h. $\det\left(A^{-1}\right) = (\det(A))^{-1}$.
i. A matrix that is its own inverse, i.e. $A = A^1$ and $A^2 = I$, is called an **involution**.

6. **Transpose matrix properties**.

   a. $(A^\intercal)^\intercal = A$.
   b. $(A + B)^\intercal = A^\intercal + B^\intercal$.
   c. $(AB)^\intercal = B^\intercal A^\intercal$.
   d. $(\alpha A)^\intercal = \alpha A^\intercal$.
   e. $\det(A^\intercal) = \det(A)$.
   f. Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ then $\sigma(A) = \sigma(A^\intercal)$, where σ denotes the spectrum of a matrix.
   g. Let $A \in \mathscr{M}_{n\times n}(\mathbb{R})$. A is called a **symmetric matrix** if $A^\intercal = A$.
   h. Let $A \in \mathscr{M}_{n\times n}(\mathbb{R})$. A is called a **skew-symmetric matrix** if $A^\intercal = -A$.
   i. Let $A \in \mathscr{M}_{n\times n}(\mathbb{R})$. A is called a **orthogonal matrix** if $A^\intercal = A^{-1}$. In addition, $AA^\intercal = A^\intercal A = I$.
   j. Let $A \in \mathscr{M}_{n\times n}(\mathbb{C})$. A is called a **Hermitian matrix** if $A^\intercal = A^*$.
   k. Let $A \in \mathscr{M}_{n\times n}(\mathbb{C})$. A is called a **skew-Hermitian matrix** if $A^\intercal = -A^*$.

7. **Symmetric and skew-symmetric matrices properties**.
   Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$.

   a. $A + A^\intercal$ is a symmetric matrix.
   b. $AA^\intercal$ is a symmetric matrix.
   c. $A - A^\intercal$ is a skew-symmetric matrix.

   If A, B are symmetric (skew-symmetric) matrices:

   a. $A + B$ is a symmetric (skew-symmetric) matrix.
   b. αA is a symmetric (skew-symmetric) matrix.
   c. AB is not necessarily a symmetric (skew-symmetric) matrix.

8. **Conjugate transpose properties**.
   Let $A^* = \left(\bar{A}\right)^\intercal = \bar{A}^\intercal$ the conjugate transpose or Hermitian transpose, where $A^\intercal$ denotes the transpose matrix and $\bar{A}$ is matrix with complex conjugated entries. Then

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