ADDITION.

Since the printing of this work, I recognized that with the help of a very simple formula we could bring back to the differential calculus the solution of several problems that I returned to the integral calculus. I will, in the first place, give this formula; then, I will indicate its main applications.

From what has been said in the seventh lecture, if we denote by $x_0,\ X$ two values of x between which the functions $f(x)$ and $f'(x)$[1] remain continuous, and by θ a number less than unity, we will have

$$\frac{f(X)-f(x_0)}{X-x_0} = f'\big[x_0+\theta(X-x_0)\big].$$

Now, it is easy to see that reasoning entirely similar to that which we used to demonstrate the preceding equation will suffice to establish the formula

$$(1) \qquad \frac{f(X)-f(x_0)}{F(X)-F(x_0)} = \frac{f'\big[x_0+\theta(X-x_0)\big]}{F'\big[x_0+\theta(X-x_0)\big]},$$

This final portion of *Calcul infinitésimal* was attached to the text by our Mr. Cauchy after his initial submission, and while it awaited printing. This important addendum includes proofs of the Generalized Mean Value Theorem for Derivatives, Cauchy's version of l'Hôpital's Rule, and his conditional differential calculus derivation of Taylor's formula. One may wonder why such an important portion of the text is left for the end of the book and included as an apparent afterthought. Unfortunately, no record exists to fully explain the added notes and proofs, but a possible explanation could have to do with the situation Cauchy found himself. He and Ampère (the two professors responsible for the analysis course at the École Polytechnique) were under constant pressure from the Conseil d'Instruction to construct a course on analysis in which they (the Conseil) would approve – one which developed the calculus quicker and included more applications. Given the importance of Taylor series expansions in the curriculum at the time, Cauchy may have felt it necessary to include these new derivations so that Taylor's formula could be presented in a course immediately following the development of differential calculus. However, the true story is not known and is likely lost to history.

[1]The 1899 text has "$f(x)$ and $f(x)$," an obvious typographical error that does not occur in the original 1823 edition.

D. M. Cates, *Cauchy's Calcul Infinitésimal*,
https://doi.org/10.1007/978-3-030-11036-9

θ still denoting a number less than unity, and $F(x)$ a new function that, always growing or decreasing from the limit $x = x_0$ up to the limit $x = X$, remains continuous, along with its derivative $F'(x)$, between these same limits.[2]

We can also demonstrate formula (1) directly with the help of principles established in the sixth lecture (page 21).[3] In fact, it results from these principles that, in the admitted hypothesis, the function $F'(x)$ will constantly retain the same sign from $x = x_0$ up to $x = X$. As a result, if A and B represent the smallest and the largest of the values that the ratio $\frac{f'(x)}{F'(x)}$ receives in this interval, the two products

$$F'(x)\left[\frac{f'(x)}{F'(x)} - A\right] = f'(x) - A\,F'(x),$$
$$F'(x)\left[B - \frac{f'(x)}{F'(x)}\right] = B\,F'(x) - f'(x)$$

will remain, one and the other, constantly positive or constantly negative between the limits x_0, X of the variable x. Therefore, the two functions

$$f(x) - A\,F(x), \qquad B\,F(x) - f(x),$$

that have these same products for derivatives, will grow or will decline simultaneously from the first limit up to the second. So, the difference between the extreme values of the first function, namely

$$f(X) - f(x_0) - A\big[F(X) - F(x_0)\big],$$

and the difference between the extreme values of the last, namely

$$B\big[F(X) - F(x_0)\big] - \big[f(X) - f(x_0)\big],$$

will be two quantities of the same sign;[4] hence, we can conclude that the difference

$$f(X) - f(x_0)$$

will be contained between the two products

$$A\big[F(X) - F(x_0)\big], \qquad B\big[F(X) - F(x_0)\big],$$

and the fraction

$$\frac{f(X) - f(x_0)}{F(X) - F(x_0)},$$

[2]Cauchy realizes this simple first proof of the General Mean Value Theorem is not adequate, as he immediately begins a second, more convincing proof. The first proof does not guarantee the two θ's are identical. It is not clear why Cauchy even includes this flawed proof in his text.

[3]Cauchy seems to be referring to Problem I of his sixth lecture where he demonstrates, among other things, that a constantly increasing or decreasing function has a nonzero derivative that does not change sign on the interval of interest.

[4]Because the two functions are known to either both be increasing or both be decreasing.

between the limits A and B. Moreover, the two functions $f'(x)$, $F'(x)$, being continuous by hypothesis between the limits $x = x_0$, $x = X$, any quantity contained between A and B will be equivalent to an expression of the form[5]

$$\frac{f'\big[x_0 + \theta(X - x_0)\big]}{F'\big[x_0 + \theta(X - x_0)\big]},$$

θ denoting a number less than unity.[6] There will exist, therefore, a number of this kind that works to satisfy equation (1); it is this that is necessary to demonstrate.[7]

If we let $X = x_0 + h$, equation (1) will become

$$\frac{f(x_0 + h) - f(x_0)}{F(x_0 + h) - F(x_0)} = \frac{f'(x_0 + \theta h)}{F'(x_0 + \theta h)}. \tag{2}$$

This latter equation, which includes as a particular case equation (6) of the seventh lecture, is open to several important applications, as we will prove in a few words.

First, consider that the functions $f(x)$ and $F(x)$ vanish, one and the other, for $x = x_0$, and let, for brevity, $\theta h = h_1$. In this case, we will derive from formula (2)

$$\frac{f(x_0 + h)}{F(x_0 + h)} = \frac{f'(x_0 + h_1)}{F'(x_0 + h_1)}, \tag{3}$$

h_1 being a quantity of the same sign as h, but a value numerically less. If the functions

$$\begin{matrix} f(x), & f'(x), & f''(x), & \ldots, & f^{(n-1)}(x), \\ F(x), & F'(x), & F''(x), & \ldots, & F^{(n-1)}(x) \end{matrix}$$

all vanish for $x = x_0$, and remain continuous, as well as those of $f^{(n)}(x)$ and $F^{(n)}(x)$, between the limits $x = x_0$, $x = x_0 + h$, then, by supposing each of the functions

$$F(x), \quad F'(x), \quad F''(x), \quad \ldots, \quad F^{(n-1)}(x)$$

always increasing or always decreasing from the first limit up to the second, and denoting by

$$h_1, \quad h_2, \quad \ldots, \quad h_n$$

quantities of the same sign, but whose numerical values are from most to least, we would obtain along with equation (3), a sequence of similar equations which

[5]It helps to recall $A < \frac{f'(x)}{F'(x)} < B$, for all x values within the interval. The 1899 version contains a typographical error here. The reprint uses x in place of the correct X. This error does not occur in the 1823 edition.

[6]These two θ's <u>are</u> identical.

[7]Cauchy has just ended his proof of what is known today as a result very close to the Generalized Mean Value Theorem, sometimes called Cauchy's Mean Value Theorem, or the Extended Mean Value Theorem. Most modern versions of the theorem restrict $F(x)$ in a similar way by requiring $F(x) \neq 0$, for all x in (x_0, X).

together would comprise the formula

$$\text{(4)}\qquad \frac{f(x_0+h)}{F(x_0+h)} = \frac{f'(x_0+h_1)}{F'(x_0+h_1)} = \frac{f''(x_0+h_2)}{F''(x_0+h_2)} = \cdots = \frac{f^{(n)}(x_0+h_n)}{F^{(n)}(x_0+h_n)}.$$

If, in formula (4), we are content to set the first fraction equal to the last, the equation to which we will arrive can be written as follows

$$\text{(5)}\qquad \frac{f(x_0+h)}{F(x_0+h)} = \frac{f^{(n)}(x_0+\theta h)}{F^{(n)}(x_0+\theta h)},$$

θ always being a number less than unity. In conclusion, if in equation (5) we substitute for the finite quantity h, an infinitely small quantity[8] denoted by i, we will have

$$\text{(6)}\qquad \frac{f(x_0+i)}{F(x_0+i)} = \frac{f^{(n)}(x_0+\theta i)}{F^{(n)}(x_0+\theta i)}.$$

When, in formulas (5) and (6), we set[9]

$$F(x) = (x-x_0)^n,$$

we find

$$F^{(n)}(x) = 1\cdot 2\cdot 3\cdots n,$$

and by consequence,

$$\text{(7)}\qquad \frac{f(x_0+h)}{h^n} = \frac{f^{(n)}(x_0+\theta h)}{1\cdot 2\cdot 3\cdots n},$$

$$\text{(8)}\qquad \frac{f(x_0+i)}{i^n} = \frac{f^{(n)}(x_0+\theta i)}{1\cdot 2\cdot 3\cdots n}.$$

These last equations are in accordance with formulas (4) and (5) of the fifteenth lecture and coincide with formulas (17), (18) of the thirty-sixth. They can be employed with advantage, not only in the study of maxima and minima, but again in the determination of the values of fractions which are presented under the form $\frac{0}{0}$. Moreover, to resolve this last problem, it will most often suffice to resort to formula (6). Suppose, in fact, that the two terms of the fraction

$$\frac{f(x)}{F(x)},$$

[8] Recall from Cauchy's Lecture One, an "infinitely small quantity" is one whose limit is zero, not necessarily one that ever becomes zero.

[9] Cauchy stops short of taking limits here, interrupting his derivation with a specific example which he will employ later.

and their successive derivatives, up to those of order $n-1$, vanish for $x = x_0$. Formula (6) will generally remain valid for very small numerical values of i, because in general, each of the functions

$$F(x), \quad F'(x), \quad F''(x), \quad \ldots, \quad F^{(n-1)}(x)$$

will grow or will decline steadily from the particular value of x represented by x_0, up to a value very close; and, we will derive from this formula, by letting i converge toward the limit zero,[10]

$$\lim \frac{f(x_0+i)}{F(x_0+i)} = \lim \frac{f^{(n)}(x_0+\theta i)}{F^{(n)}(x_0+\theta i)} = \frac{f^{(n)}(x_0)}{F^{(n)}(x_0)}. \tag{9}$$

If we replace in formula (7), x_0 by zero, and the letter f by $\mathfrak{F}$, we will infer

$$\mathfrak{F}(h) = \frac{h^n}{1\cdot 2\cdot 3\cdots n}\mathfrak{F}^{(n)}(\theta h). \tag{10}$$

This last formula assumes that the functions

$$\mathfrak{F}(h), \quad \mathfrak{F}'(h), \quad \mathfrak{F}''(h), \quad \ldots, \quad \mathfrak{F}^{(n)}(h),$$

being continuous starting from the limit $h = 0$, all vanish, with the exception of $\mathfrak{F}^{(n)}(h)$, at the same time as the quantity h.[11]

Now, let $f(x)$ be an arbitrary function of the variable x, but such that

$$f(x+h), \quad f'(x+h), \quad f''(x+h), \quad \ldots, \quad f^{(n)}(x+h)$$

remain continuous with respect to h, starting from $h = 0$. With the help of formula (10), we can easily extract from $f(x+h)$, or to what amounts to the same thing, from the difference $f(x+h) - f(x)$, a sequence of terms proportional to the integer powers of h; and first, since the difference $f(x+h) - f(x)$, considered as a function of h, vanishes with h and has $f'(x+h)$ for its first order derivative, it is clear that by substituting this function for $\mathfrak{F}(h)$, and setting $n = 1$, we will derive from formula (10)

$$f(x+h) - f(x) = \frac{h}{1} f'(x+\theta h). \tag{11}$$

When, in the second member of the preceding equation, we replace θ by zero, we obtain the term $\frac{h}{1} f'(x)$, and by subtracting this term from the first member, we find

[10] After now taking limits, Cauchy has developed his form of l'Hôpital's Rule as an intermediate result on his way to Taylor's formula. This is the final of several inquiries within this text into how to evaluate an indeterminate fraction that presents itself in the form $\frac{0}{0}$.

[11] This condition will need to be satisfied throughout this derivation.

for the remainder a new function of h, namely

$$f(x+h)-f(x)-\frac{h}{1}f'(x).$$

As this new function of h vanishes with h, along with its first order derivative, and it has $f''(x+h)$ for its second order derivative, by substituting this new function of h for $\mathfrak{F}(h)$, and setting $n=2$, we will derive from formula (10)[12]

$$f(x+h)-f(x)-\frac{h}{1}f'(x)=\frac{h^2}{1\cdot 2}f''(x+\theta h). \tag{12}$$

If, in the second member of equation (11), we replace θ by zero, we will obtain the term $\frac{h^2}{1\cdot 2}f''(x)$, and by subtracting this term from the first member, we will find for the remainder a third function of h, namely

$$f(x+h)-f(x)-\frac{h}{1}f'(x)-\frac{h^2}{1\cdot 2}f''(x).$$

As this third function of h vanishes with h, along with its first and second order derivatives, and it has $f'''(x+h)$ for its third order derivative, by substituting this third function of h for $\mathfrak{F}(h)$, and setting $n=3$, we will derive from formula (10)

$$f(x+h)-f(x)-\frac{h}{1}f'(x)-\frac{h^2}{1\cdot 2}f''(x)=\frac{h^3}{1\cdot 2\cdot 3}f'''(x+\theta h), \tag{13}$$

etc. By continuing in this same manner, we will generally establish the formula

$$\left\{\begin{aligned} &f(x+h)-f(x)-\frac{h}{1}f'(x)-\frac{h^2}{1\cdot 2}f''(x)-\cdots\\ &\qquad -\frac{h^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)}f^{(n-1)}(x)=\frac{h^n}{1\cdot 2\cdot 3\cdots n}f^{(n)}(x+\theta h),\end{aligned}\right. \tag{14}$$

which coincides with equation (12) of the thirty-sixth lecture. If, in this formula, we replace x by zero, h by x, and f by F, $F(x)$ denoting an arbitrary function of x, we will find

$$\left\{\begin{aligned} &F(x)-F(0)-\frac{x}{1}F'(0)-\frac{x^2}{1\cdot 2}F''(0)-\cdots\\ &\qquad -\frac{x^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)}F^{(n-1)}(0)=\frac{x^n}{1\cdot 2\cdot 3\cdots n}F^{(n)}(\theta x).\end{aligned}\right. \tag{15}$$

This last equation coincides with formula (11) of the thirty-sixth lecture, and we can again directly achieve this in the following manner.

[12]The 1899 reprint has a typographical error. It reads $f(x+h)-f(x)-\frac{h}{1}f'(x)=\frac{h^2}{1\cdot 2}f'(x+\theta h)$. This error is not present in the 1823 edition.

Let $F(x)$ be any function of x, and let $\varpi(x)$ be an entire polynomial of degree $n-1$, subject to satisfy the condition equations

$$\varpi(0) = F(0), \qquad \varpi'(0) = F'(0),$$

$$\varpi''(0) = F''(0), \qquad \dots, \qquad \varpi^{(n-1)}(0) = F^{(n-1)}(0).$$

$\varpi^{(n)}(x)$ then being identically null, if in formula (10) we replace h by x, and $\mathfrak{F}(x)$ by $F(x) - \varpi(x)$, we will find

$$(16) \qquad F(x) - \varpi(x) = \frac{x^n}{1 \cdot 2 \cdot 3 \cdots n} F^{(n)}(\theta x);$$

and, moreover, as we will have (*see* the nineteenth lecture)

$$(17) \quad \begin{cases} \varpi(x) = \varpi(0) + \dfrac{x}{1}\varpi'(0) + \dfrac{x^2}{1 \cdot 2}\varpi''(0) + \cdots + \dfrac{x^{n-1}}{1 \cdot 2 \cdot 3 \cdots (n-1)}\varpi^{(n-1)}(0), \\ \qquad = F(0) + \dfrac{x}{1}F'(0) + \dfrac{x^2}{1 \cdot 2}F''(0) + \cdots + \dfrac{x^{n-1}}{1 \cdot 2 \cdot 3 \cdots (n-1)}F^{(n-1)}(0), \end{cases}$$

it is clear that formula (16) will lead to equation (15).

It is important to observe that, in all the cases where the second members of equations (14) and (15) converge toward zero for increasing values of n, we immediately deduce from these formulas the Theorems of Taylor and of Maclaurin.[13]

If, in formula (8), we have $x_0 = 0$, it would simply become

$$(18) \qquad \frac{f(i)}{i^n} = \frac{f^{(n)}(\theta i)}{1 \cdot 2 \cdot 3 \cdots n}.$$

This equation above assumes that the functions

$$f(i), \quad f'(i), \quad f''(i), \quad \dots, \quad f^{(n-1)}(i), \quad f^{(n)}(i),$$

being continuous for very small numerical values of i, all vanish, with the exception of the last, for $i = 0$. In this hypothesis, the ratios

$$\frac{f(i)}{i}, \quad \frac{f(i)}{i^2}, \quad \dots, \quad \frac{f(i)}{i^{n-1}},$$

being themselves equivalent to expressions of the form

$$\frac{f'(\theta i)}{1}, \quad \frac{f''(\theta i)}{1 \cdot 2}, \quad \dots, \quad \frac{f^{(n-1)}(\theta i)}{1 \cdot 2 \cdot 3 \cdots (n-1)},$$

[13] Cauchy's earlier work on Taylor's Theorem from Lectures Thirty-Five through Thirty-Seven involves integral calculus. Here, he shows the relationships can likewise be derived through differential calculus only with the observed limitations.

will all vanish with i. By consequence, i and $f(i)$ representing two infinitely small quantities,

$$\frac{f(i)}{i^n}$$

will be the first term of the geometric progression

$$f(i),\quad \frac{f(i)}{i},\quad \frac{f(i)}{i^2},\quad \frac{f(i)}{i^3},\quad \ldots \tag{19}$$

which ceases to be an infinitely small quantity, if $f^{(n)}(0)$ is the first of the quantities

$$f(0),\quad f'(0),\quad f''(0),\quad f'''(0),\quad \ldots \tag{20}$$

which ceases to be null. Add that, in the admitted hypothesis,

$$\frac{f^{(n)}(0)}{1\cdot 2\cdot 3\cdots n}$$

will be, by virtue of formula (18), the actual value of the ratio $\frac{f(i)}{i^n}$ corresponding to $i=0$.

The preceding considerations naturally lead us to partition the infinitely small quantities into different classes. Conceive, in fact, that all the quantities of this type that enter in a calculation are functions of one from among them designated by i, and we name $f(i)$ one of these functions. Several consecutive terms of the progression (19), counted starting from the first term, may be infinitesimals; and, depending on whether the number of these terms will be $1, 2, 3, \ldots,$ we will say that the quantity $f(i)$ is an infinitesimal of first, of second, of third class, etc. This granted, $f(i)$ will be an infinitesimal of the n^{th} *class*, if $\frac{f(i)}{i^n}$ is the first term of the progression (19) which ceases to vanish with i. In the same hypothesis, $f(i)$ will become what we call an *infinitesimal of order* n, if for decreasing numerical values of i, the ratio $\frac{f(i)}{i^n}$ converges toward a finite limit different from zero.

These definitions being accepted, we will immediately deduce the following propositions from the principles established above.

THEOREM I. – *When $f(i)$ is an infinitesimal of n^{th} class, $f^{(n)}(0)$ is the first term of the series (20) which ceases to be null. In the same case, $f(i)$ will be an infinitesimal of order n, if $f^{(n)}(0)$ obtains a finite value different from zero.*

THEOREM II. – *When, $f(i)$ being an infinitesimal of n^{th} class, the function $f(x)$ and its successive derivatives, up to order n, remain continuous between the limits $x=0, x=h$, we have, in designating by m an integer number less than or equal to n,*

$$f(h)=\frac{h^m}{1\cdot 2\cdot 3\cdots m}f^{(m)}(\theta h). \tag{21}$$

If, in this last formula, we replace h by i, and m by n, we will end up with equation (18), with the help of which we can establish the theorem that we will now state.

THEOREM III. – *Let $f(i)$ be an infinitely small quantity of order n. This quantity will change its sign with i if n is an odd number, and will be constantly affected by the same sign as $f^{(n)}(0)$ if n is an even number.*

Theorem III assumes, like formula (18), that the function $f(i)$ and its successive derivatives, up to that of order n, remain continuous with respect to i in the neighborhood of the particular value $i = 0$. If this condition has not been satisfied, the quantity designated by $f^{(n)}(0)$ could admit several values, and if these values are not all of the same sign, the theorem in question would cease to exist. It is this that would happen, for example, if we were to take for $f(i)$ the infinitely small quantity $\sqrt{i^2}$. In this hypothesis, the derived function[14]

$$f'(i) = \frac{i}{\sqrt{i^2}}$$

would admit one solution of continuity[15] corresponding to $i = 0$, and it would sometimes reduce to $+1$, sometimes to -1, depending on whether the value of i was positive or negative. Moreover, it is obvious that the quantity $\sqrt{i^2}$, although we are naturally inclined to consider it as an infinitesimal of first order, constantly remains positive and does not change sign with i. The same remark applies to the infinitely small quantity $\sqrt{i^6}$ that we are naturally led to regard as an infinitesimal of third order, etc.

Theorem I provides a very simple means to recognize the class or order of an infinitely small quantity. Thus, for example, we will conclude from this theorem that the quantities

$$\frac{1}{\mathbf{l}i}, \quad \sqrt{i}, \quad i^{\frac{2}{3}}, \quad \sin i$$

are four infinitesimals of first class, the last being only of first order. We will ensure by the same manner that the four quantities

$$\frac{i}{\mathbf{l}i}, \quad i^{\frac{3}{2}}, \quad \sin^2 i, \quad 1 - \cos i$$

are infinitesimals of second class, the last two being of second order; that the three quantities

$$\frac{i^2}{\mathbf{l}i}, \quad i^3, \quad i - \sin i$$

are infinitesimals of third class, the last two being of third order; and so on.

When we multiply an infinitesimal of n^{th} class or of n^{th} order by a constant quantity or by a function of i which has for a limit a finite quantity different from zero,

[14] Recall this is Cauchy's term, adopted from Lagrange, to denote our modern derivative.

[15] Another use of Cauchy's older term for a point of discontinuity.

we obviously obtain for a product another infinitesimal of the same class or of the same order as the first.

It is again easy to prove that, among infinitely small quantities, those which pertain to upper classes end up constantly obtaining the smallest numerical values. In fact, let $\varphi(i)$, $\chi(i)$ be two infinitely small quantities, the first of n^{th} class, the second of m^{th}, m being $< n$. The first of the two fractions $\frac{\varphi(i)}{i^m}$, $\frac{\chi(i)}{i^m}$ will be the only one which converges with i toward the limit zero; and as a result, the ratio that we obtain in dividing one by the other, or the fraction $\frac{\varphi(i)}{\chi(i)}$, will equivalently converge toward zero, which it cannot do without its numerical value dropping below unity, or in other words, without the numerical value of the numerator becoming less than that of the denominator.

Finally, we will easily establish the following proposition.

THEOREM IV. – *Designate by i and by $f(i)$ two infinitely small quantities. Zero will be the single value, or one of the values, that the ratio*

$$\frac{f(i)}{f'(i)} \tag{22}$$

receives when we will cause the quantity i to vanish.

Proof. – It is obviously sufficient to demonstrate Theorem IV in the case where the derived function $f'(i)$ vanishes at the same time as $f(i)$ for $i = 0$, expecting that the limit of the ratio $\frac{f(i)}{f'(i)}$, to be null in all other hypotheses, is then presented only under the indeterminate form $\frac{0}{0}$. Now, we will achieve this without difficulty with the help of formula (18), at least when the two functions $f(i)$, $f'(i)$ are continuous with respect to i in the vicinity of the particular value $i = 0$. In fact, if this condition is fulfilled, we will derive from formula (18), by setting $n = 1$,

$$f(i) = if'(\theta i), \tag{23}$$

and we will have, by consequence,

$$\frac{f(i)}{f'(i)} = i\,\frac{f'(\theta i)}{f'(i)}, \tag{24}$$

θ always denoting a number less than unity. Let us now conceive that, in formula (24), we allow the numerical value of i to decrease indefinitely. $f'(0)$ being null by hypothesis, and θi denoting a quantity contained between zero and i, $f'(\theta i)$ will converge toward the limit zero more rapidly than $f'(i)$, where it results that the fraction $\frac{f'(\theta i)}{f'(i)}$ will obtain a multitude of numerical values less than unity, and the product $i\,\frac{f'(\theta i)}{f'(i)}$ a multitude of values essentially null. Therefore, the limit, or one of the limits, toward which this same product will converge, and the ratio it represents, will be equal to zero.

Scholium I. – Theorem IV can be easily verified with regard to the functions

$$\sin i, \quad 1 - \cos i, \quad e^{-\left(\frac{1}{i}\right)^2}, \quad i^3 \sin \frac{1}{i}, \quad \dots .$$

It remains valid even in the case where the function $f(i)$ does not remain real and infinitely small as we attribute to the variable i values affected by a certain sign, as it happens, for example, when we take for $f(i)$ one of the functions

$$\boldsymbol{l}i, \quad \sqrt{i}, \quad e^{-\frac{1}{i}}, \quad e^{-\left(\frac{1}{i}\right)^3}, \quad \dots$$

which cease to be real or infinitely small when we give negative values to i. Finally, this theorem can remain valid, even though the function $f'(i)$ becomes discontinuous for $i = 0$. Thus, by supposing

$$f(i) = i \sin \frac{1}{i}, \tag{25}$$

we will find that the function

$$f'(i) = \sin \frac{1}{i} - \frac{1}{i} \cos \frac{1}{i} \tag{26}$$

becomes indeterminate, and by consequence, discontinuous for $i = 0$; and, if we then let i converge toward the limit zero, the value of the ratio (22) derived from equations (25) and (26), namely

$$\frac{f(i)}{f'(i)} = \frac{i}{1 - \frac{1}{i} \cot \frac{1}{i}}, \tag{27}$$

will admit an infinite number of limits, one of which will be equal to zero.

Scholium II. – Suppose that, the function $f(i)$ and its successive derivatives, up to that of order $n - 1$, being continuous with respect to i in the vicinity of the particular value $i = 0$, the n quantities

$$f(0), \quad f'(0), \quad f''(0), \quad \dots, \quad f^{(n-1)}(0) \tag{28}$$

vanish; and let us conceive that the numerical value of i comes to decrease indefinitely. Zero will be the limit, or one of the limits, toward which each of the ratios

$$\frac{f(i)}{f'(i)}, \quad \frac{f'(i)}{f''(i)}, \quad \frac{f''(i)}{f'''(i)}, \quad \dots, \quad \frac{f^{(n-1)}(i)}{f^{(n)}(i)} \tag{29}$$

will converge, and by consequence, their product, or the ratio

$$\frac{f(i)}{f^{(n)}(i)}. \tag{30}$$

We can say as much of the expressions

$$(31) \qquad \frac{f'(i)}{f^{(n)}(i)}, \quad \frac{f''(i)}{f^{(n)}(i)}, \quad \dots, \quad \frac{f^{(n-2)}(i)}{f^{(n)}(i)}$$

that we obtain by multiplying, one by the others, some of the ratios in question.

This page ends the published 1823 version of Mr. Cauchy's original *Calcul infinitésimal*. However, the 1899 reprint includes an added short research paper entitled "On the Formulas of Taylor and Maclaurin," which can be found in Appendix D.

APPENDIX A.

COURS D'ANALYSE – CHAPTER II, §III.

THEOREM I.[1] – *If, for increasing values of x, the difference*

$$f(x+1) - f(x)$$

converges toward a certain limit k, the fraction

$$\frac{f(x)}{x}$$

will converge at the same time toward the same limit.

Proof. – First, we suppose that the quantity k has a finite value, and designate by ε a number as small as we will wish. Since the increasing values of x make the difference

$$f(x+1) - f(x)$$

converge toward the limit k, we can give to the number h a value considerable enough, such that, x being greater than or equal to h, the difference in question is constantly contained between the limits

$$k - \varepsilon, \quad k + \varepsilon.$$

Included in this Appendix A are excerpts from Cauchy's *Cours d'analyse* Chapter II, §III, where he proves two straightforward convergence tests. These two theorems are referenced by Cauchy in his *Calcul infinitésimal* Lecture Seven by page number. However, they are particularly important and interesting because within Cauchy's actual proofs of these theorems we clearly see how he transforms his verbal definition of the limit into an algebra of inequalities squeeze.

[1]This is the precursor of Cauchy's Theorem II in his Lecture Thirty-Eight and eventually leads to the Ratio Convergence Test for infinite series.

D. M. Cates, *Cauchy's Calcul Infinitésimal*,
https://doi.org/10.1007/978-3-030-11036-9

This granted, if we designate by n any integer number, each of the quantities

$$\begin{aligned}&f(h+1)-f(h),\\&f(h+2)-f(h+1),\\&\dots\dots\dots\dots,\\&f(h+n)-f(h+n-1),\end{aligned}$$

and as a result, their arithmetic average, namely

$$\frac{f(h+n)-f(h)}{n},$$

will be found contained between the limits $k-\varepsilon$, $k+\varepsilon$. We will have, therefore,

$$\frac{f(h+n)-f(h)}{n}=k+\alpha,$$

α being a quantity contained between the limits $-\varepsilon$, $+\varepsilon$. Now, let

$$h+n=x.$$

The previous equation will become

$$\frac{f(x)-f(h)}{x-h}=k+\alpha, \tag{1}$$

and we will infer

$$f(x)=f(h)+(x-h)(k+\alpha),$$

$$\frac{f(x)}{x}=\frac{f(h)}{x}+\left(1-\frac{h}{x}\right)(k+\alpha). \tag{2}$$

In addition, to make the value of x grow indefinitely, it will suffice to let the integer number n grow indefinitely without changing the value of h. We suppose, by consequence, that in equation (2) we consider h as a constant quantity, and x as a variable quantity which converges toward the limit ∞. The quantities

$$\frac{f(h)}{x}, \qquad \frac{h}{x}$$

contained in the second member will converge toward the limit zero, and the second member itself toward a limit of the form

$$k+\alpha,$$

α being always contained between $-\varepsilon$ and $+\varepsilon$. As a result, the ratio

$$\frac{f(x)}{x}$$

will have for a limit a quantity contained between $k-\varepsilon$ and $k+\varepsilon$. This conclusion above remains valid, regardless of the smallness of the number ε, it follows that the limit in question will be precisely the quantity k. In other words, we will have

$$\lim \frac{f(x)}{x} = k = \lim \big[f(x+1) - f(x) \big]. \tag{3}$$

In the second place, we suppose $k = \infty$. Then, by denoting with H a number as large as we will wish, we can always attribute to the number h a value considerable enough, such that, x being greater than or equal to h, the difference

$$f(x+1) - f(x),$$

which converges toward the limit ∞, becomes constantly greater than H; and, by reasoning like that above, we will establish the formula

$$\frac{f(h+n) - f(h)}{n} > H.$$

If we now set $h + n = x$, we will find, in place of equation (2), the following formula

$$\frac{f(x)}{x} > \frac{f(h)}{x} + H\left(1 - \frac{h}{x}\right),$$

from which we will deduce, by letting x converge toward the limit ∞,

$$\lim \frac{f(x)}{x} > H.$$

The limit of the ratio

$$\frac{f(x)}{x}$$

will be, therefore, greater than the number H, however large it is. This limit, greater than any assignable number, can only be that of positive infinity.

In finishing, we suppose $k = -\infty$. To return this last case to the previous one, it will suffice to observe that the difference

$$f(x+1) - f(x),$$

having for a limit $-\infty$, the following

$$\big[-f(x+1) \big] - \big[-f(x) \big],$$

will have $+\infty$ for a limit. We will conclude that the limit of $\frac{-f(x)}{x}$ is equal to $+\infty$, and as a result, that of $\frac{f(x)}{x}$ to $-\infty$.

⋮

THEOREM II.[2] – *If, the function $f(x)$ being positive for very large values of x, the ratio*

$$\frac{f(x+1)}{f(x)}$$

converges, while x grows indefinitely, toward the limit k, the expression

$$[f(x)]^{\frac{1}{x}}$$

will converge at the same time toward the same limit.

Proof. – First, we suppose that the quantity k, necessarily positive, has a finite value, and designate by ε a number as small as we will wish. Since the increasing values of x make the ratio

$$\frac{f(x+1)}{f(x)}$$

converge toward the limit k, we can give to the number h a value considerable enough, such that, x being greater than or equal to h, the ratio in question is constantly contained between the limits

$$k-\varepsilon, \qquad k+\varepsilon.$$

This granted, if we designate by n any integer number, each of the quantities

$$\frac{f(h+1)}{f(h)}, \qquad \frac{f(h+2)}{f(h+1)}, \qquad \dots, \qquad \frac{f(h+n)}{f(h+n-1)},$$

and as a result, their geometric average, namely

$$\left[\frac{f(h+n)}{f(h)}\right]^{\frac{1}{n}},$$

will be found contained between the limits $k-\varepsilon$, $k+\varepsilon$. We will have, therefore,

$$\left[\frac{f(h+n)}{f(h)}\right]^{\frac{1}{n}} = k+\alpha,$$

[2] This is the precursor of Cauchy's Theorem I in his Lecture Thirty-Eight and eventually leads to the Root Convergence Test for infinite series.

α being a quantity contained between the limits $-\varepsilon$, $+\varepsilon$. Now, let

$$h+n=x.$$

The previous equation will become

$$\left[\frac{f(x)}{f(h)}\right]^{\frac{1}{x-h}}=k+\alpha, \tag{4}$$

and we will conclude

$$f(x)=f(h)\bigl(k+\alpha\bigr)^{x-h},$$

$$\bigl[f(x)\bigr]^{\frac{1}{x}}=\bigl[f(h)\bigr]^{\frac{1}{x}}(k+\alpha)^{1-\frac{h}{x}}. \tag{5}$$

In addition, to make the value of x grow indefinitely, it will suffice to let the integer number n grow indefinitely without changing the value of h. We suppose, by consequence, that in equation (5) we consider h as a constant quantity, and x as a variable quantity which converges toward the limit ∞. The quantities

$$\bigl[f(h)\bigr]^{\frac{1}{x}}, \qquad 1-\frac{h}{x}$$

contained in the second member will converge toward the limit 1, and the second member itself toward a limit of the form

$$k+\alpha,$$

α being always contained between $-\varepsilon$ and $+\varepsilon$. As a result, the expression

$$\bigl[f(x)\bigr]^{\frac{1}{x}}$$

will have for a limit a quantity contained between $k-\varepsilon$ and $k+\varepsilon$.

This conclusion above remains valid, regardless of the smallness of the number ε, it follows that the limit in question will be precisely the quantity k. In other words, we will have

$$\lim\bigl[f(x)\bigr]^{\frac{1}{x}}=k=\lim\frac{f(x+1)}{f(x)}. \tag{6}$$

In the second place, we suppose the quantity k infinite, that is to say, since this quantity is positive, $k=\infty$. Then, in denoting by H a number as large as we will wish, we can always attribute to the number h a value considerable enough, such that, x being greater than or equal to h, the ratio

$$\frac{f(x+1)}{f(x)},$$

which converges toward the limit ∞, becomes constantly greater than H; and, by reasoning like that above, we will establish the formula

$$\left[\frac{f(h+n)}{f(h)}\right]^{\frac{1}{n}} > H.$$

If we now set $h + n = x$, we will find, in place of equation (5), the following formula

$$\left[f(x)\right]^{\frac{1}{x}} > \left[f(h)\right]^{\frac{1}{x}} H^{1-\frac{h}{x}},$$

from which we will deduce, by letting x converge toward the limit ∞,

$$\lim \left[f(x)\right]^{\frac{1}{x}} > H.$$

The limit of the relationship

$$\left[f(x)\right]^{\frac{1}{x}}$$

will be, therefore, greater than the number H, however large it is. This limit, greater than any assignable number, can only be that of positive infinity.

⋮

APPENDIX B.

COURS D'ANALYSE – NOTE II.

We now turn to theorems on averages. As we have already said (*Preliminaries,* p. 14), we call an *average* among several given quantities a new quantity contained between the smallest and the largest of those that we consider.[1] From this definition, the quantity h will be an average between the two quantities g, k, or between several quantities among which one of the two that we just mentioned would be the largest and the other the smallest, if the two differences $g - h$ and $h - k$ are of the same sign. This granted, if, to denote an average among the quantities $a, a', a'', \ldots$ we employ, as in the *Preliminaries*, the notation

$$M\big(a, a', a'', \ldots\big),$$

we will establish without difficulty the following propositions.

⋮

THEOREM XII.[2] – *Let* $b, b', b'', \ldots$ *be several quantities of the same sign,* n *in number, and* $a, a', a'', \ldots$ *any quantities equal in number to those of the first. We will have*

(17) $$\frac{a + a' + a'' + \cdots}{b + b' + b'' + \cdots} = M\left(\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}, \ldots\right).$$

Note II is a large addendum of Cauchy's *Cours d'analyse* comprised of seventeen theorems. Of particular interest to us are his definition of an average and his Theorem XII, both of which play pivotal roles in his *Calcul infinitésimal*.

[1]This unusual definition of an average is used throughout the *Calcul infinitésimal* text, where it plays a prominent role. Cauchy's definition is nothing at all like the modern notion.

[2]Cauchy's Theorem XII is a result he calls upon in his *Calcul infinitésimal* Lectures Seven and Twenty-One. It also provides a wonderful detailed example of one of his proof techniques.

D. M. Cates, *Cauchy's Calcul Infinitésimal*,
https://doi.org/10.1007/978-3-030-11036-9

Proof. – Let g be the largest and k the smallest of the quantities

$$\frac{a}{b},\ \frac{a'}{b'},\ \frac{a''}{b''},\ \ldots.$$

The differences

$$\begin{array}{lcl} g-\dfrac{a}{b} & \text{and} & \dfrac{a}{b}-k, \\ g-\dfrac{a'}{b'} & \text{and} & \dfrac{a'}{b'}-k, \\ g-\dfrac{a''}{b''} & \text{and} & \dfrac{a''}{b''}-k, \\ \ldots\ldots & \ldots & \ldots\ldots, \end{array}$$

will all be positive. In multiplying the first two by b, the following two by b', etc., we will obtain the products

$$\begin{array}{lcl} gb-a & \text{and} & a-kb, \\ gb'-a' & \text{and} & a'-kb', \\ gb''-a'' & \text{and} & a''-kb'', \\ \ldots\ldots & \ldots & \ldots\ldots, \end{array}$$

which will all be of the same sign, as well as the quantities $b, b', b'', \ldots$. As a result, the sums of these two kinds of products, namely

$$g\big(b+b'+b''+\cdots\big)-\big(a+a'+a''+\cdots\big),$$

$$a+a'+a''+\cdots-k\big(b+b'+b''+\cdots\big),$$

and the quotients of these sums by $b+b'+b''+\cdots$, namely

$$g-\frac{a+a'+a''+\cdots}{b+b'+b''+\cdots},$$

$$\frac{a+a'+a''+\cdots}{b+b'+b''+\cdots}-k,$$

will again be quantities of the same sign, from which we will conclude

$$\frac{a+a'+a''+\cdots}{b+b'+b''+\cdots}=M(g,k)=M\left(\frac{a}{b},\frac{a'}{b'},\frac{a''}{b''},\ldots\right)$$

$\big($*see* in the *Preliminaries*, Theorem I and formula (6)$\big)$.

⋮

Corollary III.[3] – If we designate by $\alpha, \alpha', \alpha'', \ldots$ new quantities which are all of the same sign, we will have, by virtue of equation (17),

$$(20)\qquad \left\{ \begin{aligned} \frac{\alpha a + \alpha' a' + \alpha'' a'' + \cdots}{\alpha b + \alpha' b' + \alpha'' b'' + \cdots} &= M\left(\frac{\alpha a}{\alpha b}, \frac{\alpha' a'}{\alpha' b'}, \frac{\alpha'' a''}{\alpha'' b''}, \ldots\right) \\ &= M\left(\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}, \ldots\right). \end{aligned} \right.$$

This latter formula is sufficient to establish Theorem III of the *Preliminaries*.

[3]This obscure result is used by Cauchy within Lecture Twenty-One in his effort to show the definite integral exists independent of the derivative.

APPENDIX C.

COURS D'ANALYSE – NOTE III.

ON THE NUMERICAL RESOLUTION OF EQUATIONS.

To *numerically* resolve one or several equations is to find the numerical values of the unknowns which they contain, which obviously requires that the constants included in the equations in question are themselves reduced to numbers. We will only occupy ourselves here with equations which contain one unknown, and we will start by establishing, to their regard, the following theorems.

THEOREM I. – *Let $f(x)$ be a real function of the variable x, which remains continuous with respect to this variable between the limits $x = x_0$, $x = X$.*[1] *If the two quantities $f(x_0)$, $f(X)$ are of different signs, we will be able to satisfy the equation*

(1) $$f(x) = 0$$

by one or several real values of x contained between x_0 and X.[2]

Cauchy's Note III is again located in the ending material of his 1821 *Cours d'analyse* and in general deals with finding the numerical solutions to equations. It is also a fairly lengthy addendum to his original work which includes multiple theorems and problems. Only his Theorem I from this Note III is included in our Appendix C, as it is here where we find his wonderful proof of the Intermediate Value Theorem (a result absolutely fundamental to the entire *Calcul infinitésimal* text).

[1]Cauchy intuitively recognizes one of the most important properties of continuous single-variable functions, namely their preservation of sequential limits. Given a sequence $\{x_k\}_{k=1}^{\infty} = \{x_1, x_2, x_3, \ldots\}$ such that $\lim_{k\to\infty} x_k = a$, if $f(x)$ is continuous, we will have $\lim_{x\to a} f(x) = f(a)$. In other words, a limit can be brought inside a continuous function.

[2]This result is today known as Bolzano's Theorem and is a special case of the Intermediate Value Theorem. It is named for Bernard Bolzano (1781–1848), a talented Bohemian monk who developed work similar to Cauchy's on this subject during roughly the same period. Bolzano's work is greatly underappreciated and not nearly as well known today as it deserves.

D. M. Cates, *Cauchy's Calcul Infinitésimal*,
https://doi.org/10.1007/978-3-030-11036-9

Proof. – Let x_0 be the smallest of the two quantities x_0, X. Allow

$$X - x_0 = h,$$

and designate by m any integer number greater than unity. Since, of the two quantities $f(x_0)$, $f(X)$, one is positive, the other is negative, if we form the sequence[3]

$$f(x_0),\quad f\left(x_0 + \frac{h}{m}\right),\quad f\left(x_0 + \frac{2h}{m}\right),\quad \dots,\quad f\left(X - \frac{h}{m}\right),\quad f(X),$$

and that, in this sequence, we successively compare the first term with the second, the second with the third, the third with the fourth, etc., we will necessarily end by finding one or several times, two consecutive terms that will have different signs. Let

$$f(x_1),\quad f(X')$$

be two terms of this type, x_1 being the smaller of the two corresponding values of x. We will obviously have

$$x_0 < x_1 < X' < X$$

and

$$X' - x_1 = \frac{h}{m} = \frac{1}{m}(X - x_0).$$

Having determined x_1 and X' like we have just said, we can do the same between these two new values of x, by placing two others, x_2 and X'', which, substituted in $f(x)$, produce the results of opposite signs, and which work to satisfy the conditions

$$x_1 < x_2 < X'' < X',$$

$$X'' - x_2 = \frac{1}{m}(X' - x_1) = \frac{1}{m^2}(X - x_0).$$

By continuing thusly, we will obtain: 1° a series[4] of increasing values of x, namely

$$x_0,\quad x_1,\quad x_2,\quad \dots; \tag{2}$$

2° a series of decreasing values

$$X,\quad X',\quad X'',\quad \dots, \tag{3}$$

which, surpassing the first set by quantities, respectively, equal to the products

$$1 \times (X - x_0),\quad \frac{1}{m} \times (X - x_0),\quad \frac{1}{m^2} \times (X - x_0),\quad \dots,$$

[3]Cauchy uses the phrase *la suite* here, meaning *sequence*.

[4]Cauchy uses the phrase *une série*, meaning *series*, to describe the lists in (2) and (3). He surely knows these are both sequences and is likely using the word "series" in an informal sense.

will eventually differ from these first values by as little as we will want. We must deduce that the general terms of the series (2) and (3) will converge toward a common limit.[5] Let a be this limit. Since the function $f(x)$ remains continuous from $x = x_0$ up to $x = X$, the general terms of the following series

$$f(x_0), \quad f(x_1), \quad f(x_2), \quad \ldots,$$
$$f(X), \quad f(X'), \quad f(X''), \quad \ldots$$

will equally converge toward the common limit $f(a)$; and, since by approaching this limit they will always keep different signs, it is clear that the quantity $f(a)$, necessarily finite, cannot differ from zero. By consequence, we will satisfy the equation

$$f(x) = 0 \tag{1}$$

by attributing to the variable x the particular value a contained between x_0 and X. In other words,

$$x = a \tag{4}$$

will be a *root* of equation (1).[6]

Scholium I. – If, after having extended the series (2) and (3) up to the terms

$$x_n \qquad \text{and} \qquad X^{(n)}$$

(n denoting any integer number), we take the half-sum of these two terms for the approximated value of the root a, the error committed will be smaller than their half difference, namely

$$\frac{1}{2}\frac{X - x_0}{m^n}.$$

[5]Cauchy is arguing this must be a common limit, otherwise there would be two distinct limits, a_1 and a_2, where it would always be possible to find an n such that $\frac{1}{m^n}(X - x_0) < |a_1 - a_2|$.

[6]Cauchy is doing something quite bold here. He is showing the existence of a quantity, a, by demonstrating one can get closer and closer to it. Cauchy is doing something extraordinary for his time, a method which is now very common in analysis, but not in his day. He is assigning a value to the limit of a converging sequence. This technique is also employed by Cauchy in a loose sense within his Lecture One when he defines e, and later when he defines his definite integral in Lecture Twenty-One. In short, with this analysis, Cauchy is attempting to prove the existence of a limit. However, it is certainly true there is a great deal Cauchy does not say here. Although he makes an attempt at showing a exists, he does not prove it rigorously by today's standards. The final details took a great deal of work in later years, culminating with what is known as the *completeness* property of the real numbers. Cauchy assumes this completeness property without knowing it (which was common for the time) through this entire text. The rigorous treatment of completeness was not dealt with properly until the late 19th century, approximately fifty years after the publication of Cauchy's *Calcul infinitésimal* text, by many mathematicians including Richard Dedekind (1831–1916) and Georg Cantor (1845–1918).

Since this latter expression decreases indefinitely as the measure of n increases, it follows that, by calculating a sufficient number of terms of the two series, we will finally obtain approximated values of the root a as close as we will want.

Scholium II. – If there exists between the limits x_0, X several real roots of equation (1), the preceding method will make known a part of them and sometimes will furnish all of them. Then, we will find for x_1 and X', as well for x_2 and X'', ... several systems of values that enjoy the same properties.

Scholium III. – If the function $f(x)$ is constantly increasing or constantly decreasing from $x = x_0$ up to $x = X$, there will only exist between these limits a single value of x that works to satisfy equation (1).

Corollary I. – If equation (1) does not have real roots contained between the limits x_0, X, the two quantities

$$f(x_0), \qquad f(X)$$

will be of the same sign.

Corollary II. – If, in the wording of Theorem I, we replace the function $f(x)$ by

$$f(x) - b$$

(b denoting a constant quantity), we will obtain precisely Theorem IV of Chapter II (§II). In the same hypothesis, by following the method indicated above, we will numerically determine the roots of the equation

$$f(x) = b \tag{5}$$

contained between x_0 and X.[7]

⋮

[7]One of his most beautiful proofs, Cauchy has just proven the Intermediate Value Theorem.

APPENDIX D.

ON THE FORMULAS OF TAYLOR AND MACLAURIN.

We easily prove that, in the case where the fraction

$$\frac{\mathfrak{F}(h)}{h^{n-1}} \tag{1}$$

vanishes for $h = 0$, we have

$$\mathfrak{F}(h) = \frac{h^n}{1 \cdot 2 \cdot 3 \cdots n} \mathfrak{F}^{(n)}(\theta h), \tag{2}$$

θ denoting an unknown number, but less than unity. Now, equation (2), with the help of which we can directly establish the theory of maxima or minima, and settle the values of the fractions which present under the form $\frac{0}{0}$, also leads very simply to the series of Taylor and to the determination of the remainder which must complete this series. In fact, we will successively derive from equation (2):

1° By setting $\mathfrak{F}(h) = f(x+h) - f(x)$, and $n = 1$,

$$f(x+h) - f(x) = \frac{h}{1} f'(x + \theta h); \tag{3}$$

then, by setting $f'(x+h) = f'(x) + H_1$,

$$H_1 = \frac{f(x+h) - f(x) - \frac{h}{1} f'(x)}{h};$$

2° By setting $\mathfrak{F}(h) = f(x+h) - f(x) - hf'(x)$, and $n = 2$,

$$f(x+h) - f(x) - hf'(x) = \frac{h^2}{1 \cdot 2} f''(x + \theta h); \tag{4}$$

This research paper is NOT included in the published edition of Cauchy's original 1823 *Calcul infinitésimal*. However, the editors of *Œuvres complètes d'Augustin Cauchy* attached it at the end of their published reprint in 1899. It is only included here for the sake of thoroughness.

D. M. Cates, *Cauchy's Calcul Infinitésimal*,
https://doi.org/10.1007/978-3-030-11036-9

then, by setting $f''(x+\theta h) = f''(x) + H_2$,

$$\frac{1}{1\cdot 2} H_2 = \frac{f(x+h) - f(x) - hf'(x) - \frac{h^2}{1\cdot 2} f''(x)}{h^2};$$

3° By setting $\mathfrak{F}(h) = f(x+h) - f(x) - hf'(x) - \frac{h^2}{1\cdot 2} f''(x)$, and $n = 3$,

$$f(x+h) - f(x) - hf'(x) - \frac{h^2}{1\cdot 2} f''(x) = \frac{h^3}{1\cdot 2\cdot 3} f'''(x+\theta h); \tag{5}$$

then, by setting $f'''(x+\theta h) = f'''(x) + H_3$,

$$\frac{1}{1\cdot 2\cdot 3} H_3 = \frac{f(x+h) - f(x) - hf'(x) - \frac{h^2}{1\cdot 2} f''(x) - \frac{h^3}{1\cdot 2\cdot 3} f'''(x)}{h^3}.$$

By continuing in the same manner, and observing that the quantities

$$H_1, \quad \frac{1}{1\cdot 2} H_2, \quad \frac{1}{1\cdot 2\cdot 3} H_3, \quad \ldots$$

all vanish along with h, we will generally establish the equation

$$\left\{\begin{aligned} &f(x+h) - f(x) - hf'(x) - \frac{h^2}{1\cdot 2} f''(x) - \cdots \\ &\qquad - \frac{h^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)} f^{(n-1)}(x) = \frac{h^n}{1\cdot 2\cdots n} f^{(n)}(x+\theta h) \end{aligned}\right. \tag{6}$$

or

$$\left\{\begin{aligned} f(x+h) = f(x) &+ \frac{h}{1} f'(x) + \frac{h^2}{1\cdot 2} f''(x) \\ &+ \cdots + \frac{h^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)} f^{(n-1)}(x) \\ &+ \frac{h^n}{1\cdot 2\cdots n} f^{(n)}(x+\theta h). \end{aligned}\right. \tag{7}$$

If we replace x by 0, and h by x, we will find

$$\left\{\begin{aligned} f(x) = f(0) &+ \frac{x}{1} f'(0) + \frac{x^2}{1\cdot 2} f''(0) \\ &+ \cdots + \frac{x^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)} f^{(n-1)}(0) \\ &+ \frac{x^n}{1\cdot 2\cdot 3\cdots n} f^{(n)}(\theta x). \end{aligned}\right. \tag{8}$$

It follows from formula (7) that the function $f(x+h)$ can be considered as composed of an entire function of h, namely

$$(9) \qquad f(x) + \frac{h}{1} f'(x) + \frac{h^2}{1 \cdot 2} f''(x) + \cdots + \frac{h^{n-1}}{1 \cdot 2 \cdot 3 \cdots (n-1)} f^{(n-1)}(x),$$

and of a remainder, namely

$$(10) \qquad \frac{h^n}{1 \cdot 2 \cdot 3 \cdots n} f^{(n)}(x + \theta h).$$

When this remainder becomes infinitely small for infinitely large values of the number n, we can assert that the series

$$(11) \qquad f(x), \quad h f'(x), \quad \frac{h^2}{1 \cdot 2} f''(x), \quad \ldots$$

is convergent, and that it has for a sum, $f(x+h)$. Therefore, we can then write the equation

$$(12) \qquad f(x+h) = f(x) + \frac{h}{1} f'(x) + \frac{h^2}{1 \cdot 2} f''(x) + \cdots,$$

which is precisely the formula of Taylor. Similarly, if the remainder

$$(13) \qquad \frac{x^n}{1 \cdot 2 \cdot 3 \cdots n} f^{(n)}(\theta x)$$

becomes infinitely small for the infinite values of n, equation (8) will lead to the following

$$(14) \qquad f(x) = f(0) + \frac{x}{1} f'(0) + \frac{x^2}{1 \cdot 2} f''(0) + \cdots,$$

which is precisely the formula of Maclaurin.

It is often useful to substitute for the expressions (10) and (13) other equivalent expressions. We can achieve this as follows.

Denote by $\varphi(z)$ that which becomes the first member of equation (6) when we replace h by $h - z$, and x by $x + z$, or in other words, the remainder that we obtain when we expand $f(x+h)$ according to ascending and integer powers of $h - z$, and that we stop at the power of degree $n - 1$; so that we would have

$$(15) \qquad \begin{cases} f(x+h) = f(x+z) + \dfrac{h-z}{1} f'(x+z) + \cdots \\ \qquad\qquad + \dfrac{(h-z)^{n-1}}{1 \cdot 2 \cdot 3 \cdots (n-1)} f^{(n-1)}(x+z) + \varphi(z). \end{cases}$$

$\varphi(0)$ will represent the common value of each of the members of equation (6). Moreover, by differentiating formula (15) with respect to z, we will find

$$\varphi'(z) = -\frac{(h-z)^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)} f^{(n)}(x+z), \tag{16}$$

and we will deduce

$$\frac{\varphi(h)-\varphi(0)}{h} = -\frac{(h-\theta h)^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)} f^{(n-1)}(x+\theta h), \tag{17}$$

or, because $\varphi(h)$ is obviously reduced to zero,

$$\varphi(0) = \frac{(h-\theta h)^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)} h f^{(n)}(x+\theta h). \tag{18}$$

The preceding value of $\varphi(0)$ is nothing other than the remainder of the series of Taylor presented in a new form. If, in this remainder, we replace x by 0, and h by x, we will obtain the remainder of the series of Maclaurin under the following form

$$x\frac{(x-\theta x)^{n-1}}{1\cdot 2\cdot 3\cdots(n-1)} f^{(n)}(\theta x). \tag{19}$$

It is sufficient, in several cases, to substitute this last product into expression (13) to establish formula (14). Suppose, for example,

$$f(x) = (1+x)^{\mu}, \tag{20}$$

μ denoting a real constant. Expressions (13) and (19) will become, respectively,

$$\frac{\mu(\mu-1)\cdots(\mu-n+1)}{1\cdot 2\cdot 3\cdots n} x^n (1+\theta x)^{\mu-n} \tag{21}$$

and

$$\frac{\mu(\mu-1)\cdots(\mu-n+1)}{1\cdot 2\cdot 3\cdots(n-1)} x^n (1-\theta)^{n-1}(1+\theta x)^{\mu-n}. \tag{22}$$

This granted, we will easily prove: 1° with the help of expression (21) that the equation

$$(1+x)^{\mu} = 1 + \frac{\mu}{1}x + \frac{\mu(\mu-1)}{1\cdot 2}x^2 + \cdots \tag{23}$$

remains valid when the numerical value of the ratio

$$\frac{x}{1+\theta x} \tag{24}$$

is less than unity; 2° with the help of expression (22), that equation (23) remains valid when the product

$$x\,\frac{1-\theta}{1+\theta x} \tag{25}$$

is contained between the limits -1 and 1. As a result, it will suffice to employ expression (21) to establish formula (23) between the limits $x = 0$, $x = 1$. But, will revert back to expression (22) if we want to extend the same formula to all values of x contained between the limits $x = -1,\ x = +1$.

APPENDIX E.

PAGINATION OF THE 1823 & 1899 EDITIONS.

D. M. Cates, *Cauchy's Calcul Infinitésimal*,
https://doi.org/10.1007/978-3-030-11036-9

One cannot help but take notice of the symmetry of the original published edition in 1823. Each and every one of Cauchy's forty lectures is exactly four pages in length.

References

1. Belhoste, B., *Augustin-Louis Cauchy: A Biography,* Trans. Frank Ragland, Springer-Verlag, New York, 1991.
2. Birkhoff, G. (editor), *A Source Book in Classical Analysis*, Harvard University Press, Cambridge, Massachusetts, 1973.
3. Bottazzini, U., *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*, Springer-Verlag, New York, 1986.
4. Boyer, C. B., *The History of The Calculus and its Conceptual Development (The Concepts of the Calculus)*, Dover Publications, Inc., New York, 1949.
5. Bradley, R. E. and Sandifer, C. E., *Cauchy's Cours d'analyse: An Annotated Translation*, Springer, New York, 2009.
6. Bressoud, D. M., *A Radical Approach to Real Analysis*, The Mathematical Association of America, Washington, DC, 1994.
7. Cauchy, A. L., *Cours d'analyse de l'École royale polytechnique*, de Bure, Paris, 1821. Reprinted in *Œuvres complètes d'Augustin Cauchy*, Series II, Volume III, Gauthier-Villars, Paris, 1897.
8. Cauchy, A. L., *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*, de Bure, Paris, 1823. Reprinted in *Œuvres complètes d'Augustin Cauchy*, Series II, Volume IV, Gauthier-Villars, Paris, 1899.
9. Dunham, W., *The Calculus Gallery: Masterpieces from Newton to Lebesgue*, Princeton University Press, Princeton, New Jersey, 2005.
10. Edwards, C. H., *The Historical Development of the Calculus*, Springer-Verlag, New York, 1979.
11. Grabiner, J. V., *The Origins of Cauchy's Rigorous Calculus*, The MIT Press, Cambridge, Massachusetts, 1981.
12. Grattan-Guiness, I., *The Development of the Foundations of Mathematical Analysis from Euler to Riemann*, M.I.T. Press, Cambridge, Massachusetts, 1970.
13. Hairer, E. and Wanner, G., *Analysis by Its History*, Springer, New York, 1996.
14. Iacobacci, R. F., *Augustin-Louis Cauchy and the Development of Mathematical Analysis*, Ph.D. Dissertation, New York University, 1965.
15. Jahnke, H. N. (editor), *A History of Analysis*, American Mathematical Society, Providence, Rhode Island, 2003.
16. Smith, D. E. (editor), *A Source Book in Mathematics*, McGraw-Hill Book Company, Inc., New York, 1929.
17. Smithies, F., *Cauchy and the Creation of Complex Function Theory*, Cambridge University Press, Cambridge, United Kingdom, 1997.
18. Spivak, M., *Calculus, Fourth Edition*, Publish or Perish, Inc., Houston, Texas, 2008.
19. Struik, D. J., *A Source Book in Mathematics, 1200–1800*, Harvard University Press, Cambridge, Massachusetts, 1969.

D. M. Cates, *Cauchy's Calcul Infinitésimal*,
https://doi.org/10.1007/978-3-030-11036-9

Index

D. M. Cates, *Cauchy's Calcul Infinitésimal*,
https://doi.org/10.1007/978-3-030-11036-9

I

J

K

L

M