Appendix A
Internal Stability Notions

The purpose of this appendix is to introduce the basic properties arising in the characterization of the long-term qualitative behavior of solutions of unforced, time invariant differential systems. Notation and terminology are those of the Introduction (Chap. 1). However, since the interest is focused on the state variable, in this appendix the observation map is ignored. From the mathematical point of view, the systems considered in this chapter reduce therefore to systems of ordinary differential equations (in general, nonlinear)[1]

$$\dot{x} = f(x) \tag{A.1}$$

where $x \in \mathbf{R}^n$. Recall that a *solution* of (A.1) is any differentiable function $x = \varphi(t)$ defined on some interval $I \subseteq \mathbf{R}$ such that $\dot{\varphi}(t) = f(\varphi(t))$ for each $t \in I$. We will assume that the function f in (A.1) is defined and continuous together with its first partial derivatives, for each $x \in \mathbf{R}^n$; moreover, we assume that it satisfies the inequality

$$||f(x)|| \leq a||x|| + b$$

for some positive constants a, b. Under these assumptions, for each initial pair (t_0, x_0) existence and uniqueness of solutions are guaranteed, and we may further take $I = \mathbf{R}$ without loss of generality [24]. Moreover, since the function f does not depend explicitly on t, according to Proposition 1.9, the system (A.1) is time invariant; therefore it is not restrictive to assume $t_0 = 0$.

The notions introduced in this appendix are often referred to as *internal stability notions*, in order to emphasize the difference with the notion of *external stability* introduced in Chap. 1 and studied in detail in Chap. 6.

[1]The notions we are going to introduce are applied in this book essentially for the case of linear systems; however, they can be better understood when referred to a general system of the type (A.1).

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A.1 The Flow Map

A solution of (A.1) can be regarded as a parameterized curve $x = \varphi(t)$ of $\mathbf{R}^n$. For each $t \in \mathbf{R}$, the tangent vector to such a curve at the point x coincides with $f(x)$. For this reason, the function $f : \mathbf{R}^n \to \mathbf{R}^n$ which defines (A.1) is also called a *vector field*.

The image of a solution $x = \varphi(t)$ of (A.1) is called an *orbit* or a *trajectory*. It is important do not confuse the graph of a solution $\varphi(t)$, which is a subset of $\mathbf{R}\times\mathbf{R}^n$, with the orbit of $\varphi(t)$, which coincides with the set $\varphi(\mathbf{R})$ and it is a subset of $\mathbf{R}^n$. We may also view the orbit of φ as the orthogonal projection of the graph of φ on $\mathbf{R}^n$, along the time axis (see Fig. A.1).

We already mentioned that under the stated assumptions, for each initial condition (A.1) has a unique solution. In order to emphasize its global validity, we my reformulate this property writing that if $x = \varphi(t)$ and $x = \psi(t)$ are two arbitrary solutions of (A.1), then

$$\exists \bar{t} : \quad \varphi(\bar{t}) = \psi(\bar{t}) \quad \Longrightarrow \quad \varphi(t) = \psi(t) \quad \forall t \in \mathbf{R} \, . \tag{A.2}$$

The geometric interpretation of (A.2) is that if the graphs of the two solutions have a common point, then they must coincide. We may also interpret the time invariance property from a geometrical point of view: the time translation of the graph of a solution is again the graph of a (in general, different) solution. All the solutions obtained as time translation of a fixed solution obviously are equivalent parametrization of the same curve, and so they define the same orbit (see again Fig. A.1). This fact admits a converse.

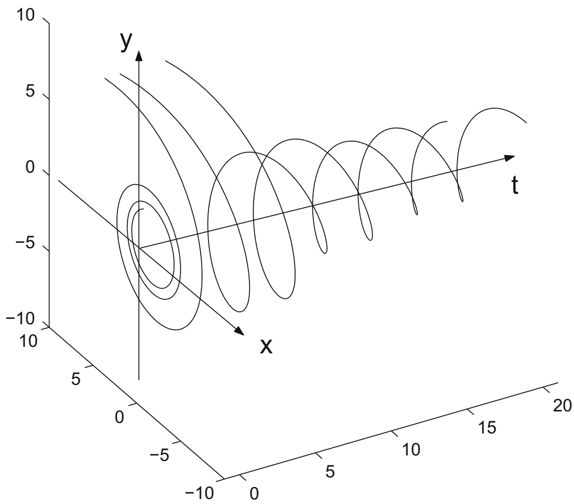

Fig. A.1 Two solutions and their projections

Lemma A.1 *Let $\varphi(t)$ and $\psi(t)$ be two arbitrary solutions of (A.1) defined for each $t \in \mathbf{R}$. Then,*

$$\exists t_1, t_2 : \quad \varphi(t_1) = \psi(t_2) \quad \Longrightarrow \quad \psi(t) = \varphi(t + T) \ \ \forall t \in \mathbf{R}\,, \tag{A.3}$$

where we set $T = t_1 - t_2$.

Proof Let $T = t_1 - t_2$ and $\chi(t) = \varphi(t + T)$. Clearly, $\chi(t)$ is a solution. It satisfies the initial condition

$$\chi(t_2) = \varphi(t_2 + t_1 - t_2) = \varphi(t_1)\,.$$

But also $\psi(t)$ is a solution which, by hypothesis, satisfies the same condition. Because of the uniqueness property, we have

$$\chi(t) = \varphi(t + T) = \psi(t) \qquad \forall t \in \mathbf{R}\,.$$

■

The meaning of Lemma A.1 is that if two orbits have a common point, then they must coincide (the reader is warned to notice the difference between (A.2) and (A.3)). In other words, there is a unique orbit passing through every point of $\mathbf{R}^n$. The orbits of the system (A.1) fill the space and are displayed in such a way to form a partition of the space. We might define an equivalence relation, saying that two points are equivalent when they lie on the same orbit. In the particular case $n = 2$, we can image that the orbits form a picture in the plane. This picture is also called *state configuration* or *phase portrait*. To denote the solution of the Cauchy problem

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \tag{A.4}$$

we use the notation[2]

$$x = x(t, x_0) \tag{A.5}$$

which has the advantage of emphasizing, beside the time variable t, also the initial state x_0. Equation (A.5) define a function from $\mathbf{R} \times \mathbf{R}^n$ to $\mathbf{R}^n$: this is called the *flow map* generated by the vector field f. It can be interpreted as a function of t for each fixed x_0, or as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$, parameterized by t.

Remark A.1 In (A.5), the variable t should be thought of not as the indication of a precise instant of time, but rather as the indication of the duration of a process, that is the length of the time interval needed to transfer the state of the system from x_0 to $x(t, x_0)$. ■

[2]Note that (A.5) is nothing else than (1.12) adapted to the case of (A.1).

Proposition A.1 *The flow map of the vector field f satisfies the following properties:*

$$x(0, x_0) = x_0 \tag{A.6}$$

$$x(t, x(\tau, x_0)) = x(t + \tau, x_0) \tag{A.7}$$

for each $t, \tau \in \mathbf{R}$ and $x_0 \in \mathbf{R}^n$.

A.2 Equilibrium Points and Stability in Lyapunov Sense

Roughly speaking, *internal stability* means that in the absence of external energy supply, the state of a system evolves remaining in a neighborhood of a rest point, and eventually approaches a rest point.

Let the unforced, time invariant differential system (A.1) be given. We say that $\bar{x} \in \mathbf{R}^n$ is an *equilibrium point* if the constant function $\varphi(t) \equiv \bar{x}$ is a solution. Sometimes, equilibrium points are also called *rest* or *singular*, or even *critical points*. If $\bar{x}$ is an equilibrium point, then the orbit issuing from $\bar{x}$ reduces to the singleton $\{\bar{x}\}$.

Proposition A.2 *The point $\bar{x}$ is an equilibrium point if and only if $f(\bar{x}) = 0$.*

We say that the equilibrium point $\bar{x}$ is *isolated* if there exists a neighborhood $\mathcal{O}$ of $\bar{x}$ such that $f(x) \neq 0$ for each $x \in \mathcal{O}$, $x \neq \bar{x}$.

Definition A.1 Let $\bar{x}$ be an equilibrium point. We say that $\bar{x}$ is *stable (in Lyapunov sense)* for the system (A.1) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0 - \bar{x}\| < \delta \quad \Longrightarrow \quad \|x(t; x_0) - \bar{x}\| < \varepsilon, \quad \forall t \geq 0.$$

Definition A.2 Let $\bar{x}$ be an equilibrium point. We say that $\bar{x}$ is *attractive* if there exists $\delta_0 > 0$ such that, for each initial state x_0 for which $\|x_0 - \bar{x}\| < \delta_0$, one has

$$\lim_{t \to +\infty} x(t; x_0) = \bar{x} \, . \tag{A.8}$$

Definition A.3 Let $\bar{x}$ be an equilibrium point. If $\bar{x}$ is both stable and attractive, we say that it is *asymptotically stable*. Moreover, if (A.8) holds for each $x_0 \in \mathbf{R}^n$, $\bar{x}$ is called *globally asymptotically stable*. Finally, if the solutions converge to $\bar{x}$ with an exponential decay i.e., there exist $M > 0$, $\alpha > 0$ such that

$$\forall x_0 \ \text{ with } \ \|x_0 - \bar{x}\| < \delta_0 \ \text{ we have } \ \|x(t; x_0) - \bar{x}\| \leq M e^{-\alpha t} \, , \tag{A.9}$$

we speak about *exponential stability*. The supremum of the numbers α such that (A.9) holds for some suitable M, is called the *decay rate*.

We end this chapter by the following notion, very useful in the analysis of the qualitative behavior of the unforced system (A.1).

Definition A.4 Let K be a closed subset of $\mathbf{R}^n$. We say that K is *dynamically invariant* for the system (A.1) if for each $x_0 \in K$ one has $x(t; x_0) \in K$ for every $t \in \mathbf{R}$.

Appendix Summary

Appendix A recalls some mathematical definitions concerning stability. The informal term "stability" actually involves the notion of stability in the sense of Lyapunov and the notion of attraction. In general, these notions are mutually independent, but in the case of linear systems the latter implies the former. Moreover, in the case of linear systems there is no way to distinguish the local and global aspects. For these reasons, in this Appendix, and only in this Appendix, we refer to general (nonlinear) systems of ordinary differential equations.

Appendix B
Laplace Transform

In this appendix we recall some basic facts about Laplace transform, that are needed for the applications considered in this book. In view of our limited goals and for sake of simplicity, the subject will not be treated with the maximal generality and mathematical rigor. In particular, the Dirac-delta function and its Laplace transform will be introduced only at heuristic level. For a more formal presentation, the reader can be addressed to one of many existing books on this topic, for instance [15].

B.1 Definition and Main Properties

Let $f : [0, +\infty) \to \mathbf{R}$ be a piecewise continuous function.

Definition B.1 We say that f is a *subexponential function* if there exist real constants $M > 0$ and α such that

$$|f(t)| \le Me^{\alpha t} \qquad \forall t \in [0, +\infty) \ . \tag{B.1}$$

To each piecewise continuous, subexponential function we can associate a real number σ_0, defined as the infimum of the numbers α for which there exists M such that (B.1) holds. This number σ_0 is called the *order* of f.

Lemma B.1 *Let f be a subexponential, piecewise continuous function, and let s be any complex number such that* $\operatorname{Re} s > \sigma_0$. *Then,*

$$\lim_{\xi \to +\infty} f(\xi)e^{-s\xi} = 0 \ .$$

Proof If $H(\xi)$ is a complex function of one real variable, $\lim_{\xi \to +\infty} H(\xi) = 0$ is equivalent to

$$\lim_{\xi \to +\infty} \operatorname{Re} H(\xi) = \lim_{\xi \to +\infty} \operatorname{Im} H(\xi) = 0 \ .$$

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But for each $z \in \mathbf{C}$, we have

$$|\mathrm{Re}\, z| \leq |z| \qquad \text{and} \qquad |\mathrm{Im}\, z| \leq |z| \, .$$

Thus, it is sufficient to prove that

$$\lim_{\xi \to +\infty} |f(\xi)e^{-s\xi}| = \lim_{\xi \to +\infty} |f(\xi)|e^{-\xi \mathrm{Re}\, s} = 0 \, .$$

Let α be a real number such that $\sigma_0 < \alpha < \mathrm{Re}\, s$. We have

$$|f(\xi)|e^{-\xi \mathrm{Re}\, s} \leq M e^{\xi(\alpha - \mathrm{Re}\, s)}$$

and the statement follows since $\alpha - \mathrm{Re}\, s < 0$.

■

The Laplace transform allows us to associate a function $F : \mathbf{C} \to \mathbf{C}$ to each piecewise continuous, subexponential function $f : [0, +\infty) \to \mathbf{R}$. Before giving the formal definition, we still need some preliminary results.

Lemma B.2 *Let f be a piecewise continuous, subexponential function, whose order is σ_0. For each complex number s such that* $\mathrm{Re}\, s > \sigma_0$, *the improper integral*

$$\int_0^{+\infty} f(t)e^{-st} dt$$

is absolutely convergent.

Proof The absolute convergence of the improper integral $\int_0^{+\infty} H(t)dt$ of a function $H : \mathbf{R} \to \mathbf{C}$ is equivalent to the convergence of both the integrals

$$\int_0^{+\infty} |\mathrm{Re}\, H(t)| \, dt \qquad \text{and} \qquad \int_0^{+\infty} |\mathrm{Im}\, H(t)| \, dt \, . \tag{B.2}$$

As already noticed in the proof of Lemma B.1, it is therefore sufficient to show that the integral

$$\int_0^{+\infty} |H(t)| dt$$

is convergent. In our case,

$$|f(t)e^{-st}| = |f(t)||e^{-st}| = |f(t)|e^{-t \mathrm{Re}\, s} \leq M e^{t(\alpha - \mathrm{Re}\, s)} \, .$$

According to the definition of σ_0, we can chose α in such a way that $\sigma_0 < \alpha < \mathrm{Re}\, s$, so that $\alpha - \mathrm{Re}\, s < 0$. The convergence of the two integrals (B.2) is guaranteed by comparison. Notice that if we set $s = \sigma + \mathrm{i}\,\omega$, we have

$$\int_0^{+\infty} |\mathrm{Re}\,(f(t)e^{-st})|dt = \int_0^{+\infty} |f(t)e^{-t\sigma}\cos(-\omega t)|dt$$

and

$$\int_0^{+\infty} |\mathrm{Im}\,(f(t)e^{-st})|dt = \int_0^{+\infty} |f(t)e^{-t\sigma}\sin(-\omega t)|dt\ .$$

■

We are finally ready to introduce the main definition of this Appendix.

Definition B.2 Let f be a piecewise continuous, subexponential function, defined in $[0, +\infty)$, whose order is σ_0. The *Laplace transform* of f is the complex function

$$s \mapsto F(s) = \int_0^{+\infty} f(t)e^{-st}dt \tag{B.3}$$

defined on the domain $\{s \in \mathbf{C} : \mathrm{Re}\, s > \sigma_0\}$.

It is convenient to remark that F could be coincident with the restriction to the half-plane $\{s \in \mathbf{C} \ : \ \mathrm{Re}\, s > \sigma_0\}$ of a function $\tilde{F} : \mathbf{C} \to \mathbf{C}$ defined in a broader domain (to this respect, see Remark B.1). We also remark that (B.3) is meaningful even if f is a complex function of one real variable.

The operator defined by (B.3) will be denoted by the symbol $\mathcal{L}$. We will also agree to denote by the same letter (respectively, small and capital) the function to be transformed and its Laplace transform. Hence, we write

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t)e^{-st}dt.$$

Next we review some important properties of the Laplace transform.

Property 1 (Linearity) *Let f and g be two piecewise continuous, subexponential functions, defined on the interval* $[0, +\infty)$*, of order respectively σ_1 and σ_2. Then, for each $a, b \in \mathbf{R}$ the function $af + bg$ is subexponential of order* $\sigma_0 = \max\{\sigma_1, \sigma_2\}$. *Moreover,*

$$aF(s) + bG(s) = \mathcal{L}[af(t) + bg(t)] \tag{B.4}$$

for $\mathrm{Re}\, s > \sigma_0$*, where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.*

Proof It is immediate to check that $af + bg$ is subexponential (take a value of α greater than both σ_1 and σ_2). Formula (B.4) is a trivial consequence of the properties of the integrals.

■

Property 2 (Rescaling) *Let f be a piecewise continuous, subexponential function, defined in* $[0, +\infty)$*, of order σ_0, and let $F(s) = \mathcal{L}[f(t)]$ for* $\mathrm{Re}\, s > \sigma_0$*. Then, for each* $a > 0$*, $g(t) = f(at)$ is a subexponential function of order $a\sigma_0$, and*

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad for\ \ \mathrm{Re}\, s > a\sigma_0\ . \tag{B.5}$$

Proof From (B.1) we have easily $|f(at)| \leq Me^{\alpha at} = Me^{\beta t}$ for $\beta = \alpha a > a\sigma_0$. Setting $\tau = at$, we therefore have

$$\int_0^{+\infty} f(at)e^{-st}\, dt = \frac{1}{a}\int_0^{+\infty} f(\tau)e^{-\frac{s}{a}\tau}\, d\tau = \frac{1}{a}\int_0^{+\infty} f(\tau)e^{-r\tau}\, d\tau = \frac{1}{a}F(r)$$

provided that $\mathrm{Re}\, r > \sigma_0$, where $r = s/a$. But $\mathrm{Re}\, r = \mathrm{Re}\,(s/a) = (\mathrm{Re}\, s)/a$, and hence requiring $\mathrm{Re}\, r > \sigma_0$ it is equivalent to require $\mathrm{Re}\, s > a\sigma_0$.

■

Property 3 (Right translation) *Let f be a piecewise continuous, subexponential function, defined in* $[0, +\infty)$*, of order σ_0, and let $F(s) = \mathcal{L}[f(t)]$ for* $\mathrm{Re}\, s > \sigma_0$. *In addition, let*

$$g(t) = \begin{cases} 0 & for\ 0 \leq t \leq c \\ f(t-c) & for\ t > c\ , \end{cases}$$

where $c > 0$. Then g is a subexponential function of order σ_0, and $\mathcal{L}[g(t)] = e^{-cs}F(s)$ for $\mathrm{Re}\, s > \sigma_0$.

Proof The reader can easily check that g is a subexponential function. Moreover, by the definition of g, we have

$$\int_0^{+\infty} g(t)e^{-st}\, dt = \int_c^{+\infty} f(t-c)e^{-st}\, dt\ .$$

Finally, the substitution $\tau = t - c$ yields

$$\int_0^{+\infty} g(t)e^{-st}\, dt = \int_0^{+\infty} f(\tau)e^{-s(\tau+c)}\, d\tau = e^{-cs}F(s).$$

■

Property 4 (Multiplication by t^n) *Let f be a piecewise continuous, subexponential function, defined in* $[0, +\infty)$*, of order σ_0, and let $F(s) = \mathcal{L}[f(t)]$ for* $\mathrm{Re}\, s > \sigma_0$. *For each $n \in \mathbf{N}$, the function $t^n f(t)$ is a subexponential function of order σ_0, and*

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}\ , \qquad \mathrm{Re}\, s > \sigma_0\ . \tag{B.6}$$

Proof The subexponential property of $t^n f(t)$ is a consequence of Lemma 2.1. Formula (B.6) can be proved by induction. The case $n = 0$ follows immediately by definition. Assuming that (B.6) holds for $n = k$, the case $n = k + 1$ can be obtained by computing the derivative[3] of both sides with respect to s.

■

[3]Here and in other following proofs, the crucial point consists in exchanging the order of certain operations like limits, derivatives, integrals. The correctness of such exchanges requires some uni-

Property 5 (Multiplication by e^{at}) *Let f be a piecewise continuous, subexponential function, defined in* $[0, +\infty)$*, of order σ_0, and let $F(s) = \mathcal{L}[f(t)]$ for* $\mathrm{Re}\, s > \sigma_0$*. Let moreover $a \in \mathbf{R}$. Then, $e^{at} f(t)$ is a subexponential function of order $\sigma_0 + a$, and*

$$\mathcal{L}[e^{at} f(t)] = F(s-a) \qquad \mathrm{Re}\, s > \sigma_0 + a\ . \tag{B.7}$$

Proof It is easy to check that $e^{at} f(t)$ is subexponential. Moreover,

$$\int_0^{+\infty} e^{at} f(t) e^{-st}\, dt = \int_0^{+\infty} f(t) e^{-t(s-a)}\, dt = \int_0^{+\infty} f(t) e^{-tr}\, dt = F(r)$$

with $\mathrm{Re}\, r > \sigma_0$, where $r = s - a$. But $\mathrm{Re}\, r = \mathrm{Re}\, s - a$ and so $\mathrm{Re}\, r > \sigma_0$ is equivalent to $\mathrm{Re}\, s > \sigma_0 + a$.

■

Property 5 extends in the case $a \in \mathbf{C}$, with $\mathrm{Re}\, s > \sigma_0 + \mathrm{Re}\, a$.

Next properties are the most important from our point of view, since they refer to the behavior of the operator $\mathcal{L}$ with respect to the operations of the differential and integral calculus.

Property 6 *Assume that f is a piecewise continuous, subexponential function defined in* $[0, +\infty)$*, of order σ_0. Assume further that its derivative f' exists and it is piecewise continuous in* $[0, +\infty)$*. If $F(s) = \mathcal{L}[f(t)]$ for* $\mathrm{Re}\, s > \sigma_0$*, then the Laplace transform of f' exists, and it is given by*

$$\mathcal{L}[f'(t)] = -f(0) + sF(s) \qquad for\ \ \mathrm{Re}\, s > \sigma_0\ . \tag{B.8}$$

Proof We want to compute

$$\lim_{\xi \to +\infty} \int_0^{\xi} f'(t) e^{-st}\, dt\ .$$

Integrating by parts we have

$$\begin{aligned} \int_0^{\xi} f'(t) e^{-st}\, dt &= f(t) e^{-st}\Big|_0^{\xi} + s \int_0^{\xi} f(t) e^{-st}\, dt \\ &= f(\xi) e^{-s\xi} - f(0) + s \int_0^{\xi} f(t) e^{-st}\, dt\ . \end{aligned}$$

The statement is proved, taking the limit for $\xi \to +\infty$, and taking into account Lemma B.1.

■

formity assumptions, which are not difficult to ensure when we work with continuous functions defined on compact intervals. In our framework (complex variables, unbounded intervals) there are some additional technical difficulties. We do not enter in these details..

Property 7 *Let f be a piecewise continuous, subexponential function, defined in $[0, +\infty)$, of order σ_0, and let $F(s) = \mathcal{L}[f(t)]$ for $\operatorname{Re} s > \sigma_0$. Then, each antiderivative of f is subexponential of order $\max\{\sigma_0, 0\}$ and we have*

$$\mathcal{L}\left[\int_0^t f(\rho)\,d\rho\right] = \frac{F(s)}{s} \qquad for \;\; \operatorname{Re} s > \max\{\sigma_0, 0\}\,. \tag{B.9}$$

Proof The first statement is left as an exercise. As far as (B.9) is concerned, we can apply again the integration by part rule:

$$\begin{aligned}\int_0^\xi \left(\int_0^t f(\rho)\,d\rho\right) e^{-st}\,dt &= \int_0^\xi h(t)e^{-st}\,dt \\ &= \frac{h(t)e^{-st}}{-s}\Big|_0^\xi + \frac{1}{s}\int_0^\xi h'(t)e^{-st}\,dt \\ &= \frac{h(\xi)e^{-s\xi}}{-s} - \frac{h(0)}{-s} + \frac{1}{s}\int_0^\xi f(t)e^{-st}\,dt\,.\end{aligned}$$

Noticing that $h(0) = 0$, the conclusion follows by taking the limit for $\xi \to +\infty$.

■

Obviously, (B.8) and (B.9) can be iterated, which gives:

$$\mathcal{L}[f^{(k)}(t)] = -f^{(k-1)}(0) - sf^{(k-2)}(0) - \cdots - s^{k-1}f(0) + s^k F(s)\,, \tag{B.10}$$

$$\mathcal{L}\left[\int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-1}} f(t_k)\,dt_k\,dt_{k-1}\ldots dt_1\right] = \frac{F(s)}{s^k}\,. \tag{B.11}$$

Property 8 *Let f and g be two piecewise continuous, subexponential functions, defined in $[0, +\infty)$, of order respectively σ_1 and σ_2. Let $F(s) = \mathcal{L}[f(t)]$ for $\operatorname{Re} s > \sigma_1$ and $G(s) = \mathcal{L}[g(t)]$ for $\operatorname{Re} s > \sigma_2$. Let moreover*

$$h(t) = \int_0^t f(t-\rho)g(\rho)\,d\rho\,. \tag{B.12}$$

Then, h is a subexponential function, and $\mathcal{L}[h(t)] = F(s)G(s)$ for $\operatorname{Re} s > \max\{\sigma_1, \sigma_2\}$.

Property 8 answers the question of finding a function $h(t)$ such that $\mathcal{L}[h(t)] = F(s)G(s)$, assuming that $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$ are known. We remark that (B.12) is well defined, since for $\rho \in [0, t]$ we have $t - \rho \geq 0$. Introducing the following extensions of the functions f and g:

$$\tilde{f}(t) = \begin{cases} 0 & \text{if } t < 0 \\ f(t) & \text{if } t \geq 0 \end{cases} \qquad \text{and} \qquad \tilde{g}(t) = \begin{cases} 0 & \text{if } t < 0 \\ g(t) & \text{if } t \geq 0\,, \end{cases}$$

we can write

$$h(t) = \int_0^t f(t-\rho)g(\rho)\,d\rho = \int_{-\infty}^{+\infty} \tilde{f}(t-\rho)\tilde{g}(\rho)\,d\rho\,.$$

Now let $p(\cdot)$ and $q(\cdot)$ be two piecewise continuous arbitrary functions defined on the whole of **R**. The *convolution* between p and q is defined by

$$(p*q)(t) = \int_{-\infty}^{+\infty} p(t-\rho)q(\rho)\,d\rho\,,$$

provided that the integral is convergent. Thus, we may reformulate (B.12) by writing

$$\mathcal{L}[(\tilde{f}*\tilde{g})(t)] = F(s)G(s)\,.$$

Proof of Property 8 To prove that $h(t)$ is a subexponential function of order $\max\{\sigma_1, \sigma_2\}$ is a simple exercise. With the notation above, we may also write $h(t) = \int_0^{+\infty} \tilde{f}(t-\rho)g(\rho)\,d\rho$. Thus

$$\begin{aligned}\mathcal{L}[h(t)] &= \int_0^{+\infty}\left(\int_0^{+\infty} \tilde{f}(t-\rho)g(\rho)\,d\rho\right)e^{-st}\,dt\\ &= \int_0^{+\infty} g(\rho)\left(\int_0^{+\infty} \tilde{f}(t-\rho)e^{-st}\,dt\right)d\rho\,.\end{aligned}$$

By virtue of Property 3, we finally get

$$\mathcal{L}[h(t)] = \int_0^{+\infty} g(\rho)e^{-\rho s}F(s)\,d\rho = F(s)\int_0^{+\infty} g(\rho)e^{-\rho s}\,d\rho = F(s)G(s)\,.$$

■

B.2 A List of Laplace Transforms

B.2.1 Elementary Functions

We now compute the Laplace transform of some elementary functions.

Proposition B.1 *Let $f(t) \equiv 1$ for $t \geq 0$. Then*

$$\mathcal{L}[f(t)] = \frac{1}{s} \qquad for\ \ \mathrm{Re}\, s > 0\,.$$

Proof We have

$$\mathcal{L}[f(t)] = \int_0^{+\infty} e^{-st}\,dt = \lim_{\xi\to+\infty}\int_0^{\xi} e^{-st}\,dt = \lim_{\xi\to+\infty}\frac{e^{st}}{-s}\Big|_0^{\xi} = \lim_{\xi\to+\infty}\frac{e^{-s\xi}}{-s} + \frac{1}{s}\,.$$

The conclusion follows from the remark that if $\operatorname{Re} s > 0$, then we have $\lim_{\xi\to+\infty} e^{-s\xi} = 0$.

■

Proposition B.2 *The Laplace transform of the restrictions to the interval* $[0, +\infty)$ *of the power functions, the exponential function, and the trigonometric functions are given by:*

$$\mathcal{L}[at] = \frac{a}{s^2} \qquad for\ \operatorname{Re} s > 0\,; \tag{B.13}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad for\ \operatorname{Re} s > 0 \qquad (n \in \mathbf{N})\,; \tag{B.14}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad for\ \operatorname{Re} s > a\,; \tag{B.15}$$

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2} \qquad for\ \operatorname{Re} s > 0\,; \tag{B.16}$$

$$\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2} \qquad for\ \operatorname{Re} s > 0\,. \tag{B.17}$$

Proof We will prove formulæ (B.13), (B.14) and (B.15) as applications of Proposition B.1 and Properties 1, 2, 4 and 5. We begin with (B.13). We have

$$\mathcal{L}[at] = a\mathcal{L}[t] = a\mathcal{L}[t\cdot 1]\,.$$

By applying Property 4 and recalling that $\mathcal{L}[1] = 1/s$, we conclude

$$\mathcal{L}[at] = \frac{a}{s^2}\,.$$

Now consider formula (B.14). We have

$$\mathcal{L}[t^n] = \mathcal{L}[t^n\cdot 1] = (-1)^n\frac{d^nF(s)}{ds^n} \qquad \text{where} \qquad F(s) = \frac{1}{s}\,.$$

From this, by mathematical induction, we get $F^{(n)}(s) = (-1)^n n! s^{-(n+1)}$. As far as (B.15) is concerned, we just need to remark that

$$\mathcal{L}[e^{at}] = \mathcal{L}[e^{at}\cdot 1] = \frac{1}{s-a}\,.$$

Now we consider the Laplace transform of the trigonometric functions. As already noticed, the operator $\mathcal{L}$ applies to functions $f : \mathbf{R} \to \mathbf{C}$, as well. Hence we can compute $\mathcal{L}[\cos \omega t]$ making use of the Euler formula

$$\cos \omega t = \frac{e^{\mathrm{i}\,\omega t} + e^{-\mathrm{i}\,\omega t}}{2} \ .$$

We have

$$\mathcal{L}[\cos \omega t] = \frac{1}{2}\left(\mathcal{L}[e^{\mathrm{i}\,\omega t}] + \mathcal{L}[e^{-\mathrm{i}\,\omega t}]\right) = \frac{1}{2}\left(\frac{1}{s + \mathrm{i}\,\omega} + \frac{1}{s - \mathrm{i}\,\omega}\right)$$

for $\mathrm{Re}\, s > 0$. This yields

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2} \ .$$

The proof of (B.17) is similar. ■

B.2.2 Discontinuous functions

From Propositions B.1 and B.2 it is possible to deduce the Laplace transform of some functions which are commonly used in signal theory. For instance,

$$U(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

is called the *unit step* or also the *Heaviside function*. It represents a signal which instantaneously jumps from zero to 1 (switch-on). The function $f(t) \equiv 1$ considered in Proposition B.1 coincides with the restriction of $U(t)$ to $[0, +\infty)$. Taking into account the definition of Laplace transform, with a little abuse of notation we will write

$$\mathcal{L}(U(t)) = \frac{1}{s} \qquad (\mathrm{Re} s > 0) \ .$$

Remark B.1 Let us remark that the complex function of a complex variable which associates s to its inverse $1/s$ is defined for each $s \neq 0$. Nevertheless, it is not correct to say that such a function is the Laplace transform of $U(t)$. Indeed the identity $\mathcal{L}[U(t)] = 1/s$ holds only for $\mathrm{Re}\, s > 0$. In other words, $\mathcal{L}[U(t)]$ coincides with the restriction to the positive complex half plane of the function $1/s$.

■

The function $U(t)$ allows us to represent other types of discontinuous signals, whose Laplace transform can be easily computed by applying Properties 1 and 3. For instance,

(i) *switch-on* for $t = c$

$$f(t) = \begin{cases} 0 & \text{for } t < c \\ 1 & \text{for } t \geq c \end{cases}$$

is equivalent to $f(t) = U(t - c)$;

(ii) *switch-off* for $t = c$

$$f(t) = \begin{cases} 1 & \text{for } t \leq c \\ 0 & \text{for } t > 0 \end{cases}$$

is equivalent to $f(t) = 1 - U(t - c)$ (or $= U(c - t)$);

(iii) *rectangular impulse*

$$f(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } a \leq t \leq b \\ 0 & \text{for } t > b \end{cases}$$

is equivalent to $f(t) = U(t - a) - U(t - b)$.

The function $U(t)$ is also useful to represent piecewise elementary functions. For instance the function

$$f(t) = \begin{cases} t & \text{for } t < 1 \\ t^2 & \text{for } t \geq 1 \end{cases}$$

can be written as

$$f(t) = t[1 - U(t - 1)] + t^2 U(t - 1) .$$

With the same abuse of notation as above, we can think of (B.13) as the Laplace transform of a signal of the form

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ at & \text{for } t \geq 0 \end{cases}$$

while (B.16), (B.17) provide the Laplace transform of signal of sinusoidal shape (but vanishing for $t < 0$).

B.2.3 Dirac Delta Function

One of the most important signals typically employed in system theory is the *unit impulse* function, denoted by the symbol $\delta(t)$ and also called Dirac δ function.

$$\delta(t) = \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \,. \end{cases} \tag{B.18}$$

This definition is an ideal representation of a signal of very large energy, concentrated at a single point. Of course, (B.18) is nonsense from a rigorous point of view, since it really does not define a function $\mathbf{R} \to \mathbf{R}$. There is a theory, called *distribution theory*, based on a generalization of the notion of function, which allows us to formally introduce and study objects like (B.18). To our purposes, it is sufficient to think of the function $\delta(t)$ as the limit of suitable sequences; for instance

$$\delta(t) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \left(U(t + \varepsilon) - U(t - \varepsilon) \right) \,. \tag{B.19}$$

From (B.19) we infer in particular that

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = \int_{-k}^{k} \delta(t)\, dt = 1 \qquad \forall k > 0 \,. \tag{B.20}$$

It is possible to define the sum and the multiplication between generalized functions like $\delta(t)$. It is also possible to give a sense to certain operators of the differential calculus for generalized functions, but this is not required in this book. We limit ourselves to recall some facts and properties related to the Dirac delta function.

An impulse of intensity k concentrated at a point $a \in \mathbf{R}$ is represented by $k \cdot \delta(t - a)$. We have

$$\int_{-\infty}^{+\infty} k \cdot \delta(t - a) dt = k \tag{B.21}$$

and

$$\int_{-\infty}^{+\infty} f(t)\delta(t - a) dt = f(a) \tag{B.22}$$

provided that the function f is continuous at the point $t = a$. Finally,

$$\int_{-\infty}^{t} \delta(\tau) d\tau = U(t) \tag{B.23}$$

and

$$\mathcal{L}[\delta(t)] = 1 \,. \tag{B.24}$$

Formulæ (B.20)–(B.24) can be formally proved in the context of distribution theory. They can be also justified heuristically on the base of (B.19). For instance, concerning (B.24) we suggest the following argument.

$$\begin{aligned}\mathcal{L}[\delta(t)] &= \mathcal{L}\left[\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\big(U(t)-U(t-\varepsilon)\big)\right]\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\,\{\mathcal{L}[U(t)]-\mathcal{L}[U(t-\varepsilon)]\} = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left\{\frac{1}{s}-\frac{e^{-\varepsilon s}}{s}\right\}\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{s}\cdot\frac{1-e^{-\varepsilon s}}{\varepsilon} = -\lim_{\varepsilon\to 0^+}\frac{e^{-\varepsilon s}-1}{\varepsilon s} = 1\,.\end{aligned}$$

B.3 Inverse Transform

The following proposition guarantees the existence of the inverse transformation $\mathcal{L}^{-1}$.

Proposition B.3 *Let f and g be two continuous functions defined on $[0,+\infty)$, and let $F(s)=\mathcal{L}[f(t)]$, $G(s)=\mathcal{L}[g(t)]$, both defined for $\operatorname{Re} s>\sigma_0$. If $F(s)=G(s)$ for all s such that $\operatorname{Re} s>\sigma_0$, then $f(t)=g(t)$ for all $t\geq 0$.*

This proposition states that $f(t)$ can be uniquely reconstructed from $F(s)$, and so it allows us to define $\mathcal{L}^{-1}$. However, it should be noted that if $f(t)$ is the restriction to $[0,+\infty)$ of a function $\varphi(t)$ defined on a larger interval $[a,+\infty)$ with $-\infty\leq a<0$, in general it is not possible to reconstruct $\varphi(t)$ on the interval $(a,0)$ by applying $\mathcal{L}^{-1}$ to $F(s)$. Nevertheless, in many applications this problem can be overcome if it is known a priori that $f(t)$ is real analytic.

The inverse transformation $\mathcal{L}^{-1}$ can be explicitly represented by a suitable formula, but the use of this formula is not needed in this book. We limit ourselves to remark that $\mathcal{L}^{-1}$ is, like $\mathcal{L}$, a linear operator.

B.4 The Laplace Transform of a Vector Function

The extension of the Laplace transform to vector functions

$$f : [0,+\infty)\to\mathbf{R}^n \quad\text{or}\quad f : [0,+\infty)\to\mathbf{C}^n$$

where $f=(f_1,\ldots,f_n)$, is straightforward: under the assumption that each component is a subexponential function, we set $\mathcal{L}[f]=(\mathcal{L}[f_1],\ldots,\mathcal{L}[f_n])$.

The aforementioned properties of the Laplace transform can be easily extended, as well. In addition, we have

$$\mathcal{L}[Mf(t)] = M\mathcal{L}[f(t)] \tag{B.25}$$

for each matrix M with real or complex constant entries. We are especially interested in the transform of the exponential matrix and in the convolution product formula (Property 8).

Proposition B.4 *Let A be a square matrix (real or complex), c be a constant vector and $b(t)$ be a vector function whose components are piecewise continuous, subexponential functions. Let σ_0 be the maximal real part of the eigenvalues of A. Then, for each $s \in \mathbf{C}$ such that $\mathrm{Re}\, s > \sigma_0$, we have:*

$$\mathcal{L}[e^{tA}c] = -(A - sI)^{-1}c\ , \tag{B.26}$$

$$\mathcal{L}\left[\int_0^t e^{(t-\tau)A}b(\tau)d\tau\right] = -(A - sI)^{-1}B(s)\ . \tag{B.27}$$

Proof By definition,

$$\mathcal{L}[e^{tA}c] = \int_0^{+\infty} e^{-st}e^{tA}cdt = \int_0^{+\infty} e^{t(A-sI)}c\,dt = \lim_{\xi\to\infty}\int_0^{\xi} e^{t(A-sI)}c\,dt\ .$$

If $\mathrm{Re}\, s > \sigma_0$, then det $(A - sI) \neq 0$ and the inverse $(A - sI)^{-1}$ exists. On the other hand, it is well known that for each square matrix M, the exponential matrix e^{tM} admits a derivative and $(e^{tM})' = Me^{tM} = e^{tM}M$. This implies that if M is invertible, $\int e^{tM} = M^{-1}e^{tM} = e^{tM}M^{-1}$. We can therefore proceed in the following way:

$$\begin{aligned}\mathcal{L}[e^{tA}c] &= \lim_{\xi\to\infty}(A - sI)^{-1}e^{t(A-sI)}c\Big|_0^{\xi} \\ &= \lim_{\xi\to\infty}\left[(A - sI)^{-1}e^{\xi(A-sI)}c - (A - sI)^{-1}c\right]\ .\end{aligned}$$

The assumption that $\mathrm{Re}\, s > \sigma_0$ also implies that all the eigenvalues of $A - sI$ have negative real part. Indeed, it is clear that the eigenvalues μ of $A - sI$ have the form $\mu = \lambda - s$ where λ is an eigenvalue of A . But then $\mathrm{Re}\,\mu = \mathrm{Re}\,\lambda - \mathrm{Re}\, s < 0$.

We know that if all the eigenvalues of a matrix M have negative real part, then for each c we have $\lim_{\xi\to+\infty} e^{\xi M}c = 0$. In conclusion,

$$\mathcal{L}[e^{tA}c] = -(A - sI)^{-1}c$$

as required. As far as (B.27) is concerned, we remark that

$$\begin{aligned}\mathcal{L}\left[\int_0^t e^{(t-\tau)A}b(\tau)d\tau\right] &= \int_0^{+\infty} e^{-st}\left(\int_0^t e^{(t-\tau)A}b(\tau)d\tau\right)dt \\ &= \int_0^{+\infty} e^{-\tau A}\left(\int_{\tau}^{+\infty} e^{t(A-sI)}\,dt\right)b(\tau)\,d\tau\ .\end{aligned}$$

Note the change of the integration interval due to the change of integration order. Making use of the assumption that $\mathrm{Re}\, s > \sigma_0$, we finally conclude

$$\mathcal{L}\left[\int_0^t e^{(t-\tau)A}b(\tau)d\tau\right] = -\int_0^{+\infty} e^{-\tau A}(A-sI)^{-1}e^{\tau(A-sI)}b(\tau)d\tau$$
$$= -\int_0^{+\infty} (A-sI)^{-1}e^{-s\tau}b(\tau)d\tau$$
$$= -(A-sI)^{-1}\int_0^{+\infty} e^{-s\tau}b(\tau)d\tau$$
$$= -(A-sI)^{-1}B(s)\ .$$

■

Appendix Summary

Appendix B recalls the definition of Laplace transform and its main properties. Moreover, we give a list of the Laplace transforms of some elementary functions.

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