Comments, Further Results, and Open Problems

The first and basic contribution on the problem of spectral synthesis for the space $C(\mathbb{R}^1)$ is that of Schwartz in 1947 [188]. Another proof of the spectral synthesis theorem of Schwartz was provided by Kahane [133, 134] (see also Lyubich and Tkachenko [149]). The main tool they used was the Karleman transform.

The question whether Schwartz's theorem can be extended from $\mathbb{R}^1$ to $\mathbb{R}^n$ received a lot of attention. By the Malgrange approximation theorem, the space of solutions of an arbitrary convolution equation $f * T = 0$, $T \in \mathcal{E}'(\mathbb{R}^n)$, admits spectral synthesis. Gurevich [102] constructed a system of two convolution equations in $\mathbb{R}^2$ for which spectral synthesis does not hold. Positive results on the problem of spectral synthesis for systems of convolution equations were obtained by Delsarte [55], Brown, Schreiber, and Taylor [42], Yger [265], Berenstein and Taylor [22], and others. In particular, it was proved that the analog of Schwartz theorem is true for slowly decreasing systems of convolution equations in the sense of [22].

It follows from the Brown–Schreiber–Taylor theorem that if $V \subset C(\mathbb{R}^n)$ is a nontrivial subspace invariant under translations and rotations, then V contains a function

$$(\lambda|x|)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\lambda|x|)$$

for some $\lambda \in \mathbb{C}$. A similar result for noncompact symmetric spaces was established by Bagchi and Sitaram [8]. The cases of the phase space and the reduced Heisenberg group were considered by Thangavelu [210]. In all these cases it was proved that the appropriate subspace V contains an elementary spherical function.

Analogues of Schwartz theorem for other groups were treated by various authors. For example, Ehrenpreis and Mautner [71] proved that every two-sided translation-invariant closed space of continuous functions on $\mathrm{SL}(2, \mathbb{R})$ admits spectral synthesis, and characterized the minimal and indecomposable two-sided translation-invariant subspaces. Similar results were obtained by Weit [256] for two-sided translation invariant spaces of continuous functions on M(2), the motion group of the plane. It turns out that the theory of one-sided translation-invariant spaces is completely different from that of two-sided translation-invariant spaces. In particular, there is a continuous function f on $\mathrm{SL}(2, \mathbb{R})$ so that $V(f)$ contains no minimal

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

right-invariant subspace. ($V(f)$ denotes the right-invariant subspace of $C(\mathrm{SL}(2,\mathbb{R}))$ generated by f.) This phenomena was discovered by Weit [260]. Further results on spectral analysis on groups can be found in [257, 259].

For a relatively long time, it was not clear how to generalize the Brown–Schreiber–Taylor theorem to the case of systems of convolution equations on bounded domains in $\mathbb{R}^n$. The Berenstein–Gay theorem (see the beginning of Chap. 19) is the important first step in this direction. Their proof is based on tools related to the Hörmander approximation theorem. However, the Berenstein–Gay approach is not applicable in more delicate situations considered in Sects. 19.1 and 19.3 (see also Sects. 18.1–18.3 for one-dimensional analogues). The main results in these sections were obtained by V.V. Volchkov. Although our study concerning the classes $\mathcal{D}'_{\mathcal{T}}(-R,R)$ and $\mathcal{D}'_{\mathcal{T}}(B_R)$ is rather complete, the following questions remain open.

Problem 1. *Let $\mathcal{T}\in\mathfrak{T}_2(\mathbb{R}^1)$ with $\mathcal{Z}(\mathcal{T})=\varnothing$. Prove that $C^\infty_{\mathcal{T}}(-R,R)=\{0\}$ for $R=R_{\mathcal{T}}$ or give a counter-example.*

Together with Theorem 13.12, some of ideas in Leont'ev [144, Part 1, Chap. 4] may be useful.

Problem 2. *Give a sensible description of the sets $\mathcal{W}_{\mathcal{T},R}$ and $\mathcal{V}_{\mathcal{T},R}$.*

Problem 3. *Is an analogous to Theorem 18.7 result true when*

$$T_1\notin(\mathrm{Inv}_+\cap\mathrm{Inv}_-)\big(\mathbb{R}^1\big)?$$

The same question for assertion (ii) *of Theorem 18.6.*

Problem 4. *Assume that $T_1,T_2\in\mathcal{E}'(\mathbb{R}^1)$ and $r_\nu=r(T_\nu)>0$, $\nu=1,2$. Let $\mathcal{Z}(\widehat{T_1})\cap\mathcal{Z}(\widehat{T_2})=\varnothing$, $R=r_1+r_2$, and $\mathcal{D}'_{\mathcal{T}}(-R,R)=\{0\}$, where $\mathcal{T}=\{T_1,T_2\}$. Does there exist a sequence $\zeta_1,\zeta_2,\dots$ of complex numbers such that conditions* (18.19) *and* (18.20) *hold?*

Observe that in the special case $T_1\in\mathfrak{N}(\mathbb{R}^1)$, we already have an affirmative answer to this question (see Theorem 18.8(ii)).

Problem 5. *Characterize the family $\mathcal{F}\subset\mathcal{E}'(\mathbb{R}^1)$ such that assertion* (ii) *in Theorem 18.13 is valid for $P\in\mathcal{F}$.*

It follows by Theorem 18.13(ii) that $\mathfrak{N}(\mathbb{R}^1)\subset\mathcal{F}$. We expect that $\mathfrak{N}(\mathbb{R}^1)\neq\mathcal{F}$.

The notion of sparse set, arising in the course of our study of the class $\mathcal{D}'_{\mathcal{T}}(-R,R)$, $\mathcal{T}=\{T_1,T_2\}$, is of interest in its own right. It also leads to some unanswered questions.

Problem 6. *Which sets $E\subset\mathbb{C}$ are sparse sets?*

Certain necessary and sufficient conditions follow from the analysis in Sect. 18.2. The situation becomes more difficult for sets E close to the real line.

Problem 7. *Given $c \geqslant 0$, describe sparse sets*

$$E \subset \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leqslant c \log(2 + |z|)\}.$$

Problem 8. *Let $E \subset \mathbb{R}^1$, and let $\lim_{R\to+\infty} c(R, E)/R = 0$, where $c(R, E)$ is the cardinality of the set $E \cap (-R, R)$. Does this imply that E is sparse?*

If the answer is in the affirmative, this, together with Proposition 18.1(iii), would allow one to obtain a quite satisfactory description of sparse sets $E \subset \mathbb{R}^1$.

We note also that Problems 6–8 are closely connected with the question about the completeness of a system of exponential functions (see Levin [145], Appendix III).

The results in Sects. 20.1, 20.2, and 20.4 are a joint research of V.V. Volchkov and Vit.V. Volchkov [231]. In light of the results in Sect. 20.2, it is natural to ask the following question.

Problem 9. *Let X be a symmetric space of noncompact type, and let $\mathrm{rank}\, X \geqslant 2$. Find a class F (as large as possible), $\mathcal{E}'_{\natural\natural}(X) \subset F \subset \mathcal{E}'_{\natural}(X)$, such that if $T_1, T_2 \in F$, then*

$$\mathcal{D}'_{T_1}(B_R) \cap \mathcal{D}'_{T_2}(B_R) = \{0\}, \tag{1}$$

provided that

$$R > r(T_1) + r(T_2) \quad \text{and} \quad \left\{\xi \in \mathfrak{a}^*_{\mathbb{C}} : \widetilde{T}_1(\zeta) = \widetilde{T}_2(\zeta) = 0\right\} = \varnothing. \tag{2}$$

Concerning the exponential representation problem we only note that the study of many questions in harmonic analysis, like finding all distribution solutions to a system of linear partial differential equations with constant coefficients (or, more generally, convolution equations) in $\mathbb{R}^n$, can be translated into interpolation problems in spaces of entire functions with growth conditions. This idea, which one can trace back to Euler, is the basis of Ehrenpreis's Fundamental Principle for partial differential equations [68], and has been explicitly stated, for convolution equations, in the work of Berenstein and Taylor [22]. For other references, see the comments to Part III. The results in Sects. 19.2 and 20.3 are due to V.V. Volchkov [229], Parts I and II. Some analogues of Theorem 20.13 for $\mathbb{H}^n_{\mathbb{C}}$ and $\mathbb{H}^n_{\mathbb{Q}}$ were established by Vit.V. Volchkov [234, 239].

The problem of deconvolution has been studied by many authors. It is closely related to the so called *Bezout identities*. The Bezout equation is simply an equation of the form

$$h_1 g_1 + \cdots + h_n g_n = 1, \tag{3}$$

where $h_1, \ldots, h_n$ are, for example, polynomials (or entire functions with growth conditions), and one looks for polynomial solutions $g_1, \ldots, g_n$ (or entire functions with growth conditions similar to those satisfied by the h_i). According to Hörmander [125], for compactly supported distributions $\{T_i\}_{i=1}^n$ on $\mathbb{R}^1$, there exist compactly supported distributions $\{U_i\}_{i=1}^n$ such that

$$\delta = T_1 * U_1 + \cdots + T_n * U_n$$

if and only if there exist positive constants A and B and a positive integer N such that

$$\left(\sum_{i=1}^{n}\left|\widehat{T_i}(z)\right|^2\right)^{1/2} \geqslant A\mathrm{e}^{-B|\mathrm{Im}z|}(1+|z|)^{-N}, \quad z \in \mathbb{C}. \tag{4}$$

In other words, a set of distributions $\{T_i\}_{i=1}^n \in \mathcal{E}'(\mathbb{R}^1)$ that satisfy (4) is precisely a set for which there exist $\{U_i\}_{i=1}^n \in \mathcal{E}'(\mathbb{R}^1)$ such that (3) holds with $h_i = \widehat{T_i}$ and $g_i = \widehat{U_i}$. Thus, the problem of recovering f from $f * T_i$ can be solved by constructing explicit formulae for $\widehat{U_i}$. This approach to deconvolution comes out of the fundamental work of Berenstein et al. on residue theory and division problems, which produced the needed solutions to the Bezout equation (see [16, 24, 25, 27, 28]). The results in Sect. 18.4 were proved by V.V. Volchkov. For a wide variety of applications of the deconvolution techniques, see Casey and Walnut [49] and the references therein.

The Pompeiu problem extensively studied by several authors is very closely related to the theory of mean periodic functions. For a comprehensive review of the history of the Pompeiu and related problems, we refer the reader to the survey papers of Zalcman [273] and Berenstein and Struppa [19]. More recent references and results can be found in Zalcman [274], Berenstein, Chang, and Eby [31], V.V. Volchkov [225], V.V. Volchkov and Vit.V. Volchkov [228], and Vit.V. Volchkov [240, 243, 246]. Some results pertaining to related problems for the classical Radon transform are contained in Helgason [122] and Zalcman [270].

The results in Sect. 19.4 were obtained by V.V. Volchkov [231]. Theorem 19.16 and Corollary 19.3 solve a problem posed in [225, p. 334]. More special result (under the condition of hyperbolicity for some point in ∂E) was obtained by Berenstein and Gay [16] in a different way. In view of Corollary 19.3, the following problem arises (see [225, p. 220]).

Problem 10. *Let* $E \in \mathrm{Pomp}(\mathbb{R}^n)$. *Find*

$$\mathcal{R}(E) = \inf\{r > r_E : E \in \mathrm{Pomp}(B_r)\}$$

and investigate when the value $\mathcal{R}(E)$ *is attainable, that is,* $E \in \mathrm{Pomp}(B_r)$ *for* $r = \mathcal{R}(E)$.

For many E, these questions have been studied in [225]. However, our knowledge concerning $\mathcal{R}(E)$ remains very incomplete. Among the unattackable cases, we point out the following:

(1) $E = \{x \in \mathbb{R}^n : \sum_{j=1}^n |x_j|^p \leqslant 1\}$, $p > 0$, $p \neq 2$;
(2) $E = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 \leqslant r^2, 0 \leqslant x_n \leqslant h\}$;
(3) $E = \{x \in \mathbb{R}^n : \alpha\,(x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leqslant x_n \leqslant h\}, \alpha > 0$;
(4) E is a regular simplex in $\mathbb{R}^n$, $n \geqslant 4$;
(5) $E = \mathrm{Cl}\,\Omega$, where Ω is the von Koch snowflake domain in $\mathbb{R}^2$, see [228, Corollary 3.1].

We now discuss the case $E = \mathrm{Cl}\,\Omega$, where Ω is a bounded domain in $\mathbb{R}^n$ with real analytic boundary. Apart from the case where E is an ellipsoid, there has so far been no value $\mathcal{R}(E)$ for any $E \in \mathrm{Pomp}(\mathbb{R}^n)$ whatever. Let us state a conjecture.

For $R > r_E$, we set

$$M(E, R) = \{g \in \mathrm{M}(n) : \ gE \subset B_R\}.$$

Introduce the set $\mathcal{S}_g(E, R) \subset (0, R)$, $g \in M(E, R)$, defined as follows: a point $r \in (0, R)$ belongs to $\mathcal{S}_g(E, R)$ if and only if the boundary of the set gE touches the sphere S_r. It is easy to see that $\mathcal{S}_g(E, R) = (0, R)$ if $R > 2r_E$.

Conjecture 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with real analytic boundary, let $E = \mathrm{Cl}\,\Omega \in \mathrm{Pomp}(\mathbb{R}^n)$, and assume that the set $\mathbb{R}^n \backslash E$ is connected. Then*

$$\mathcal{R}(E) = \inf\left\{R > r_E : \bigcup_{g \in M(E,R)} \mathcal{S}_g(E, R) = (0, R)\right\}.$$

In the case where E is an ellipsoid, the proof of this conjecture is included in some recent results by V.V. Volchkov [225, Part IV]. The important ingredient used in [225, Part IV] is microlocal analysis, i.e., analysis of analytic wave front sets (see Hörmander [126, Chap. 8] and Quinto [170–176]).

Next, for each convex $E \in \mathrm{Pomp}(\mathbb{R}^n)$, it can be shown that

$$\mathcal{R}(E) \geqslant \mu_*(E), \quad \text{where } \mu_*(E) = \min_{x \in \partial E} \max_{y \in \partial E} |x - y|$$

(see [225, Part IV, Theorem 1.6]). The estimate is attainable for a broad class of sets E close to a ball, but this does not hold in general. The following question remains open.

Problem 11. *Characterize those convex sets $E \in \mathrm{Pomp}(\mathbb{R}^n)$ for which $\mathcal{R}(E) = \mu_*(E)$.*

A symmetric space analogue of Theorem 19.16 is contained in [231]. The local Pompeiu property and some analogues of Problem 10 on a sphere were considered in [228].

Surprisingly, the connection between the Pompeiu and the Morera properties seems not to have been noticed before Zalcman's paper [267]. As regards various results related to the Morera property, see, for instance, [225], which contains an extensive bibliography. Among the papers not listed in [225], we can point out Berenstein, Chang, and Tie [29], Berenstein, Chang, Eby, and Zalcman [30], Globevnik et al. [85–95], and Grinberg and Quinto [100].

Zalcman's result mentioned in Sect. 20.4 is an analog of Delsarte's theorem for harmonic functions [56]. Similar "two-radii theorems" have been proved for rank one symmetric spaces by Berenstein and Zalcman [26]. The first local versions of Zalcman's theorem are due to Smith [200] and Berenstein and Gay [14]. In the case of the ball means, exact analogues of Theorem 20.17 were obtained for $\mathbb{R}^n$ and $\mathbb{H}^n_{\mathbb{R}}$ by V.V. Volchkov [222, 224, 225, Part II], and for $\mathbb{H}^n_{\mathbb{C}}$ and $\mathbb{H}^n_{\mathbb{Q}}$, by Vit.V. Volchkov [233, 238]. For rank one symmetric spaces of noncompact type, they have been established by V.V. Volchkov in [227]. For the matter of reconstructing f from the knowledge of its integrals over balls, see Berenstein, Gay, and Yger [28], El Harchaoui [72], Berkani, El Harchaoui and Gay [33], and Vit.V. Volchkov and N.P. Volchkova [249].

Let us now consider some open problems concerning these results.

Problem 12. *Assume that X is as in Problem 9. Let T_1 and T_2 be the indicators of geodesic balls in X, and let* (2) *hold. Must* (1) *be valid?*

Assertion (ii) in Theorem 20.17 remains valid if we assume that $f \in V_{r_1,r_2}(B_R)\cap C^\infty(B_r)$ for some $r > \min\{r_1, r_2\}$. The proof of this fact is analogous to that of Theorem 20.17(ii), only instead of Theorem 20.2 one applies Theorem 20.9(i). In this connection the following question arises.

Problem 13. *Let $r_1, r_2 > 0$, $r_1/r_2 \notin E_X$, and $R = r_1 + r_2$. Assume that $0 < r \leqslant \min\{r_1, r_2\}$ and $f \in V_{r_1,r_2}(B_R) \cap C^\infty(B_r)$. For what r does it follow that $f = 0$?*

Next, assertions (i)–(iii) in Theorem 20.17 mean that under given conditions the problem of recovering a function f if the integrals

$$\int_{B_{r_i}} f(gx)\,\mathrm{d}\mu(x), \quad g \in G, d(o, go) < R - r_i, i = 1, 2, \tag{5}$$

are known has a unique solution. The problem, then, is to reconstruct f by given integrals (5).

Furthermore, for $r_1/r_2 \in E_X$, it is natural to pose the problem of describing of the class $V_{r_1,r_2}(X)$.

Conjecture 2. *Let $r_1, r_2 > 0$, $r_1/r_2 \in E_X$, and $c = \lambda_1/r_1 = \lambda_2/r_2$, where $\lambda_1, \lambda_2 \in \mathcal{Z}(\mathbf{I}_{l/2})$, $l = (\dim X)/2$. Then $f \in V_{r_1,r_2}(X)$ if and only if $(L + |\rho|^2 + c^2)f = 0$.*

The main difficulty in proving that is a very incomplete knowledge concerning the algebraic independence of the zeroes of $\mathbf{I}_{l/2}$. The results in Sect. 20.4 and [225, Part II, Sect. 1.7] ensure that Conjecture 2 is true if the condition $\alpha/\beta = \gamma/\delta \neq 1$, where $\alpha, \beta, \gamma, \delta \in \mathcal{Z}(\mathbf{I}_{l/2}) \cap (0, +\infty)$, implies that $\alpha = \gamma$ and $\beta = \delta$. The only result in this direction was obtained by Siegel (see, for instance, Shidlovskiĭ [196]): the zeroes of $\mathbf{I}_\alpha$ are transcendental whenever α is rational.

The following fairly distinct questions also seem to be interesting (we keep the assumption on $X = G/K$ from Sect. 20.4).

Problem 14. *Let $r_1, r_2 > 0$, $r_1/r_2 \notin E_X$, $R > \max\{r_1, r_2\}$, and $f \in L^{1,\mathrm{loc}}(B_R)$. Assume that*

$$\int_{B_{r_1}} f(gx)\,\mathrm{d}\mu(x) \leqslant 0, \quad g \in G, d(o, go) < R - r_1$$

and

$$\int_{B_{r_2}} f(gx)\,\mathrm{d}\mu(x) \geqslant 0, \quad g \in G, d(o, go) < R - r_2.$$

For what R can we assert that $f = 0$?

Related results for the Euclidean case can be found in V.V. Volchkov [225], Part II, Chap. 2.1.

Problem 15. *Let $f \in L^{1,\mathrm{loc}}(X)$. Characterize the compact sets $K \subset X$ for which we may conclude from the condition*

$$\int_K f(gx)\,\mathrm{d}\mu(x) = 0, \quad g \in G,$$

that $f = 0$.

Generalizations of Theorem 20.17 for the case of several radii are of great interest. This would complicate things considerably, and existing techniques are not quite satisfactory. "Freak theorems" for the sphere $\mathbb{S}^2$ have roots in the work of Radon [180] and Ungar [219]. Theorem 21.1 is the analogue of the Berenstein–Zalcman theorem (see the summary to Chap. 21) for systems of convolution equations. For generalizations of "freak theorems" to locally symmetric spaces, see Badertscher [7]. Local versions of the Berenstein–Zalcman theorem for compact symmetric spaces of rank one were obtained by Vit.V. Volchkov [242, 244, 228, 232]. The results in Sect. 21.2 are also due to him.

The technique developed in Chaps. 12 and 18 makes it possible to obtain analogues of the main results of Chap. 19 for the phase space and the reduced Heisenberg group H^n_{red}. Here we only state some of them (see V.V. Volchkov and Vit.V. Volchkov [231]).

We shall use the notation established in Sects. 12.4 and 17.3. Furthermore, we denote by d_a ($a > 0$) the operator mapping $f \in \mathcal{D}'(\mathbb{C}^n)$ into the distribution $f(az)$.

Theorem 1. *Let $\{T_i\}_{i\in\mathcal{I}}$ be a family of* $\mathrm{U}(n)$*-invariant compactly supported distributions on H^n_{red}. Suppose that the following conditions hold:*

(1) $(T_i)_k \neq 0$ *and* $r(T_i) = r((T_i)_k)$ *for all* $i \in \mathcal{I}$, $k \in \mathbb{Z}$*;*
(2) $r(T_\nu) + \inf_{i\in\mathcal{I}} r(T_i) < R \leqslant +\infty$ *for every* $\nu \in \mathcal{I}$*;*
(3)

$$\bigcup_{k\in\mathbb{Z}\setminus\{0\}} \bigcap_{i\in\mathcal{I}} \mathcal{Z}\big(\mathcal{F}_1^{(0,0)}\big(d_{1/\sqrt{|k|}}(T_i)_k\big)\big) = \bigcap_{i\in\mathcal{I}} \mathcal{Z}\big(\widetilde{(T_i)_0}\big) = \varnothing, \tag{6}$$

where $\widetilde{(T_i)_0}$ is the Euclidean spherical transform of the distribution $(T_i)_0$ on $\mathbb{C}^n$.

Then

$$\big\{f \in \mathcal{D}'(C_R) : f * T_i = 0 \text{ in } C_{R-r(T_i)}\ \forall i \in \mathcal{I}\big\} = \{0\}.$$

Analogues of Theorem 1 for spherical means on H^n_{red} were obtained by Vit.V. Volchkov in [245] and Agranovsky and Narayanan in [4] (see also Thangavelu [209] for the case where $R = +\infty$ and f is a continuous function of tempered growth). We note that Vit.V. Volchkov has established Theorem 1 in a refined version, along with proving the sharpness of the conditions. The Agranovsky–Narayanan method is based on the wave equation, which of course is not applicable in the general situation.

If $R = +\infty$, Theorem 1 is true for an arbitrary family $\{T_i\}_{i\in\mathcal{I}}$ of $\mathrm{U}(n)$-invariant compactly supported distributions on H^n_{red} satisfying (6). This leads to the following result.

Theorem 2. *Let V be a nontrivial subspace of $C(H^n_{\mathrm{red}})$ invariant under translations and the action of the group $\mathrm{U}(n)$. Then V contains at least one of the functions*

$$(\lambda|z|)^{-n+1}J_{n-1}(\lambda|z|), \quad \phi_{\lambda,0,0,0}\big(\sqrt{|k|}|z|\big)e^{ikt}$$

for some $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}\backslash\{0\}$.

For subspaces V generated by functions of tempered growth, analogues of Theorem 2 are contained in Thangavelu [210].

Finally, let us consider the Pompeiu problem with a twist. Let $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$ be a family of nonzero compactly supported distributions on $\mathbb{C}^n$ such that supp $\psi_i \subset \mathring{B}_{r(\psi_i)}$ and $r(\psi_i) < R \leqslant +\infty$ for all $i \in \mathcal{I}$. We say that Ψ has the *twisted Pompeiu property* in the ball B_R if there is no nontrivial function $f \in C^\infty(B_R)$ satisfying

$$(f \circ \tau) \star \psi_i = 0 \quad \text{in } B_{R-r(\psi_i)} \quad \text{for all } \tau \in \mathrm{U}(n),\, i \in \mathcal{I}.$$

Theorem 3. *Let $r(\psi_\nu) + \inf_{i\in\mathcal{I}} r(\psi_i) < R \leqslant +\infty$ for all $\nu \in \mathcal{I}$. Then the following statements are equivalent.*

(i) *The family Ψ has the twisted Pompeiu property in the ball B_R.*
(ii) *The family Ψ has the twisted Pompeiu property in $\mathbb{C}^n$.*

This result is due to the second author [231]. He also proved the following analogue of the Brown–Schreiber–Taylor theorem on the Pompeiu property [42].

Theorem 4. *The family Ψ fails to have the twisted Pompeiu property if and only if there exists $\lambda \in \mathbb{C}$ such that $\phi_{\lambda,0,0,0,1} \star \psi_i = 0$ for all $i \in \mathcal{I}$.*

A similar criterion for distributions with the weak twisted Pompeiu property was established by Thangavelu [210]. For further results concerning the Pompeiu property on the Heisenberg group, see Berenstein, Chang, and Tie [29], Berenstein, Chang, and Eby [31], Thangavelu [209, 210], and the references therein.