Comments, Further Results, and Open Problems

It will be convenient to break up our discussion under various headings, according to the items mentioned in the summary to Part III.

(i) The main results in Sects. 13.1, 14.1, 15.1, and 17.1 are essentially contained in V.V. Volchkov and Vit.V. Volchkov [231]. The material in Sect. 16.1 is due to Vit.V. Volchkov [248]. The operators A_j (see (16.1)–(16.6)) can often be used when translation invariant questions are concerned. We present for further reference analogues of Propositions 16.2 and 16.4 in the noncompact case.

Let X be a noncompact symmetric space of rank one. We shall assume that the minimum sectional curvature of X is equal to -4 and that X is realized in the same way as in Chap. 2 (by $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ we understand $B^n_{\mathbb{R}}$ with the Poincaré metric). Put $\mathcal{X} = \overline{\mathbb{R}^n}$, $\mathbb{P}^n_{\mathbb{C}}$, $\mathbb{P}^n_{\mathbb{Q}}$, or $\mathbb{P}^2_{\mathbb{C}a}$ as $X = B^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{Q}}$, or $\mathbb{H}^2_{\mathbb{C}a}$, respectively. Let $\mathcal{A}_j$ be the differential operator on X obtained from the operator A_j on $\mathcal{X}$ under the simple change of variables $x_\nu \to \mathrm{i}x_\nu$ in the coefficients of A_j. In the same manner as in Sect. 16.1 we have the following:

Proposition 1. *Let $T \in \mathcal{E}'_\natural(X)$, and let $\mathcal{O}$ be an open subset of X such that $\mathcal{O}_T \neq \varnothing$ (see (15.1)). Suppose that $f \in \mathcal{D}'_T(\mathcal{O})$. Then $\mathcal{A}_j f \in \mathcal{D}'_T(\mathcal{O})$ for all $1 \leqslant j \leqslant a_{\mathcal{X}}$.*

For a nonempty open set $E \subset (0,1)$ and $\varphi \in C^1(E)$, we set

$$\big(\mathfrak{D}(\alpha,\beta)\varphi\big)(\varrho) = \frac{(1-\varrho^2)^{\beta+1}}{\varrho^\alpha}\frac{\mathrm{d}}{\mathrm{d}\varrho}\bigg(\frac{\varrho^\alpha}{(1-\varrho^2)^\beta}\varphi(\varrho)\bigg).$$

Proposition 2. *Let $T \in \mathcal{E}'_\natural(X)$, and let $\mathcal{O}$ be an open $K_{\mathcal{X}}$-invariant subset of X such that $\mathcal{O}_T \neq \varnothing$. Let $s \in \mathbb{N}$ and $Y \in \mathcal{H}^{k,m}_{\mathcal{X}}\backslash\{0\}$ for some $k \in \mathbb{Z}_+$ and $m \in \{0,\dots,M_{\mathcal{X}}(k)\}$, and assume that $\varphi(\varrho)Y(\sigma) \in C^s_T(\mathcal{O})$. Then the following statements hold.*

(a) *For all $j \in \{1,\dots,d^{k+1,m}_{\mathcal{X}}\}$,*

$$\big(\mathfrak{D}(-k,m-k)\varphi\big)(\varrho)Y^{k+1,m}_j(\sigma) \in C^{s-1}_T(\mathcal{O}).$$

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

(b) *If* $m \leqslant M_{\mathcal{X}}(k+1)-1$, *then*

$$\big(\mathfrak{D}(-k, \beta_{\mathcal{X}} - m)\varphi\big)(\varrho) Y_j^{k+1,m+1}(\sigma) \in C_T^{s-1}(\mathcal{O})$$

for all $j \in \{1, \dots, d_{\mathcal{X}}^{k+1,m+1}\}$.

(c) *If* $k \geqslant 1$ *and* $m \leqslant M_{\mathcal{X}}(k-1)$, *then*

$$\big(\mathfrak{D}(k + 2\alpha_{\mathcal{X}}, k - m + \gamma_{\mathcal{X}})\varphi\big)(\varrho) Y_j^{k-1,m}(\sigma) \in C_T^{s-1}(\mathcal{O})$$

for all $j \in \{1, \dots, d_{\mathcal{X}}^{k-1,m}\}$.

(d) *If* $m \geqslant 1$, *then*

$$\big(\mathfrak{D}(k + 2\alpha_{\mathcal{X}}, \alpha_{\mathcal{X}} + m)\varphi\big)(\varrho) Y_j^{k-1,m-1}(\sigma) \in C_T^{s-1}(\mathcal{O})$$

for all $j \in \{1, \dots, d_{\mathcal{X}}^{k-1,m-1}\}$.

The proof of Proposition 2 is similar to that of Proposition 16.4. Special cases of Propositions 1 and 2 can be found in V.V. Volchkov [225], Part II, Chap. 2, and Vit.V. Volchkov [236, 241].

(ii)–(iv) Zero sets of mean periodic functions on the real line were studied by Leont'ev [144, Chap. 5], Lyubich [148], Kargaev [136], and others. Special cases of Theorem 13.1(i) follow from the Titchmarsh theorem on supports of convolutions (see Corollary 6.1). Support theorems for the spherical means in $\mathbb{R}^3$ go back to John [128, 129]. For a connection of John's theorem with microlocal analysis, see Quinto [172].

Zaraisky [275] has established Theorem 14.5 and the analog of Theorem 14.10(i) for $r = r(T)$ without employment of the distribution $\zeta_{T,k,j}$. His ideas are also available in the one-dimensional case (see Theorems 13.3 and 13.4(i) for $r = r(T)$). The rest of the results in Sects. 13.2, 14.2, 14.3, 15.3, and 15.4 are due to the first author (see [225–227, 229] for particular cases). The analog of Theorem 14.1 for other classes of the distributions T can be found in Zaraisky [275]. A weakened version of Theorem 15.1 for $\mathbb{H}^n_{\mathbb{C}}$ and $\mathbb{H}^n_{\mathbb{Q}}$ was obtained by Vit.V. Volchkov [236, 241]. Section 15.2 represents joint work of V.V. Volchkov and Vit.V. Volchkov [231], Sect. 4. The results of Sects. 16.2 and 17.2 are owed to the second author (see Vit.V. Volchkov [245, 247, 248] and V.V. Volchkov and Vit.V. Volchkov [228, 229, 231]).

Theorem 17.1 for the twisted spherical means was obtained by Vit.V. Volchkov in [245, 228]. The same statement for C^∞-functions was established almost simultaneously and by a very different method in Agranovsky and Narayanan [4]. Unlike our approach, their proof is based on general properties of the wave equation. However, the wave equation method is unsuitable for general convolution equations.

In addition to their intrinsic interest, uniqueness theorems for mean periodic functions are used in other areas of mathematics (see V.V. Volchkov [225, 226]). As an example, we show how Theorem 15.1 can be applied to the theory of series with gaps.

A sequence $\{r_q\}_{q=1}^{\infty}$ of positive numbers is called a ζ *sequence* ($\zeta > 0$) if there exists (depending on ζ and $\{r_q\}_{q=1}^{\infty}$) an increasing function $\varphi : [0, +\infty) \to [1, +\infty)$ such that

$$\sum_{q=1}^{\infty} \frac{1}{q\varphi(q)} < \infty \quad \text{and} \quad \max_{1 \leqslant m \leqslant q} \left| r_m - \frac{\pi m}{\zeta} \right| \leqslant \gamma \frac{q}{\varphi^2(q)},$$

where the constant $\gamma > 0$ is independent of q.

Theorem 1. *Let $X = G/K$ be a symmetric space of noncompact type, and let $\mathcal{O}$ be a ζ domain in X for some $\zeta > 0$. Assume that $(\lambda_m, b_m) \in \mathfrak{a}_{\mathbb{C}}^* \times B$, $m = 1, 2, \ldots,$ and there is a ζ sequence $\{r_q\}_{q=1}^{\infty}$ such that $\lambda_m \in \bigcup_{q=1}^{\infty} \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \langle \lambda, \lambda \rangle = r_q^2\}$ for all m. Suppose that the sequence*

$$f_\nu(x) = \sum_{m=1}^{\nu} c_{m,\nu} e^{(i\lambda_m + \rho)(A(x, b_m))}, \quad c_{m,\nu} \in \mathbb{C}, \nu = 1, 2, \ldots,$$

converges in $\mathcal{D}'(\mathcal{O})$ to f and let $f = 0$ in some open geodesic ball of radius $r > \zeta$. Then $f = 0$ in $\mathcal{O}$.

This result is no longer valid for $r \leqslant \zeta$. The proof of Theorem 1 can be obtained by using Theorem 15.1 and the argument in V.V. Volchkov [225], Part V, Sect. 3.1.

Let us formulate some unanswered questions concerning uniqueness for convolution equations.

Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and $U \subset \mathcal{D}'_T(\mathbb{R}^n)$. An open set $\mathcal{O} \subset \mathbb{R}^n$ is said to be an *uniqueness set* for the class U if $f \in U$ and $f|_{\mathcal{O}} = 0$ imply $f = 0$ on $\mathbb{R}^n$.

The present knowledge of the properties of uniqueness sets is rather insufficient.

Problem 1. *Given $U \subset \mathcal{D}'_T(\mathbb{R}^n)$, determine subsets $\mathcal{O}$ of $\mathbb{R}^n$ which are uniqueness sets for U.*

As is evident from Sects. 13.2 and 14.2, this kind of problems seems quite difficult. One can ask the following easier question.

Problem 2. *Let $n \geqslant 2$ and $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$. Given $U \subset \mathcal{D}'_T(\mathbb{R}^n)$, describe* O($n$)-*invariant uniqueness sets for U.*

We note that a study of this problem seems to be interesting in connection with applications to the extreme versions of the Pompeiu problem (see V.V. Volchkov [225]).

Theorem 13.5 solves Problem 1 for the class $U = \mathcal{D}'_T(\mathbb{R}^1)$. However, it would be nice to obtain a description of uniqueness sets formulated in the simple geometrical terms.

Also, it would be interesting to prove the following conjecture, a result which includes as special cases both the "hemisphere theorem" and Theorem 14.1.

Conjecture 1. *Let $\nu, m \in \mathbb{Z}$, $m \geqslant \max\{0, 2[(1-\nu)/2]\}$, $T \in \mathcal{M}_\natural^\nu(\mathbb{R}^n)$, and let $R > r(T) > 0$. Assume that $\mathcal{U}$ is an open subset of B_R such that $S_{r(T)} \subset \mathcal{U} \cup \{x \in \mathbb{R}^n : (-x) \in \mathcal{U}\}$. Suppose that $f \in \mathcal{D}'_T(B_R)$ and $f = 0$ in $B_{r(T)}$. Then the following assertions hold.*

(i) *If $f \in L_m^{1,\mathrm{loc}}(\mathcal{U})$, then $f^{k,j} = 0$ in B_R for $k \leqslant m + \nu + 1$ and $j \in \{1, \dots, d(n,k)\}$.*
(ii) *If $f \in C^m(\mathcal{U})$, then $f^{k,j} = 0$ in B_R for $k \leqslant m + \nu + 2$ and $j \in \{1, \dots, d(n,k)\}$.*

Zaraisky proved Conjecture 1(ii) for $m = \infty$ (unpublished). His proof is based on two main ingredients. The first is microlocal analysis, and the second is Theorem 14.2. For microlocal techniques, see Quinto [170–176] and Hörmander [126], Chap. 16.

Analogous problems for symmetric spaces of higher ranks become much harder. We are inclined to believe that the conclusion of Theorem 15.1(ii) can be obtained under considerably weaker hypothesis, and it therefore raises some interesting questions.

Problem 3. *For which distributions $T \in \mathcal{E}'_\natural(X)$ is assertion* (ii) *of Theorem 15.1 valid?*

The answer is obvious for the rank one case where $\mathcal{E}'_\natural(X) = \mathcal{E}'_{\natural\natural}(X)$. However, if rank $X > 1$, then it is not known what happens even for the case where T is the indicator (characteristic function) of a ball.

The case of the Heisenberg group is of special interest. The technique of taking the Fourier transform in the t variable, together with Theorem 17.2, allows one to obtain uniqueness theorems for mean periodic functions in the cylinder $B_R \times \mathbb{R}^1 \subset H^n$ which satisfy some growth conditions (see the proof of Theorem 17.10). The natural question is now: find sharp growth conditions for validity of these theorems. Another interesting open problem is to obtain analogues of Theorem 17.2 on bounded domains in H^n.

There are many works investigating support theorems of a different nature. Here we give only a few references on the subject; see, e.g., Helgason [123], Chap. I, Sect. 3, Quinto [177–179], Gonzalez and Quinto [98], Globevnik [88], Narayanan and Thangavelu [160], and Zhou [278].

(v) A series development for a mean periodic function $f \in C^\infty(\mathbb{R}^1)$ in terms of the polynomial-exponential functions which are solutions of the same convolution equation as f was first obtained by Schwartz [188]. Some modifications and improvements of Schwartz's result are due to Ehrenpreis [62–66], Leont'ev [144], Chap. 5, Ehrenpreis and Malliavin [70], and Berenstein and Taylor [20].

There is an extensive literature on developments of solutions of the convolution equation $f * T = 0$ in terms of integrals over the zero set of $\widehat{T}$. These questions are closely related to several topics in multidimensional complex analysis; see, e.g., Ehrenpreis [68, 69], Palamodov [165], Kaneko [135], Berenstein and Taylor [20–22], Berenstein and Struppa [18, 19], Struppa [205], Gay [80], Berndtsson and Passare [34], Hansen [103, 104], Meril [152], Meril and Struppa [153], and

Kuchment [143]. Here we only state a sample result from [143]. We need some notation.

Let $X = G/K$ be a symmetric space of noncompact type. A function f on X is called *K-finite* if the span of its orbit under the action of K is finite-dimensional. For $T \in \mathcal{E}'(X)$, we can define an entire function $\widehat{T}(\lambda)$ of exponential type on $\mathfrak{a}^*_{\mathbb{C}}$ by $\widehat{T}(\lambda) = \langle T, \varphi_\lambda \rangle$, where φ_λ is an elementary spherical function. We will use the concept of a function $\widehat{T}(\lambda)$ which is slowly decreasing with respect to the weight $p(\lambda) = |\mathrm{Im}\,\lambda| + \log(1 + |\lambda|)$ without explanation (because of its inconvenience) (see Berenstein and Taylor [22]). In the case where $\dim_{\mathbb{R}} \mathfrak{a} = 1$, this means that there exist constants $\varepsilon, \alpha > 0$ such that the connected components of the set $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : |\widehat{T}(\lambda)| < \varepsilon \exp(-\alpha p(\lambda))\}$ are relatively compact. Finally, we put

$$\varphi_{i,j,\delta}(x, \lambda) = \int_K \mathrm{e}^{\lambda(A(x,kM))} \delta_{i,j}\big(k^{-1}\big)\, \mathrm{d}k,$$

where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\delta \in \widehat{K}$, and $\delta_{i,j}$ are the matrix elements of δ. The functions $\varphi_{i,j,\delta}(x, \lambda)$ are exact analogues of the exponential polynomials.

Theorem 2. *Suppose that a distribution $T \in \mathcal{E}'(X)\backslash\{0\}$ is such that $\widehat{T}(\lambda)$ is a slowly decreasing function and let $V = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \widehat{T}(\lambda) = 0\}$. Then there exist a locally finite covering V by means of closed subsets $V_k \subset V$, differential operators ∂_k with respect to λ, and a partition of the set $\mathcal{J} = \{k\}$ of indices into finite subsets $\mathcal{J}_l$ such that any K-finite solution $f \in C^\infty(X)$ of the equation $f \times T = 0$ is representable in the form*

$$f(x) = \sum_{i,j,\delta} \sum_{l} \sum_{k \in \mathcal{J}_l} \int_{V_k} \partial_k \varphi_{i,j,\delta}(x, \lambda)\, \mathrm{d}\mu_{i,j,\delta,k}(\lambda),$$

where the sum with respect to i, j, and δ is finite, $\mu_{i,j,\delta,k}$ are Radon measures with supports in V_k, the series with respect to l, the integrals converge in $C^\infty(X)$, and the integrands are solutions of the same equation as f.

The proof of Theorem 2 is based on results of Helgason on Fourier transforms on X (see Helgason [123]) and of Berenstein and Taylor [22] in harmonic analysis. The special case of spherical functions which are mean periodic is treated in Bagchi and Sitaram [8]. It is possible to obtain analogous results for K-finite mean periodic functions on the tangent space to X at zero. In this case the functions $\varphi_{i,j,\delta}$ are replaced by generalized Bessel functions (see Helgason [123], Chap. III, Sect. 7).

Motivated by the problem of recovering a function f in $\mathbb{R}^n$ from the knowledge of its average over balls, Berenstein and Gay [14] proved the following:

Theorem 3. *Let $\mathcal{O}$ be an open convex subset in $\mathbb{R}^n$ $(n \geqslant 2)$, $T \in \mathcal{E}'(\mathbb{R}^n)$ an invertible distribution, and Ω the convex hull of* $\mathrm{supp}\, T$. *Any function $f \in C^\infty(\mathcal{O} + \Omega)$, mean periodic with respect to T, can be written as*

$$f(x) = \sum_{j \geqslant 1} P_j(x) \quad (x \in \mathcal{O} + \Omega)$$

with P_j exponential polynomials, also mean periodic with respect to T, and the series is convergent in the C^∞-topology of $\mathcal{O} + \Omega$. Furthermore, given a sequence $\{s_j\}_{j\geqslant 1}$ of positive numbers, letting $P_0 = 0$, we can choose the P_j so that the absolute value of all frequencies in P_{j+1} exceeds the largest absolute value of the frequencies in P_j by at least s_{j+1}.

In connection with this result we note that there are analogues of Theorem 2 for mean periodic functions on convex domains in $\mathbb{R}^n$ (see Berenstein and Struppa [18], Sect. 2). However, the characteristic functions of balls do not satisfy the conditions required in [18].

The main results in Sects. 13.3, 13.4, 14.4, 14.5, 15.5, and 15.6 are due V.V. Volchkov. Some of them can be found in [225, 226, Part III] and [229, Parts I and II]. Certain cases of Theorem 15.15 were proved by Vit.V. Volchkov [237, 239]. The results in Sects. 16.3 and 17.3 are a modernization of results of Vit.V. Volchkov [229, 245, 247, 248, Parts III and IV].

Let us mention the following question, which seems to be closely related to Problem 5 in the comments to Part II.

Problem 4. *Can Theorem 14.24 be strengthened so as to allow $T \in \mathfrak{M}(\mathbb{R}^n)$?*

Theorem 14.14 allows us to give a new one-radius characterization of harmonic functions. The classical mean value theorem stated that $f \in C(\mathbb{R}^n)$ is harmonic if and only if

$$f(x) = \frac{n}{\omega_{n-1} r^n} \int_{|y|\leqslant r} f(x+y)\,\mathrm{d}y \tag{1}$$

for all $x \in \mathbb{R}^n$ and all $r > 0$. Consider the following problem. Let $f \in C(\mathbb{R}^n)$, and let (1) be satisfied for some fixed $r > 0$ and all $x \in \mathbb{R}^n$. Is f a harmonic function? In the general case the answer is in the negative; however, under some additional assumptions, f is indeed harmonic (see V.V. Volchkov [225], Part V, Chap. 5). As a consequence of Theorem 14.14, we have the following:

Theorem 4. *Let $\xi_n = \min\{|z| : 2^{n/2}\Gamma(n/2+1)\mathbf{I}_{n/2}(z) = 1, z \neq 0\}$.*

(a) *Let $f \in C^\infty(\mathbb{R}^n)$, and let it satisfy (1) for one fixed $r > 0$ and all $x \in \mathbb{R}^n$. Assume that*

$$\liminf_{m\to+\infty} \xi_n^{-2m} \int_{|x|\leqslant r} \left|\left(\Delta^m f\right)(x)\right| \mathrm{d}x = 0.$$

Then f is harmonic.

(b) *There exists a nonharmonic function $f \in C^\infty(\mathbb{R}^n)$ satisfying (1) for one fixed $r > 0$ and all $x \in \mathbb{R}^n$ and such that*

$$\int_{|x|\leqslant r} \left|\left(\Delta^m f\right)(x)\right| \mathrm{d}x = O\left(\xi_n^{2m}\right) \quad \text{as } m \to +\infty.$$

Let us add that some information about zeroes of the function

$$2^{n/2}\Gamma(n/2+1)\mathbf{I}_{n/2}(z) - 1$$

can be found in V.V. Volchkov [225], Part V, Sect. 5.1.

We point out that Theorems 14.14 and 15.12 allow one to obtain some generalizations of Theorem 4, including extensions to more differential equations and rank one symmetric spaces.

As regards other results relating to mean value characterization of various classes of functions, see Zalcman [268, 269, 273, 274], Weit [258], and V.V. Volchkov [225], Part V, containing the extensive bibliography.

(vi) The questions of mean periodic continuation for functions from various spaces have long been under study. The case of the space L^2 was considered by Golovin [97] and Kahane [133]. Conditions for mean periodic extendability of analytic functions and functions of some quasi-analytic classes are given in Leont'ev [144, Chap. 4, Sect. 4.3] and Kahane [132], respectively. There are many works devoted to the extension of convergence of Dirichlet polynomials in complex domains (see Leont'ev [144] and the references there). Sedletskii [190, 191] has studied mean periodic continuation for functions in C and L^p (see also Lyubich [147] and Leont'ev [144, Chap. 5]). As an illustration, we present a result from [191].

Let

$$\Lambda = \{\underbrace{\lambda_0, \dots, \lambda_0}_{m_0}, \dots, \underbrace{\lambda_n, \dots, \lambda_n}_{m_n}, \dots\} \quad (\lambda_n \in \mathbb{C}, |\lambda_{n+1}| \geqslant |\lambda_n|, m_n \in \mathbb{N}).$$

Denote by $L^p_\Lambda(a,b)$ $(1 \leqslant p < \infty, -\infty < a < b < +\infty)$ the closed subspace of $L^p(a,b)$ that is generated by the system $\{e^{i\lambda_n x}, x e^{i\lambda_n x}, \dots, x^{m_n-1} e^{i\lambda_n x}\}_{n=0}^{\infty}$.

Theorem 5. *Suppose that Λ is the set of zeros of the function*

$$\int_{-\pi}^{\pi} e^{izt}\, d\sigma(t) \tag{2}$$

with regard for multiplicity, where $\sigma(t)$ has a jump at the point $t = \pi$. Let $f \in L^p_\Lambda(-\pi,\pi)$. Then for an arbitrary $a > 0$, there exists $F \in L^p_\Lambda(-\pi, \pi + a)$ such that $F = f$ in $(-\pi,\pi)$.

If Λ satisfies the condition of Theorem 5 and $\sigma(t)$ has a jump at the point $t = -\pi$, then we have the following properties (see Sedletskii [192], Lemma 1.31):

(1) $\sup_n |\operatorname{Im} \lambda_n| < +\infty$; 2) $\sup_n m_n < +\infty$; 3) for each $\delta > 0$, there exists $C_\delta > 0$ such that

$$\left| \int_{-\pi}^{\pi} e^{izt}\, d\sigma(t) \right| \geqslant C_\delta e^{\pi |\operatorname{Im} z|},$$

provided that $|z - \lambda_n| \geqslant \delta$ for all $n \in \mathbb{Z}_+$. We also note that a sequence $\{\zeta_n\}_{n=-\infty}^{\infty}$ is the set of zeroes of the function (2), where $\sigma(t)$ has jumps at the points $t = \pm\pi$, if

$$\zeta_n = n + (-1)^n \int_{-\pi}^{\pi} e^{int}\, ds(t), \quad n \in \mathbb{Z};\ \operatorname{Var} s(t) < \infty$$

(see Sedletskii [191], Theorem 4).

Some information concerning mean periodic continuation on multidimensional spaces can be found in Berenstein and Struppa [19, Sect. 2] (see also Zaraisky [276]). All the results in Sect. 13.5 and their analogues for Euclidean spaces and symmetric spaces of noncompact type (see Sects. 14.5 and 15.6) are due to V.V. Volchkov. Using the techniques developed in Chaps. 16 and 17, one can obtain also similar theorems on compact symmetric spaces and on the phase space $\mathbb{C}^n$. In light of these results, it is natural to ask the following question.

Problem 5. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$, and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be domains in $\mathbb{R}^n$ such that $\mathcal{O}_1 \subset \mathcal{O}_2$ and $\{x \in \mathbb{R}^n : x - y \in \mathcal{O}_1 \text{ when } y \in \operatorname{supp} T\} \neq \varnothing$. Suppose that $f \in \mathcal{D}'_T(\mathcal{O}_1)$. Under what circumstances can f be extended to a distribution in the class $\mathcal{D}'_T(\mathcal{O}_2)$?*

In the general case this seems to be extremely difficult. A more modest problem is as follows.

Problem 6. *Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $R > r(T)$. Find a criterion that every $f \in \mathcal{D}'_T(-R, R)$ can be extended to a distribution in $\mathcal{D}'_T(\mathbb{R}^1)$.*

Notice that there are only certain sufficient conditions and other necessary conditions which are apart. These conditions are described in Sect. 13.5.

(vii) Many authors examined the precise conditions on decrease at infinity under which functions $f \in (L^{1,\mathrm{loc}} \cap \mathcal{D}'_T)(\mathbb{R}^n)$ are zero. The first precise result for ball means in $\mathbb{R}^n$ is due to Smith [200], who proved that, if a function $f \in C(\mathbb{R}^n)$ such that

$$\lim_{|x|\to\infty} f(x)|x|^{\frac{n-1}{2}} = 0 \tag{3}$$

satisfies

$$\int_{|x|\leqslant 1} f(x+y)\,\mathrm{d}x = 0, \qquad y \in \mathbb{R}^n,$$

then $f = 0$. Condition (3) cannot be replaced by the condition $f(x) = O(|x|^{(1-n)/2})$ as $|x| \to \infty$. For the first time, a similar phenomenon was mentioned by John [129] for functions with zero spherical means in $\mathbb{R}^3$. Thangavelu [209] has established that the spherical mean value operator of a fixed radius on $L^p(\mathbb{R}^n)$ is injective if and only if $1 \leqslant p \leqslant 2n/(n-1)$. A generalization of Thangavelu's theorem was obtained by Rawat and Sitaram [181]. Their result follows.

Theorem 6. *If $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)\backslash\{0\}$ and f belongs to $(L^p \cap \mathcal{D}'_T)(\mathbb{R}^n)$ for some $p \in [1, 2n/(n-1)]$, then $f = 0$.*

It turns out that the same is true for an arbitrary distribution $T \in \mathcal{E}'(\mathbb{R}^n)\backslash\{0\}$ (see Agranovsky and Narayanan [3]).

The problem under consideration becomes substantially more complicated for mean periodic functions on unbounded domains $\mathcal{O} \subset \mathbb{R}^n$, $n \geqslant 2$. These questions have been little studied. All existing results are concerned with the cases where $\mathcal{O}$ contains the exterior of a ball or a half-space (see Sitaram [197], V.V. Volchkov [225], Part II, and Ochakovskaya [162, 163]).

There are important open questions relating to this work. Let $\mathcal{O}$ be an unbounded domain in $\mathbb{R}^n$, $n \geqslant 2$, and $T \in \mathcal{E}'_\natural(\mathbb{R}^n)\backslash\{0\}$, $\mathcal{O}_T \neq \varnothing$ (see (15.1)).

Problem 7. *What growth conditions at infinity on* $f \in (L^{1,\text{loc}} \cap \mathcal{D}'_T)(\mathcal{O})$ *will force* $f = 0$ *in the set* $\bigcup_{x\in\mathcal{O}_T} \dot{B}_{r(T)}(x)$*?*

The following special cases are of great interest:

1. $\mathcal{O} = \{x \in \mathbb{R}^n : |x_n| < a\},\ a > r(T)$;
2. $\mathcal{O} = \{x \in \mathbb{R}^n : x_j > 0,\ j = 1, \dots, n\}$;
3. $\mathcal{O} = \{x \in \mathbb{R}^n : x_1^2 + \dots + x_{n-1}^2 < R^2\},\ R > r(T)$;
4. $\mathcal{O} = \{x \in \mathbb{R}^n : x_n > \alpha(x_1^2 + \dots + x_{n-1}^2)^{1/2}\},\ \alpha > 0$;
5. $\mathcal{O} = \{x \in \mathbb{R}^n : x_n > (x_1^2 + \dots + x_{n-1}^2)^{\alpha}\},\ \alpha > 0$;
6. $\mathcal{O} = \mathbb{R}^n\backslash\{x \in \mathbb{R}^n : x_1 \geqslant 0, x_2 = \dots = x_n = 0\}$.

Problem 8. *Suppose that* $f \in (L^{1,\text{loc}} \cap \mathcal{D}'_T)(\mathcal{O})$ *and, for almost all* $x \in \mathcal{O}$,

$$|f(x)| \leqslant F(|x|), \tag{4}$$

where F *is given positive function on* $[0, +\infty)$*. For what* F*,* $\mathcal{O}$*, and* T *can one assert that* $f = 0$ *in* $\mathcal{O}$*?*

The results of Sects. 13.6 and 14.6 are due to V.V. Volchkov [229]. For $\mathcal{O} = \mathbb{R}^n$, analogues of Problem 8 in which the behavior of the function on the right-hand side of (4) depends differently on different variables were also considered (see Ochakovskaya [163]).

This is interesting to compare with the counterpart of Problem 8 in the complex plane $\mathbb{C}$, in which the class $(L^{1,\text{loc}} \cap \mathcal{D}'_T)(\mathcal{O})$ is replaced by the class of holomorphic functions on the domain $\mathcal{O} \subset \mathbb{C}$. For nonzero holomorphic functions on $\mathcal{O}$, there is a number of results on the relation between the admissible rate of decrease at infinity of a function and that of the size of $\mathcal{O}$ (see Evgrafov [74], Sect. 3.5).

First results on the asymptotic behavior of mean periodic functions on symmetric spaces $X = G/K$ of noncompact type are due to Sitaram [199] and Shahshahani and Sitaram [193]. It was shown in Sitaram [198] that if $\Omega \subset X$ is a set of positive finite measure and $f \in L^1(X)$, then

$$\int_{g\Omega} f(x)\,\mathrm{d}x = 0 \quad \text{for all } g \in G$$

implies $f = 0$ almost everywhere. On the other hand, without the integrability assumption on f, the above conclusion is not justified. The results from [193] read as follows:

Theorem 7. *Assume that* rank $X = 1$*,* $p \in [1, 2]$*, and* $f \in L^p(X)$ *is such that*

$$\int_{B_r(y)} f(x)\,\mathrm{d}x = 0 \quad \textit{for } y \notin B_R. \tag{5}$$

Then $\operatorname{supp} f \subset B_{R+r}$. *In addition, given* $r, \varepsilon > 0$, *there exists a nontrivial function* $\varphi \in L^{2+\varepsilon}(X)$ *such that* $\int_{gB_r} \varphi(x)\,\mathrm{d}x = 0$ *for every* $g \in G$.

Theorem 8. *Let* X *be any symmetric space of noncompact type and* f *a "very rapidly decreasing" function on* X, *i.e.,* $f \in L^{1,\mathrm{loc}}(X)$ *and*

$$\operatorname*{ess\,sup}_{x\in X} \mathrm{e}^{kd(o,x)}|f(x)| < \infty, \quad \forall k \geqslant 0.$$

If (5) *holds, then* $\operatorname{supp} f \subset B_{R+r}$.

For Euclidean spaces, the analogue of Theorem 7 is proved in Sitaram [197]. The results in Sect. 15.7 were obtained by the first author (for special cases and other relating results, see V.V. Volchkov [223, 229], Ben Natan and Weit [9], and Cho-Ho Chu and Chi-Wai Leung [52]).

Theorems 15.29(ii) and 15.30 considerably strengthen Theorem 7 in various directions.

It will be needless to say that analogues of Problems 7 and 8 will be interesting for X.

The passage to the Heisenberg group H^n involves new features. If f is a continuous function on H^n and $f * \mu_r$ is the group convolution of f with the normalized surface measure on $S_r = \{(z,0) \in H^n : |z| = r\}$, we may ask whether the map taking f into $f*\mu_r$ is injective. By considering functions of the form $f(z,t) = \mathrm{e}^{\mathrm{i}t} g(z)$, the equation $f*\mu_r = 0$ reduces to the twisted convolution equation $g \star \mu_r = 0$. So if $\varphi_k|_{S_r} = 0$, then the function $h(z,t) = \mathrm{e}^{\mathrm{i}t}\varphi_k(z)$ satisfies $h * \mu_r = 0$ (see Example III in the summary to Part III). On the other hand, the function h is not in any $L^p(H^n)$, $p < \infty$. Therefore, if we assume that $f \in L^p(H^n)$, $p < \infty$, then we may expect that f going to $f * \mu_r$ is injective. It turns out that this is indeed true. Moreover, the following results are valid (see Thangavelu [209]).

Theorem 9. *Let* μ *be a compactly supported rotation invariant probability measure with no mass at the center of* H^n. *If* f *belongs to* $L^p(H^n) \cap C(H^n)$, $1 \leqslant p < \infty$, *and satisfies* $f * \mu = 0$, *then* $f = 0$.

Theorem 10. *Let* μ *be as in the previous theorem. Assume that* f *is a continuous function on* H^n, $f(z,\cdot) \in L^p(\mathbb{R}^1)$, $1 \leqslant p \leqslant 2$, *and for almost all* λ, *the integral*

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t} f(z,t)\,\mathrm{d}t$$

is of tempered growth on $\mathbb{C}^n$. *If* f *satisfies* $f * \mu = 0$, *then* $f = 0$.

Concerning twisted mean periodic functions on the phase space, Thangavelu [210] proved the following.

Theorem 11. *Let* f *be a continuous function of tempered growth on* $\mathbb{C}^n$, *and let* T *be a compactly supported rotation invariant measure on* $\mathbb{C}^n$. *If* $f \star T = 0$ *and* $\varphi_k \star T \not\equiv 0$ *for all* $k \in \mathbb{Z}_+$, *then* $f = 0$.

Note also [185], where the injectivity of the operator $f \to f \star \mu_r$ was established on the space of continuous functions $f \in C(\mathbb{C}^n)$ such that $f(z)e^{|z|^2/4} \in L^p(\mathbb{C}^n)$ $(1 \leqslant p \leqslant \infty)$.

The argument in the proof of Theorem 14.27 and the results of Chap. 17 make it possible to obtain a refinement of Theorem 11 with $f \in C(\mathbb{C}^n)$ satisfying the estimate

$$f(z) = O\left(e^{|z|^2/4-\gamma|z|}\right) \tag{6}$$

for some $\gamma > r(T)/2$. In the case under consideration it is reasonable to expect that (6) may be replaced by the condition

$$f(z) = O\left(\frac{e^{|z|^2/4}}{(1+|z|)^N}\right) \quad \text{as } z \to \infty \quad \text{for each } N > 0$$

(see Theorem 17.8(ii) and Erdélyi (ed.) [73, 6.13 (3)]). Using Theorem 17.8(ii) and some estimates of the functions $\phi_{\lambda,\eta,p,q,l}$, one can prove this conjecture for a broad class of radial distributions T. However, the case of an arbitrary $T \in \mathcal{E}'_\natural(\mathbb{C}^n)$ remains open. In our opinion, this question is of considerable interest.

(viii) For approximation theorems for solutions of linear partial differential equations, see Hörmander [126], Theorem 4.4.5. For $\mathcal{O} = \mathbb{R}^n$, Theorem 14.35 is due to Malgrange [150]. In the general case Theorem 14.35 was first established by Hörmander (see [126], Chap. 16). Another proof of this result is found in Napalkov [159, Theorem 20.1]. Theorems 14.36–14.40 were obtained by Zaraisky [276]. His method can be extended to symmetric spaces of rank one.

A number of questions suggest themselves for further investigation; we wish here to point out only a few of these.

Problem 9. *Given $T \in \mathcal{E}'(\mathbb{R}^n)\backslash\{0\}$, characterize those domains $\mathcal{O} \subset \mathbb{R}^n$ for which* $\mathrm{span}_{C^\infty(\mathcal{O})} E_T = N_T(\mathcal{O})$.

Problem 10. *It would be nice to know whether Theorems 14.36–14.38 are valid for each $T \in \mathcal{E}'_\natural(\mathbb{R}^n)\backslash\{0\}$.*

Problem 11. *Is it true that Theorem 14.40 holds for $0 \leqslant a < b$, $b - a \leqslant 2r(T)$?*

In conclusion we note that there are many other interesting results concerning mean periodicity and related topics. For some of them, see Berenstein and Dostal [12, 13], Berenstein and Gay [17], J.J. Betancor, J.D. Betancor and Méndez [35], Delsarte [54], Dickson [57], Ehrenpreis [67], Malgrange [151], Shapiro [194], Székelyhidi [207], and Yger [266].