Comments, Further Results, and Open Problems

Functions satisfying the assumptions of Proposition 6.1 have been studied by many authors. For a detailed discussion of these questions, we recommend the monograph of Levin [146].

In the case where $0 \notin \mathcal{Z}(f)$, the construction of $a_j^{\lambda,\eta}(f)$ (see Sect. 6.1) is from V.V. Volchkov [225], Part III, Chap. 1. The elaborated version in (6.21) and remaining results in Sect. 6.1 are also due to V.V. Volchkov.

For a systematic exposition of distribution theory on $\mathbb{R}^n$ and on manifolds, see Schwartz [189], Trèves [215], Hörmander [126], Bremermann [41], and Vladimirov [221]. A concise, but nevertheless self-contained, treatment of the basics is given in Hörmander [124]. The notion of invertible distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ was introduced by Ehrenpreis [62–66]. He showed that it is equivalent to $T * \mathcal{D}'(\mathbb{R}^n) = \mathcal{D}'(\mathbb{R}^n)$. A detailed study of the class $\mathrm{Inv}(\mathbb{R}^n)$ and its generalizations is given in Hörmander [126], Chap. 16.

The main references for the Bessel functions are Watson [250] and Erdélyi (ed.) [73]. For certain values of α and β, there are many group-theoretic interpretations of Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}(t)$. Almost all properties of $\varphi_\lambda^{(\alpha,\beta)}(t)$ in Sect. 7.2 can be found in Koornwinder [138] and Flensted-Jensen [76]. Propositions 7.4 and 7.6 and Theorem 7.1 were proved by V.V. Volchkov. In Sect. 7.3 we have followed the exposition of Vit.V. Volchkov [248]. Formula (7.47) generalizes the classical Mehler–Dirichlet formula [73, 3.7 (27)]. The treatment in Sect. 7.4 is based on Erdélyi (ed.) [73]. Proposition 7.11 was obtained by Vit.V. Volchkov [231]. In connection with representation (7.80), it is natural to pose the following question.

Problem 1. *To express the kernel $k_b(x,t)$ in* (7.80) *in terms of known special functions.*

For further study of hypergeometric functions, see Erdélyi (ed.) [73], Prudnikov, Brychkov, and Marichev [169], and Koornwinder [139].

The relation between the classes of distributions introduced in Part II has not been sufficiently explained yet. Among many open problems concerning properties of these classes, let us formulate the following ones.

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

Problem 2. *Let $T \in \mathcal{E}'(\mathbb{R}^1)$, $T \neq 0$, $\lambda \in \mathcal{Z}(\widehat{T})$. Can one construct upper bounds for* $|\mathrm{Im}\lambda|$ *and* $n_\lambda(\widehat{T})$ *in terms of* $\sigma_\lambda(\widehat{T})$*?*

The following special cases seem to be interesting.

Problem 3. *Suppose that*

$$\sup_{\lambda\in\mathcal{Z}(\widehat{T})} \frac{\log\big(2+\sigma_\lambda(\widehat{T})\big)}{\log(2+|\lambda|)} < +\infty. \tag{1}$$

Does this imply that

$$\sup_{\lambda\in\mathcal{Z}(\widehat{T})} \frac{|\mathrm{Im}\,\lambda|}{\log(2+|\lambda|)} < +\infty \tag{2}$$

and that

$$\sup_{\lambda\in\mathcal{Z}(\widehat{T})} \frac{n_\lambda(\widehat{T})}{\log(2+|\lambda|)} < +\infty? \tag{3}$$

Problem 4. *Let* (1) *and* (2) *hold. Must $T \in \mathfrak{M}(\mathbb{R}^1)$? The same question when* (1) *and* (3) *are satisfied.*

Problem 5. *Is it true that $\mathfrak{M}(\mathbb{R}^n) = \mathfrak{I}(\mathbb{R}^n)$ for $n \geqslant 2$?*

We know from Theorem 8.5 that $\mathfrak{M}(\mathbb{R}^1) \subset (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$. In addition, for each $T \in (\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$, properties (2) and (3) are valid (see Corollary 13.7 below). However, we expect that $(\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1) \neq \mathfrak{M}(\mathbb{R}^1)$.

Problem 6. *Find a criterion that a distribution $T \in \mathcal{E}'(\mathbb{R}^1)$ belongs to the class* $(\mathrm{Inv}_+ \cap \mathrm{Inv}_-)(\mathbb{R}^1)$ *in terms of the Fourier transform of T.*

All the results in Part II related to biorthogonal systems and biorthogonal decompositions (see Chap. 8 and Sects. 9.5 and 10.9) are due to V.V. Volchkov [231, 229]. For the case where all the zeros of $\widehat{T}$ are simple, the distribution ζ_T is considered in Levin [145], Appendix II. A solution of the Lyubich problem was first obtained by Kargaev [136] by a different method. Further applications of biorthogonal decompositions to problems of mean periodic functions are studied in Part III later (see also Sedletskii [192] and Leont'ev [144]).

The pioneering papers in the theory of spherical functions are those of Gel'fand [81], Berezin–Gel'fand [32], Godement [96], and Harish-Chandra [105, 106].

If rank $X = 1$, formula (7.19) gives an integral representation for spherical functions in the form of Fourier transforms of L^1-functions. Flensted-Jensen and Ragozin [77] wrote a note on the analogue of (7.19) for spherical functions on non-compact symmetric spaces of arbitrary rank. Similar representations for compact two-point homogeneous spaces and the phase space $\mathbb{C}^n$ are contained in Propositions 11.4 and 12.8.

The following problem goes in the same direction as Problem 1.

Problem 7. *To find an explicit form of the kernel* $\mathfrak{K}_{n,p,q}$ *in Proposition 12.8.*

The integral formula (10.31) is due to Harish-Chandra [105, 106]. Helgason proved the symmetry formula (10.32) and representation (10.38) [117, p. 116; 118, Chap. II]. The characterization of spherical functions on $\mathfrak{p}$ was obtained by Gindikin [83] and Korányi [140].

The generalized spherical function $\Phi_{\lambda,\delta}$ is a special case of Eisenstein integrals considered by Harish-Chandra in [107, 109]. The functions $\Phi_{\lambda,\eta,\delta,j}$, $\Psi_{\lambda,\eta,\delta,j}$, and their analogues in $\mathbb{R}^n$ were defined and studied by V.V. Volchkov in connection with significant applications in the theory of mean periodic functions. Similar results for functions (11.23), (11.28), (12.24), and (12.26) and Lemma 9.1 are due to Vit.V. Volchkov (see V.V. Volchkov and Vit.V. Volchkov [229, 231, 232]).

For further studies of the Fourier transform in $\mathbb{R}^n$, see Stein and Weiss [203], Edwards [60], and Hörmander [126]. An analog of classical Fourier analysis for Riemannian symmetric spaces of noncompact type was discovered by Helgason [116, 117, 123]. The function $\mathbf{c}(\lambda)$ was introduced by Harish-Chandra [105, 106]. The explicit formula for $\mathbf{c}(\lambda)$ in Sect. 10.3 is due to Gindikin and Karpelevič [84] (for special cases, see Harish-Chandra [105, 106] and Bhanumurthy [36, 37]). Some analogues of the Hausdorff–Young inequality, Hardy–Littlewood–Paley inequality, the Wiener Tauberian theorem, and some uncertainty principles for a Helgason-type Fourier transform can be found in Sarkar and Sitaram [186], and Mohanty, Ray, Sarkar, and Sitaram [155].

The original Paley–Wiener theorem describes the image by the Fourier transform of the space of compactly supported L^2-functions on $\mathbb{R}^n$. With distribution theory, the interest shifted to similar results for the spaces $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ (Schwartz [189], Hörmander [124]). The Paley–Wiener theorem in C^∞-category for the symmetric space X was established by Helgason [119]. The extension to $\mathcal{E}'(X)$ in Theorem 10.7(ii) was proved by Helgason [119] and by Eguchi, Hashizume, and Okamoto [61]. The results in Sect. 10.5 on the δ-spherical transform are due to Helgason [123]. Note also [211], where the Paley–Wiener theorem was established for the inverse Fourier transform on symmetric spaces and the Heisenberg group.

If X has rank one, the inversion formula and the Plancherel theorem for the spherical transform can be viewed as parts of the spectral theory of the singular ordinary differential operator (see, for instance, Dunford and Schwartz [59], Chap. 13). Flensted-Jensen [76] generalized these rank-one results and the Paley-Wiener theorem, allowing the multiplicities m_γ, $m_{2\gamma}$ to be arbitrary positive numbers. Further generalizations were given, for example, by Chébli [50], Koornwinder [138], and Trimèche [217] (see Koornwinder [139] for a survey).

Harmonic analysis on compact symmetric spaces U/K was developed by Cartan [48]. He interpreted the spherical harmonics group-theoretically and generalized the classical expansion (4.11) to U/K. Further generalization was given by Weyl (see (1.69)). Another approach, more in line with Helgason's Fourier transform on X, was initiated by Sherman [195].

The material in Sect. 11.3 is from V.V. Volchkov and Vit.V. Volchkov [229, 232]. The modified version of Theorem 11.2 was established by Koornwinder [138]. In the case of the sphere $\mathbb{S}^2$, the theorem of Koornwinder is due to Beurling (unpublished, see [138]). There is an important open question relating to this theme.

Problem 8. *Generalize Theorem 11.2 to an arbitrary Riemannian symmetric space U/K of the compact type.*

For known results concerning Problem 8, see Ólafsson and Schlichtkrull [164]. In particular, in [164] the Paley–Wiener theorem is proved for K-invariant smooth functions on U/K supported in a neighborhood of the origin.

The transform $\mathcal{F}_l^{(p,q)}$ was introduced and studied by Vit.V. Volchkov [229, 231]. Another transformation which is well adapted to the twisted convolution on $\mathbb{C}^n$ is the Fourier–Weyl transform (see, for example, Folland [78], Chap. 1, and Thangavelu [208], Chap. 1). For the Paley–Wiener theorem for the Fourier–Weyl transform, see Thangavelu [210] and the references there.

The theory of transmutation operators given here is for the most part based on the authors' papers [230–232] and [248]. The study of properties of these operators leads to a series of unsolved questions of an interesting sort, which we want to mention.

Problem 9. *Let $k \in \mathbb{Z}_+$ and $j \in \{1, \dots, d(n,k)\}$. For what $l, m \in \mathbb{Z}_+$ and $R \in (0, +\infty]$, can we assert that the transform $\mathfrak{A}_{k,j}$ sets up a homeomorphism between $C^l_{k,j}(B_R)$ and $C^m_\natural(-R, R)$?*

This question makes sense for other reasonable function spaces. Some partial information about this is provided by Theorems 9.3(iv) and 9.5(iv).

Problem 10. *Establish a best possible form of the inequalities* (9.64) *and* (9.81).

The same questions for other transmutation operators introduced in Part II also remain open.

As established in Theorem 10.12(iii), the mapping

$$\mathfrak{A}_\delta : \mathcal{D}'_{\check{\delta}}(B_R) \to \mathcal{D}'_W\big(\mathcal{B}_R, \mathrm{Hom}\big(V_\delta, V_\delta^M\big)\big)$$

is injective if $\delta \in \widehat{K}_M$ and $R \in (0, +\infty]$. One can ask the following questions in this connection.

Problem 11. *What is the range of the operator $\mathfrak{A}_\delta$ and how can one reconstruct $f \in \mathcal{D}'_{\check{\delta}}(B_R)$ from $\mathfrak{A}_\delta(f)$?*

Problem 12. *Find a natural analogue of Theorem 10.15 for the mapping $\mathfrak{A}_\delta$ instead of $\mathfrak{A}$.*

Also, we are interested in the following problem.

Problem 13. *Introduce and study an analog of the operator $\mathfrak{A}_\delta$ for compact symmetric spaces of arbitrary rank.*

Notice that for the rank one case, this question solves by the results in Sect. 10.8.

The material in Sect. 11.5 and Lemma 11.1 are due to Zaraisky (unpublished). The following questions arise in connection with these results.

Problem 14. *To extend Theorem 11.7 to* $\mathbb{P}^n_{\mathbb{R}}$.

Problem 15. *To find an analog of Theorem 11.7 for the noncompact case.*

For a discussion of other aspects concerning transmutation operators, the reader may look up Koornwinder [139], Trimèche [218], and Mourou and Trimèche [158].