Comments, Further Results, and Open Problems

The definition of octaves goes back to Graves. The proof of properties (1.1)–(1.8) can be found in Postnikov [167], Lectures 14–16. The example in (1.9) occurs in Bredon [40], Chap. 7, Sect. 4. The coincidence of Aut $\mathbb{C}a$ with G_2 is noted in Cartan [46]. The decomposition (1.10) is discussed in Johnson [130]. Equality (1.11) was first proved by Chevalley and Schafer [51]. It is also established in the paper by Freudenthal [79] using differential techniques. For further information about the algebras $\mathbb{C}a$ and $\mathbb{A}1$, see Postnikov [167], Cantor and Solodovnikov [45], Jacobson [127], Freudenthal [79], Johnson [130], and references there.

Relations (1.16) and (1.17) were obtained by Vit.V. Volchkov [244, 248]. The modified version of the form $\Phi_{\mathbb{C}a}(\cdot,\cdot)$ is contained in Mostow [157], Chap. 20. Most of the classical groups in Sect. 1.1 are described in Helgason [115], Chap. 9, Sect. 4. The definition of the group $\mathrm{O}_{\mathbb{C}a}(2)$, Examples 1.1–1.3, and Proposition 1.1 are from Vit.V. Volchkov [244]. Proposition 1.2 is classical (see, for example, Stein and Weiss [203], Chap. 4, Sect. 2). Theorem 1.3 is stated in Vit.V. Volchkov [244]. Concerning results in the spirit of Theorem 1.3, see Erdélyi (ed.) [73], Chap. 11, Sect. 11.4.

A systematic exposition of foundations of differential geometry is given in Helgason [115], Kobayashi and Nomizu [137], and in many other books, including Sternberg [204], Bishop and Crittenden [38], Gromoll, Klingenberg, and Meyer [101], Wolf [263], and Dubrovin, Novikov, and Fomenko [58].

The notion of a ζ domain was initiated by V.V. Volchkov [225] in connection with the study of convolution equations. The following problem seems to be interesting.

Problem 1. *Let $\mathcal{O}$ be a ζ domain ($\zeta > 0$) on a Riemannian space M. Does this imply that $\mathcal{O}$ is a ζ_1 domain for each $\zeta_1 \in (0,\zeta)$?*

The answer is unknown even for $M = \mathbb{R}^2$.

In Sects. 1.3 and 1.4 we have used Helgason [115, 122] and Berenstein and Zalcman [26]. The treatment of Sect. 1.5 is partly based on Helgason [122], Chap. 4. A more general version of the expansion (1.69) is given in Harish-Chandra [108]. For formula (1.80), see V.V. Volchkov [222, 227].

V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group,* Springer Monographs in Mathematics,

The automorphisms of $B^2_{\mathbb{C}}$ occur in Poincaré [166]. Proposition 2.10 is contained in Rudin [184], Chap. 2. Analogues of mapping (2.30) and Proposition 2.10 for the Poincaré model of the real hyperbolic space may be found in Ahlfors [5], Chap. 2. As the present description of the involutions σ_a in terms of inner products and projections suggests, many properties of $\mathrm{Aut}\,(B^n_{\mathbb{C}})$ extend from $B^n_{\mathbb{C}}$ to the unit ball of an arbitrary Hilbert space. See Hayden and Suffridge [113] and, for even more far-reaching extensions, Harris [110–112].

The constructions in Sects. 2.3 and 2.4 are from Vit.V. Volchkov [235, 244]. For the Beltrami–Klein model and other models of the hyperbolic spaces, see also Mostow [157], Chaps. 19, 20, and Springer and Veldkamp [202].

An extensive exposition of the projective spaces is given in Besse [10], Chap. 3, and Busemann [44], Chap. 6, Sect. 53. A very geometrical characterization of the Cayley projective plane has been investigated by Tits [213]. The identification $F_4/\,\mathrm{Spin}(9)$ with the set of primitive idempotents of the algebra $\mathbb{A}1$ was first obtained by Freudenthal [79]. The algebraic Freudenthal's approach has been widely generalized to other Cayley algebras. See Faulkner [75] for references. The treatment of Chap. 3 given here is similar to Vit.V. Volchkov [244, 248] and V.V. Volchkov and Vit.V. Volchkov [229], Sect. 3.1.

The ways in which a sphere can be written as a homogeneous space were essentially classified by Montgomery–Samelson [156] (completed by Borel [39]). See also Gorbacevič and Oniščik [99] for a survey. Concerning harmonic analysis on spheres, Kostant [141] proved the following:

Theorem 1. *Let $\mathcal{H}$ be the space of complex-valued harmonic polynomials on the tangent space $\mathfrak{p}$ to the rank one symmetric space G/K. If the dimension of the 2-root space $\mathfrak{g}^{2\alpha}$ is > 1, K decomposes $\mathcal{H}$ under the adjoint representation as a 2-parameter family of irreducible representations, each occurring with multiplicity one. More precisely,*

$$\mathcal{H} = \bigoplus_{i,j\in\mathbb{Z}_+} \mathcal{H}_{ij},$$

and the subspace of homogeneity k decomposes as

$$\mathcal{H}^k = \bigoplus_{2i+j=k} \mathcal{H}_{ij}.$$

Kostant's general theory in the case $\dim\mathfrak{g}^{2\alpha} = 0$ or 1 gives the cases $\mathrm{SO}(n)$ and $\mathrm{U}(n)$.

Various explicit characterizations of the above spaces were obtained by many authors. The material in Sects. 4.1 and 4.2 is well known (see Vilenkin [220], Chap. 9, Stein and Weiss [203], Chap. 4, and Rudin [184], Chap. 12). In Sects. 4.3–4.6 we have followed the exposition of Vit.V. Volchkov [235, 244, 248]. Alternate characterizations of the irreducibles for $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ and the exceptional case of $\mathrm{Spin}(9)$ have been given by Johnson and Wallach [131], Johnson [130], and Smith [201]. The approach in [130, 131] involves considering invariant differential operators other than the Laplacian.

The results of Chap. 4 show that every closed K-invariant subspace H of $C(S)$ (S denotes the unit sphere in $\mathfrak{p}$) has the form

$$H = \mathcal{H}_\Omega, \quad \Omega \subset \{(i, j) \colon i, j \in \mathbb{Z}_+\},$$

where $\mathcal{H}_\Omega$ is the uniform closure of the linear span of the spaces $\mathcal{H}_{ij}$ with $(i, j) \in \Omega$. In this connection the following problem arises.

Problem 2. *To find those sets Ω for which the corresponding K-space $\mathcal{H}_\Omega$ is an algebra relative to pointwise multiplication.*

Several remarks are in order here. Let $\mathcal{H}_{i_1 j_1}\mathcal{H}_{i_2 j_2}$, $i_1, i_2, j_1, j_2 \in \mathbb{Z}_+$, be the vector space of finite sums

$$\sum f_\alpha g_\alpha$$

with $f_\alpha \in H_{i_1 j_1}$, $g_\alpha \in \mathcal{H}_{i_2 j_2}$. It is clear that each $\mathcal{H}_{i_1 j_1}\mathcal{H}_{i_2 j_2}$ is a finite-dimensional K-space and is therefore a sum of finitely many $\mathcal{H}_{ij}$'s according to the foregoing. In addition, $\mathcal{H}_\Omega$ is an algebra if and only if $\mathcal{H}_{i_1 j_1}\mathcal{H}_{i_2 j_2} \subset \mathcal{H}_\Omega$ whenever $(i_1, j_1) \in \Omega$ and $(i_2, j_2) \in \Omega$. In this way Problem 2 reduces to the following question.

Problem 3. *Let*

$$\mathcal{H}_{i_1 j_1}\mathcal{H}_{i_2 j_2} = \sum_{(i,j)\in\Omega} \mathcal{H}_{ij}$$

(*as usual, we identify $\mathcal{H}^k$ with the space of restrictions of its elements to S*). *To describe the set Ω.*

The solution of Problem 3 for the spaces $\mathcal{H}_2^{n,p,q}$, $n \geqslant 3$, is given in the following:

Theorem 2 ([184], Chap. 12). *If $n \geqslant 3$, $p, q, r, s \in \mathbb{Z}_+$, and*

$$\mu = \min\{p, s\} + \min\{r, q\},$$

then

$$\mathcal{H}_2^{n,p,q}\mathcal{H}_2^{n,r,s} = \sum_{\nu=0}^{\mu} \mathcal{H}_2^{n,p+r-\nu,q+s-\nu}.$$

The case $n = 2$ is rather different and more complicated. We note only that

$$\mathcal{H}_2^{2,p,q}\mathcal{H}_2^{2,r,s} \subset \sum_{\nu=0}^{\mu} \mathcal{H}_2^{2,p+r-\nu,q+s-\nu}$$

and for $p \geqslant q$,

$$\mathcal{H}_2^{2,p,q}\mathcal{H}_2^{2,p,q} = \sum_{\nu=0}^{q} \mathcal{H}_2^{2,2p-2\nu,2q-2\nu}.$$

For further results in this direction, see Rudin [184], Chap. 12, Sect. 12.4.

The integrals in (5.9), (5.21), (5.23), (5.24), (5.26), (5.40), and (5.42) are an explicit form of the Eisenstein integrals for the rank one case (see Helgason [123], Chap. 3, Sect. 11). Relations (5.21) and (5.23) are given in El Harchaoui [72]. Lemmas 5.1–5.3 also occur there. The proof in the text is a modification of El Harchaoui's proof. Equalities (5.26) and (5.40) are from Vit.V. Volchkov and N.P. Volchkova [249], and Berkani, El Harchaoui, and Gay [33]. Note, however, that in [249] formula (5.26) is proved up to a constant factor. Lemma 5.4 is contained in [33]. Lemma 5.5 was established in [249]. Representation (5.42) and other results in Sect. 5.4 are due to Vit.V. Volchkov (see [244], and [229], Part III, Sect. 3.2). For the description of all eigenfunctions of the Laplacian by means of some "Poisson integral" of hyperfunctions, see Helgason [123], Chap. 5, Sect. 6.1.