Appendix A
Elementary Homological Algebra

Homological algebra is the branch of algebra that arose out of the necessity to provide algebraic foundations for homology theory. In this appendix we do not propose to systematically develop homological algebra. Rather, we just wish to develop it far enough to provide for our needs in this book. Thus, we do not advise the reader to read it straight through – it will seem like a curious collection of unmotivated results – but rather to refer to it as necessary.

The basic method of proof here is "diagram chasing". We do some representative proofs in detail but leave most of them for the reader.

Throughout this appendix R denotes an arbitrary commutative ring with 1. We will specialize to $R = \mathbb{Z}$, or a field, as necessary.

A.1 Modules and Exact Sequences

Definition A.1.1. A sequence of R-modules

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\varphi_{i-1}} A_i \xrightarrow{\varphi_i} A_{i+1} \xrightarrow{\varphi_{i+1}} A_{i+2} \longrightarrow \cdots$$

is *exact* if for each i, $\operatorname{Ker}(\varphi_{i+1}) = \operatorname{Im}(\varphi_i)$. ◊

Remark A.1.2. In an exact sequence, $A_i = 0$ is equivalent to $\varphi_{i-2} : A_{i-2} \to A_{i-1}$ being a surjection and $\varphi_{i+1} : A_{i+1} \to A_{i+2}$ being an injection. ◊

Remark A.1.3. A sequence $0 \to A \to 0$ is exact if and only if $A = 0$. A sequence $0 \to A \xrightarrow{\varphi} B \to 0$ is exact if and only if φ is an isomorphism. ◊

Definition A.1.4. An exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is a *short exact sequence*. ◊

S.H. Weintraub, *Fundamentals of Algebraic Topology*, Graduate Texts in Mathematics 270, DOI 10.1007/978-1-4939-1844-7

Remark A.1.5. This is equivalent to: φ is an injection, ψ is a surjection, and $\mathrm{Im}\,(\varphi) = \mathrm{Ker}\,(\psi)$. ◊

Definition A.1.6. A short exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is *split* if $\mathrm{Im}\,(\varphi) = \mathrm{Ker}\,(\psi)$ is a direct summand of B. ◊

Lemma A.1.7. *Given a short exact sequence of R-modules*

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

the following are equivalent:

(1) *There exists a homomorphism (necessarily a surjection)* $\alpha : B \to A$ *such that* $\alpha \circ \varphi = id_A$.
(2) *There exists a homomorphism (necessarily an injection)* $\beta : C \to B$ *such that* $\psi \circ \beta = id_C$.
(3) *This sequence is split.*

If these equivalent conditions hold then α *and* β *are said to split (or be a splitting of) the sequence, and*

$$\begin{aligned} B &\cong \mathrm{Im}\,(\varphi) \oplus \mathrm{Ker}\,(\alpha) \\ &\cong \mathrm{Ker}\,(\psi) \oplus \mathrm{Im}\,(\beta) \\ &\cong A \oplus C. \end{aligned}$$

Lemma A.1.8 (The five lemma). *Given a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

If f_1, f_2, f_4, *and* f_5 *all isomorphisms, then so is* f_3.

Proof. First we show f_3 is an injection. Let $x \in A_3$ with $f_3(x) = 0$, i.e., x "goes to 0" in B_3. Then x goes to 0 in B_4. Now x goes to some element y in A_4. By commutativity y goes to 0 in B_4. But f_4 is an isomorphism, so $y = 0$. Thus x goes to 0 in A_4, so by exactness x comes from some z in A_2. Then z goes to some w in B_2. By commutativity w goes to 0 in B_3, so w comes from some v in B_1. Since f_1 is an isomorphism v comes from some u in A_1. Then u goes to some t in A_2 and by commutativity t also goes to w in B_2. But f_2 is an isomorphism, so $t = z$. Thus u in A_1 goes to z in A_2 which goes to x in A_3. By the exactness of the top row, $x = 0$.

Next we show f_3 is a surjection. Let $x \in B_3$. Then x goes to some y in B_4. Since f_4 is an isomorphism, y comes from some z in A_4. By exactness, y goes to 0 in B_5, so z also goes to 0 in B_5. But f_5 is an isomorphism, so z goes to 0 in A_5. By exactness, z comes from some w in A_3. Then w goes to $v = f_3(w)$ in B_3. By commutativity, v also goes to y in B_4, so $x - v$ goes to 0 in B_4. Hence $x - v$ comes from some u in B_2. Since f_2 is an isomorphism, u comes from some t in A_2. Then t goes to some s in A_3, and by commutativity s goes to $x - v$ in B_3, i.e., $f(s) = x - v$. But then $f(s+w) = x - v + v = x$. □

Corollary A.1.9 (The short five lemma). *Given a commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \\ 0 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & 0 \end{array}$$

If f_2 and f_4 are isomorphisms, then so is f_3.

Recall the following basic definition.

Definition A.1.10. An R-module M is *free* if there is some subset $S = \{m_i\}_{i \in I}$ of M such that every $m \in M$ can be expressed uniquely as a finite sum

$$m = \sum_{i \in I} r_i m_i, \quad r_i \in R, \text{ only finitely many nonzero.}$$

In this case, S is a *basis* of M. ◊

Lemma A.1.11. (a) *Let C be a free R-module. Then every short exact sequence of R-modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is split.

(b) *If $R = \mathbb{F}$ is a field, every short exact sequence of R-modules is split.*

Proof. (a) Let C have basis $\{c_i\}_{i \in I}$. For each i, let $b_i \in B$ with $\psi(b_i) = c_i$. Then there is a unique map $\beta : C \to B$ defined by $\beta(c_i) = b_i$ for each i, and this gives a splitting by Lemma A.1.7.

(b) If $R = \mathbb{F}$ is a field, an R-module is an $\mathbb{F}$-vector space, so has a basis, and so is free.

□

We remind the reader of the construction of the dual of a module and the dual of a map, which are at the core of cohomology.

Definition A.1.12. Let M be an R-module. Its *dual module* M^* is the R-module

$$M^* = \mathrm{Hom}(M,R),$$

the module of R-homomorphisms from M to R.

If $f : M \to N$ is a map of R-modules, the *dual map* $f^* : N^* \to M^*$ is the map defined by

$$(f^*(\alpha))(m) = \alpha(f(m)) \quad \text{for } \alpha \in N^*,\ m \in M. \qquad \diamondsuit$$

We now generalize the notions of ring and module to the graded case.

Definition A.1.13. (1) A *graded commutative ring* $\mathscr{S}$ is a ring $\mathscr{S}$ with 1 such that the additive group of $\mathscr{S}$ decomposes as $\mathscr{S} = \bigoplus_{i\in\mathbb{Z}} S_i$ and the multiplication has the property that of $x \in S_j$ and $y \in S_k$ then $xy \in S_{j+k}$. Also, the ordinary commutative law for multiplication is replaced by the law

$$yx = (-1)^{jk}xy \quad \text{for } x \in S_j,\ y \in S_k.$$

(2) A *left module* $\mathscr{N}$ over a graded commutative ring $\mathscr{S}$ is a left $\mathscr{S}$-module $\mathscr{N}$ such that $\mathscr{N} = \bigoplus_{i\in\mathbb{Z}} N_i$ and the module structure has the property that if $s \in S_j$ and $n \in N_{j+k}$ then $sn \in N_k$.

(3) A *ring homomorphism* $\varphi : \mathscr{S} \to \mathscr{T}$, where $\mathscr{N} = \bigoplus_{i\in\mathbb{Z}} S_i$ and $\mathscr{T} = \bigoplus_{i\in\mathbb{Z}} T_i$ are graded commutative rings, is a homomorphism of rings that satisfies the additional property that if $s_i \in S_i$, then $t_i = \varphi(s_i) \in T_i$.

(4) A *graded commutative R-algebra* $\mathscr{S}$ is a graded commutative ring that is an R-algebra with the property that if $r \in R$ and $s_i \in S_i$, then $rs_i \in S_i$.

(5) An *algebra homomorphism* $\varphi : \mathscr{S} \to \mathscr{T}$ between graded commutative R-algebras is a ring homomorphism that is a map of algebras. $\diamondsuit$

(Of course, any graded commutative ring is a graded commutative $\mathbb{Z}$-algebra.)

A.2 Chain Complexes

Definition A.2.1. A *chain complex* over R is $\mathscr{A} = \{A_i, d_i\}_{i\in\mathbb{Z}}$ a set of R-modules A_i, and R-module homomorphisms $d_i : A_i \to A_{i-1}$, with the property that $d_{i-1}d_i : A_i \to A_{i-2}$ is the 0 map for every i. $\diamondsuit$

Often we abbreviate d_i by d, and write the relation $d_{i-1}d_i = 0$ as $d^2 = 0$. Note that this condition implies $\mathrm{Im}(d_{i+1}) \subseteq \mathrm{Ker}(d_i)$ for every i.

Definition A.2.2. Let $\mathscr{A}$ be a chain complex. Then the i-th *homology group* $H_i(\mathscr{A})$ is the R-module

$$H_i(\mathscr{A}) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}). \qquad \diamondsuit$$

Definition A.2.3. Let $a \in \mathrm{Ker}(d_i)$. Then $[a]$ denotes the image of a in $H_i(\mathscr{A})$ under the quotient map $\mathrm{Ker}(d_i) \to \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1})$, and $[a]$ is the *homology class represented by a*. If $\alpha \in H_i(\mathscr{A})$ and $a \in \mathrm{Ker}(d_i)$ with $[a] = \alpha$, then *a represents* (or is a *representative* of) α. ♢

Definition A.2.4. An element $a \in A_i$ is called a *chain*. An element $a \in \mathrm{Ker}(d_i)$ is called a *cycle*. An element $a \in \mathrm{Im}(d_{i+1})$ is called a *boundary*. ♢

Definition A.2.5. Let $\mathscr{A} = \{A_i, d_i^A\}$ and $\mathscr{B} = \{B_i, d_i^B\}$ be chain complexes. A *map* of chain complexes $F : \mathscr{A} \to \mathscr{B}$ is a collection of homomorphisms $F = \{f_i : A_i \to B_i\}_{i\in\mathbb{Z}}$ such that for each i, the following diagram commutes:

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle d_i^A}\downarrow & & \downarrow{\scriptstyle d_i^B} \\ A_{i-1} & \xrightarrow{f_{i-1}} & B_{i-1}. \end{array}$$

♢

Definition A.2.6. Let $F : \mathscr{A} \to \mathscr{B}$ be a map of chain complexes. The *induced map on homology* $F_* : H_*(\mathscr{A}) \to H_*(\mathscr{B})$ is defined as follows: $F_* = \{f_i : H_i(\mathscr{A}) \to H_i(\mathscr{B})\}_{i\in\mathbb{Z}}$ where $f_i([a]) = [f_i(a)]$. ♢

Lemma A.2.7. *The induced map on homology F_* is well-defined.*

Proof. We must show it is independent of the choice of representative a. Thus suppose $[a] = [a']$. Then $a = a' + a''$ where $a'' \in \mathrm{Im}(d_{i+1})$, i.e., $a'' = d_{i+1}(a''')$ for some a'''. But then

$$\begin{aligned} f_i(a) &= f_i(a' + a'') = f_i(a') + f_i(a'') \\ &= f_i(a') + f_i(d_{i+1}(a''')) \\ &= f_i(a') + d_{i+1}(f_i(a''')) \end{aligned}$$

so $[f_i(a)] = [f_i(a')]$. □

Definition A.2.8. Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{A} \to \mathscr{B}$ be maps of chain complexes. A *chain homotopy* between F and G is a collection of maps $\Phi = \{\varphi_i : A_i \to B_{i+1}\}_{i\in\mathbb{Z}}$ such that

$$d_{i+1}^B \varphi_i + \varphi_{i-1} d_i^A = f_i - g_i : A_i \longrightarrow B_i \quad \text{for each } i$$

♢

Lemma A.2.9. *Suppose that there is a chain homotopy Φ between F and G. Then $F_* = G_*$, i.e., $f_i = g_i$ for each i.*

Proof. Let $a \in A_i$ be a cycle. Then

$$[f_i(a)] = [(g_i + d_{i+1}^B \varphi_i + \varphi_{i-1} d_i^A)(a)]$$

$$
\begin{aligned}
&= [g_i(a)] + [d_{i+1}^B \varphi_i(a)] + [\varphi_{i-1} d_i^A(a)] \\
&= [g_i(a)]
\end{aligned}
$$

as $d_{i+1}^B(\varphi_i(a))$ is a boundary and $d_i^A(a) = 0$ since a is a cycle. □

Theorem A.2.10. *Let $0 \to \mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C} \to 0$ be a short exact sequence of chain complexes, i.e., suppose that for each i, $0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$ is exact. Then there is a long exact sequence in homology*

$$\cdots \longrightarrow H_i(\mathscr{A}) \xrightarrow{f_i} H_i(\mathscr{B}) \xrightarrow{g_i} H_i(\mathscr{C}) \xrightarrow{\partial_i} H_{i-1}(\mathscr{A}) \longrightarrow \cdots .$$

Proof. We show how to define ∂ by a “diagram chase”. The remainder of the proof is a further diagram chase, which we omit.

We have:

$$
\begin{CD}
0 @>>> A_i @>f_i>> B_i @>g_i>> C_i @>>> 0 \\
@. @VV d_i^A V @VV d_i^B V @VV d_i^C V @. \\
0 @>>> A_{i-1} @>f_{i-1}>> B_{i-1} @>g_{i-1}>> C_{i-1} @>>> 0
\end{CD}
$$

Let $c_i \in C_i$ be a cycle. Since g_i is onto, there is an element $b_i \in B_i$ with $g_i(b_i) = c_i$. Let $b_{i-1} = d_i^B(b_i)$. Then $g_{i-1}(b_{i-1}) = g_{i-1} d_i^B(b_i) = d_i^C g_i(b_i)$ by commutativity $= d_i^C(c_i) = 0$ since c_i is a cycle. By exactness, there is a unique $a_{i-1} \in A_{i-1}$ with $f_{i-1}(a_{i-1}) = b_i$. Define

$$\partial_i([c_i]) = [a_{i-1}] \in H_{i-1}(\mathscr{A}).$$

□

Theorem A.2.11. *Suppose there is a commutative diagram of exact sequences*

$$
\begin{CD}
\cdots @>>> A_i @>\alpha_1>> B_i @>\gamma_1>> C_i @>>> A_{i-1} @>>> \cdots \\
@. @VV\alpha_2V @VV\beta_1V @VV\varepsilon V @VV\alpha_2V @. \\
\cdots @>>> D_i @>\beta_2>> E_i @>\gamma_2>> F_i @>>> D_{i-1} @>>> \cdots
\end{CD}
$$

and suppose further that ε is an isomorphism, for each i.

Then there is a long exact sequence

$$\cdots \longrightarrow A_i \xrightarrow{\alpha} B_i \oplus D_i \xrightarrow{\beta} E_i \xrightarrow{\Delta} A_{i-1} \longrightarrow \cdots$$

where the maps α, β, Δ are defined by:

$$\alpha(q) = (\alpha_1(q), \alpha_2(q))$$
$$\beta(r,s) = \beta_1(r) - \beta_2(s)$$
$$\Delta(t) = \partial_1 \varepsilon^{-1} \gamma_2(t).$$

Proof. We shall chase this diagram to show that the given sequence is exact.

(i) $\mathrm{Im}(\alpha) \subseteq \mathrm{Ker}(\beta)$: Let $(r,s) = \alpha(q) = (\alpha_1(q), \alpha_2(q))$ for some q. Then $\beta(r,s) = \beta_1(r) - \beta_2(s) = \beta_1\alpha_1(q) - \beta_2\alpha_2(q) = 0$ by the commutativity of the diagram.

(ii) $\mathrm{Im}(\beta) \subseteq \mathrm{Ker}(\Delta)$: Let $t = \beta(r,s) = \beta_1(r) - \beta_2(s)$ for some r,s. Then $\Delta(t) = \partial_1\varepsilon^{-1}\gamma_2(t) = \partial_1\varepsilon^{-1}\gamma_2(\beta_1(r)) - \partial_1\varepsilon^{-1}\gamma_2(\beta_2(s))$. But, by commutativity, $\gamma_1 = \varepsilon^{-1}\gamma_2\beta_1$ so $\Delta(t) = \partial_1\gamma_1(r) - \partial_1\varepsilon^{-1}(\gamma_2\beta_2(s)) = 0 - 0 = 0$ by exactness.

(iii) $\mathrm{Im}(\Delta) \subseteq \mathrm{Ker}(\alpha)$: Let $q = \partial_1\varepsilon^{-1}\gamma_2(t)$. Then $\alpha(q) = (\alpha_1\partial_1\varepsilon^{-1}\gamma_2(t), \alpha_2\partial_1\varepsilon^{-1}\gamma_2(t))$. But, by commutativity, $\partial_2 = \alpha_2\partial_1\varepsilon^{-1}$ so $\alpha(q) = (\alpha_1\partial_1(\varepsilon^{-1}\gamma_2(t)), \partial_2\gamma_2(t)) = (0,0) = 0$.

(iv) $\mathrm{Ker}(\beta) \subseteq \mathrm{Im}(\alpha)$: Suppose $\beta(r,s) = \beta_1(r) - \beta_2(s) = 0$. Let $u = \beta_1(r) = \beta_2(s)$. Then $\gamma_2(u) = \gamma_2\beta_1(r) = \gamma_2\beta_2(s) = 0$ by exactness. By commutativity, $\gamma_2\beta_1(r) = \varepsilon\gamma_1(r)$ so $\varepsilon\gamma_1(r) = 0$. But ε is an isomorphism, so $\gamma_1(r) = 0$. Then, by exactness, $r = \alpha_1(q_0)$ for some q_0. Let $s_0 = \alpha_2(q_0)$. Then $\beta_2(s_0) = \beta_2\alpha_2(q_0) = \beta_1\alpha_1(q_0) = \beta_1(r) = \beta_2(s)$, so $\beta_2(s - s_0) = 0$. Then, by exactness, $s - s_0 = \partial_2(v)$ for some v. Let $w = \varepsilon^{-1}(v)$ and $x = \partial_1(w)$. Then $\alpha_2(x) = \alpha_2\partial_1(w) = \partial_2\varepsilon(w) = \partial_2\varepsilon(\varepsilon^{-1}(v)) = \partial_2(v) = s - s_0$. Set $q = q_0 + x$. Then $\alpha_1(q) = \alpha_1(q_0 + x) = \alpha_1(q_0) + \alpha_1(x) = \alpha_1(q_0) + \alpha_1\partial_1(w) = \alpha_1(q_0) = r$, and $\alpha_2(q) = \alpha_2(q_0 + x) = \alpha_2(q_0) + \alpha_2(x) = s_0 + (s - s_0) = s$. Thus $(r,s) = \alpha(q)$.

(v) $\mathrm{Ker}(\Delta) \subseteq \mathrm{Im}(\beta)$: Suppose $\Delta(t) = \partial_1\varepsilon^{-1}\gamma_2(t) = 0$. Then $0 = \partial_1(\varepsilon^{-1}\gamma_2(t))$ so $\varepsilon^{-1}\gamma_2(t) = \gamma_1(r)$ for some r. Then $\varepsilon\gamma_1(r) = \gamma_2(t)$, and then $\gamma_2\beta_1(r) = \varepsilon\gamma_1(r) = \gamma_2(t)$. Hence $\gamma_2(\beta_1(r) - t) = 0$ so $\beta_1(r) - t = \beta_2(s)$ for some s. But then $\beta_1(r) - \beta_2(s) = t$, so $t = \beta(r,s)$.

(vi) $\mathrm{Ker}(\alpha) \subseteq \mathrm{Im}(\Delta)$: Suppose $\alpha(q) = 0$. Then $\alpha_1(q) = \alpha_2(q) = 0$. Since $\alpha_1(q) = 0$, $q = \partial_1(p)$ for some p. Set $n = \varepsilon(p)$. Then $\partial_2(n) = \partial_2\varepsilon(p) = \alpha_2\partial_1(p) = \alpha_2(q) = 0$, so $n = \gamma_2(t)$ for some t. But then $\Delta(t) = \partial_1\varepsilon^{-1}\gamma_2(t) = \partial_1\varepsilon^{-1}(n) = \partial_1(p) = q$.

□

Theorem A.2.12. *Given a commutative diagram of exact sequences*

$$\begin{array}{ccccccccccccc}
\cdots \to & B_i & \to & A_i & \to & R_i & \to & B_{i-1} & \to & A_{i-1} & \to & R_{i-1} & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & B_i & \to & X_i & \to & S_i & \to & B_{i-1} & \to & X_{i-1} & \to & S_{i-1} & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & A_i & \to & X_i & \to & T_i & \to & A_{i-1} & \to & X_{i-1} & \to & T_{i-1} & \to \cdots
\end{array}$$

where the vertical maps $B_i \to B_i$ and $X_i \to X_i$ are both identity maps, the vertical map $B_i \to A_i$ agrees with the horizontal map $B_i \to A_i$, and the vertical map $A_i \to X_i$ agrees with the horizontal map $A_i \to X_i$, for all i, there is a long exact sequence

$$\cdots \longrightarrow R_i \longrightarrow S_i \longrightarrow T_i \xrightarrow{\partial} A_{i-1} \longrightarrow \cdots$$

where the map ∂ is the composition of $T_i \to A_{i-1}$ on the bottom row, followed by the identity map from A_{i-1} on the bottom row to A_{i-1} on the top row, followed by $A_{i-1} \to R_{i-1}$ on the top row.

Proof. We shall observe that under the hypotheses of the theorem, the diagram continues to commute if we insert the diagonal arrows from A_i on the bottom row to A_i on the top row, with these maps being the identity maps.

Otherwise, the proof is a very elaborate diagram chase, which we omit. □

A finitely generated abelian group A is isomorphic to $F \oplus T$ where F is a free abelian group of well-defined rank r (i.e., F is isomorphic to $\mathbb{Z}^r$) and T is a torsion group. In this case we define the rank of A to be r.

Theorem A.2.13. *Let $0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ be a chain complex with each C_i a finitely generated free abelian group of rank d_i. Let H_i be the i-th homology group of this chain complex, $i = 0, \ldots, n$. Then*

$$\sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i = \sum_{i=0}^{n} (-1)^i \operatorname{rank} C_i.$$

Proof. Although, with a little care, this theorem can be proven directly, it is easiest to tensor everything with $\mathbb{Q}$. We then obtain a chain complex

$$0 \longrightarrow V_n \longrightarrow V_{n-1} \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0$$

with $V_i = C_i \otimes \mathbb{Q}$ a rational vector space of dimension d_i. It is easy to check that if $\{K_i\}$ are the homology groups of the new chain complex, then $K_i = H_i \otimes \mathbb{Q}$ for each i. In particular, if $r_i = \operatorname{rank} H_i$, then K_i is a rational vector space of dimension r_i. Hence it suffices to prove that

$$\sum_{i=0}^{n} (-1)^{r_i} = \sum_{i=0}^{n} (-1)^{d_i}.$$

We prove this by induction on n. If $n = 0$, this is trivial. The chain complex is then $0 \to V_0 \to 0$ which has the single nonzero homology group $K_0 = V_0$, so certainly $r_0 = d_0$.

Suppose it is true for $n-1$, and all chain complexes.

We have $\partial_n : V_n \to \operatorname{Im}(V_n) = V'_{n-1} \subseteq \operatorname{Ker}(\partial_{n-1}) \subseteq V_{n-1}$. Let V''_{n-1} be a complement of V'_{n-1} in $\operatorname{Ker}(\partial_{n-1})$ and let V'''_{n-1} be a complement of $\operatorname{Ker}(\partial_{n-1})$ in V_{n-1}. Then

$$V_{n-1} = V'_{n-1} \oplus V''_{n-1} \oplus V'''_{n-1}.$$

Let these three subspaces have dimensions d'_{n-1}, d''_{n-1}, d'''_{n-1}, respectively.

Then $\partial_{n-1}|V'''_{n-1}$ is an injection and $\partial_{n-1}(V'''_{n-1}) = \partial_{n-1}(V_{n-1})$. Thus we have a chain complex

$$0 \longrightarrow V'''_{n-1} \longrightarrow V_{n-2} \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0$$

whose homology in dimension $n-1$ is 0, and whose homology in dimension $i < n-1$ is K_i. By the $n-1$ case, we have

$$\sum_{i=0}^{n-2}(-1)^i r_i = \left(\sum_{i=0}^{n-2}(-1)^i d_i\right) + (-1)^{n-1}\left(d'''_{n-1}\right).$$

Now $K_{n-1} = \mathrm{Ker}\,(\partial_{n-1})/\,\mathrm{Im}\,(\partial_n)$ is isomorphic to V''_{n-1}, so $r_{n-1} = d''_{n-1}$. Hence

$$\sum_{i=0}^{n-1}(-1)^i r_i = \left(\sum_{i=0}^{n-2}(-1)^i d_i\right) + (-1)^{n-1}\left(d''_{n-1} + d'''_{n-1}\right).$$

Now $K_n = \mathrm{Ker}\,(\partial_n)$. Since $\mathrm{Im}\,(\partial_n)$ has dimension d'_{n-1}, $\mathrm{Ker}\,(\partial_n)$ has dimension $d_n - d'_{n-1}$, i.e., $r_n = d_n - d'_{n-1}$. Hence

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i r_i &= \left(\sum_{i=0}^{n-2}(-1)^i d_i\right) + (-1)^{n-1}\left(d''_{n-1} + d'''_{n-1}\right) + (-1)^n\left(d_n - d'_{n-1}\right)\\ &= \left(\sum_{i=0}^{n-2}(-1)^i d_i\right) + (-1)^{n-1}\left(d'_{n-1} + d''_{n-1} + d'''_{n-1}\right) + (-1)^n d_n\\ &= \sum_{i=0}^{n}(-1)^i d_i\end{aligned}$$

as claimed.

Thus by induction we are done. □

A.3 Tensor Product, Hom, Tor, and Ext

In this section we recapitulate the basic properties of the tensor product and Hom, and introduce Tor and Ext. These constructions depend on the ring R, but we assume R is fixed, and in the interests of simplicity suppress R from our notation. *Except in* Lemmas A.3.1 *and* A.3.2, *we assume that* R *is a* PID.

We begin with the easiest situation.

Lemma A.3.1. *Let* $0 \to A \to B \to C \to 0$ *be a split short exact sequence of R-modules. Then for any R-module M,*

(a) *The sequence* $0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ *is exact, and*
(b) *The sequence* $0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M) \to 0$ *is exact.*

Proof. This follows from Lemma A.1.7 and the facts that $(A \oplus C) \otimes M \cong (A \otimes M) \oplus (C \otimes M)$ and $\mathrm{Hom}(A \oplus C, M) \cong \mathrm{Hom}(A,M) \oplus \mathrm{Hom}(C,M)$. □

Lemma A.3.2. *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of R-modules. Then for any R-module M*

(a) *The sequence* $A \otimes M \to B \otimes M \to C \otimes M \to 0$ *is exact*
(b) *The sequence* $0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M)$ *is exact.*

Note the difference between the split and the nonsplit case: In the nonsplit case we do not in general obtain five-term exact sequences. We now introduce Tor and *Ext*, which measure inexactness. But first we must introduce free resolutions.

Lemma A.3.3. *Let M be an R-module. Then there is a short exact sequence*

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\psi} M \longrightarrow 0$$

where F_0 *and* F_1 *are free R-modules.*

Proof. We can certainly find a free module F_0 and an epimorphism $\psi : F_0 \to M$, as follows: Let $\{m_i\}_{i \in I}$ generate M. (We can certainly find a set of generators. For example, we could choose this set to be all the elements of M.) Let F_0 be the free R-module with basis $\{f\}_{i \in I}$ and let $\psi : F_0 \to M$ be defined by $\psi(f_i) = m_i$ for each $i \in I$.

Let $F_1 = \mathrm{Ker}(\psi)$ and let $\varphi : F_1 \to F_0$ be the inclusion. Then we certainly have a short exact sequence

$$0 \longrightarrow F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\psi} M \longrightarrow 0.$$

Furthermore, *since R is a* PID, every submodule of a free module is free, so F_1 is free as well. □

Remark A.3.4. Note we have crucially used the hypothesis that R is a PID. (It would take us far afield to consider what happens when R is not.) ◊

Definition A.3.5. A sequence of R-modules as in Lemma A.3.3 is a *free resolution* of M. ◊

Lemma A.3.6. *Let M and N be R-modules. Then there is a well-defined R-module T (i.e., independent of the choice of* F_1 *and* F_0*) such that, if* $0 \to F_1 \to F_0 \to M \to 0$ *is a free resolution of M, the sequence*

$$0 \longrightarrow T \longrightarrow F_1 \otimes N \longrightarrow F_0 \otimes N \longrightarrow M \otimes N \longrightarrow 0$$

is exact.

Definition A.3.7. The module T of Lemma A.3.6 is the *torsion product* of M and N,

$$T = \operatorname{Tor}(M,N). \qquad \diamondsuit$$

We summarize the properties of the torsion product.

Lemma A.3.8. (a) *If M is free,* $\operatorname{Tor}(M,N) = 0$ *for every N.*
(b) $\operatorname{Tor}(M_1 \oplus M_2, N) \cong \operatorname{Tor}(M_1,N) \oplus \operatorname{Tor}(M_2,N)$.
(c) $\operatorname{Tor}(M, N_1 \oplus N_2) \cong \operatorname{Tor}(M,N_1) \oplus \operatorname{Tor}(M,N_2)$.
(d) *If N is free,* $\operatorname{Tor}(M,N) = 0$ *for every M.*
(e) *Let M be a cyclic R-module, $M \cong R/I_M$, I_M an ideal of R, and let N be a cyclic R-module, $N \cong R/I_N$, I_N an ideal of R. Let $I_M = (r_M)$, i.e., let I_M be the principal ideal consisting of the multiples of the element r_M of R and similarly let $I_N = (r_N)$. If $r_M = 0$ or $r_N = 0$, then* $\operatorname{Tor}(M,N) = 0$. *Otherwise,* $\operatorname{Tor}(M,N) \cong R/I_T$ *where $I_T = (r_T)$ and $r_T = \gcd(r_M, r_N)$.*
(f) *If M is torsion-free,* $\operatorname{Tor}(M,N) = 0$ *for every N.*
(g) *If N is torsion-free,* $\operatorname{Tor}(M,N) = 0$ *for every M.*
(h) $\operatorname{Tor}(M,N) \cong \operatorname{Tor}(N,M)$.
(i) *If $R = \mathbb{F}$ is a field,* $\operatorname{Tor}(M,N) = 0$ *for every M,N.*

Lemma A.3.9. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then for any R-module N, the sequence*

$$0 \to \operatorname{Tor}(A,N) \to \operatorname{Tor}(B,N) \to \operatorname{Tor}(C,N) \to A \otimes N \to B \otimes N \to C \otimes N \to 0$$

is exact.

Lemma A.3.10. *Let M and N be R-modules. Then there is a well-defined R-module E (i.e., independent of the choice of F_1 and F_0) such that, if $0 \to F_1 \to F_0 \to M \to 0$ is a free resolution of M, the sequence*

$$0 \longrightarrow \operatorname{Hom}(M,N) \longrightarrow \operatorname{Hom}(F_0,N) \longrightarrow \operatorname{Hom}(F_1,N) \longrightarrow E \longrightarrow 0$$

is exact.

Definition A.3.11. The module E of Lemma A.3.10 is the *extension product* of M and N,

$$E = \operatorname{Ext}(M,N). \qquad \diamondsuit$$

We summarize the properties of the extension product.

Lemma A.3.12. (a) *If M is free,* $\operatorname{Ext}(M,N) = 0$ *for every N.*
(b) $\operatorname{Ext}(M_1 \oplus M_2, N) \cong \operatorname{Ext}(M_1,N) \oplus \operatorname{Ext}(M_2,N)$.
(c) $\operatorname{Ext}(M, N_1 \oplus N_2) \cong \operatorname{Ext}(M,N_1) \oplus \operatorname{Ext}(M,N_2)$.
(d) *Let M be a cyclic R-module, $R \cong R/I_M$ with $I_M = (r_M)$. Then for any R-module N,*

$$\operatorname{Ext}(M,N) \cong N/r_M N$$

where for $r \in R$, $rN = \{rn \mid n \in N\} \subseteq N$. *In particular,*

$$\mathrm{Ext}(M,R) \cong R/r_M R \cong M.$$

If N *is a cyclic* R*-module,* $N \cong R/I_N$ *with* $I_N = (r_N)$*, then*

$$\mathrm{Ext}(M,N) \cong R/r_E R$$

where $r_E = \gcd(r_M, r_N)$. *As a special case*

$$\mathrm{Ext}(M,N) = 0$$

if r_M *and* r_N *are relatively prime.*

Lemma A.3.13. *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence of* R*-modules. Then for any* R*-module* N*, the sequences*

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(C,N) &\longrightarrow \mathrm{Hom}(B,N) \longrightarrow \mathrm{Hom}(A,N) \\ &\longrightarrow \mathrm{Ext}(C,N) \longrightarrow \mathrm{Ext}(B,N) \longrightarrow \mathrm{Ext}(A,N) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(N,A) &\longrightarrow \mathrm{Hom}(N,B) \longrightarrow \mathrm{Hom}(N,C) \\ &\longrightarrow \mathrm{Ext}(N,A) \longrightarrow \mathrm{Ext}(N,B) \longrightarrow \mathrm{Ext}(N,C) \longrightarrow 0 \end{aligned}$$

are both exact.

(We need almost all of the results of this section to compute Tor and Ext, which enter into computations of homology and cohomology. We do not need Lemmas A.3.9 and A.3.13, but have given them to complete the picture.)

Remark A.3.14. Note that if $R = \mathbb{F}$ is a field, this section is entirely superfluous. For in this case every short exact sequence of R-modules is split (Lemma A.1.11), and so $\mathrm{Tor}(M,N) = 0$ and $\mathrm{Ext}(M,N) = 0$ for any two R-modules M and N. ♢

Appendix B
Bilinear Forms

In this appendix we introduce bilinear forms and we state some of the basic classification theorems.

B.1 Definitions

Definition B.1.1. Let R be a commutative ring and let M be a free R-module. A *bilinear form*

$$\langle\ ,\ \rangle : M \times M \longrightarrow R$$

is a function that is linear in both arguments, i.e., with the property that

(1) $\langle r_1 m_1 + r_2 m_2, m\rangle = r_1\langle m_1, m\rangle + r_2\langle m_2, m\rangle$
(2) $\langle m, r_1 m_1 + r_2 m_2\rangle = r_1\langle m, m_1\rangle + r_2\langle m, m_2\rangle$

for all $r_1, r_2 \in R$ and all $m_1, m_2, m \in M$. ◇

Note that bilinearity is precisely the condition we need to obtain

$$\langle\ ,\ \rangle : M \otimes M \longrightarrow R.$$

The appropriate equivalence relation on bilinear forms is that of isometry.

Definition B.1.2. Let $\langle\ ,\ \rangle$ be a bilinear form on M and $\langle\langle\ ,\ \rangle\rangle$ be a bilinear form on N. An *isometry* between these two forms is an isomorphism $\varphi : M \to N$ such that

$$\langle\langle \varphi(m_1), \varphi(m_2)\rangle\rangle = \langle m_1, m_2\rangle \quad \text{for all } m_1, m_2 \in M.$$

In this situation, the two forms are said to be *isometric*. ◇

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Definition B.1.3. Let $\langle\ ,\ \rangle : M \otimes M \to R$ be a bilinear form. Then $\langle\ ,\ \rangle$ defines a map $\alpha : M \to M^* = \mathrm{Hom}(M,R)$ by

$$\alpha(m_1)(m_2) = \langle m_1, m_2\rangle \quad \text{for all } m_2 \in M$$

and a map $\beta : M \to M^*$ by

$$\beta(m_2)(m_1) = \langle m_1, m_2\rangle \quad \text{for all } m_1 \in M.$$

The form $\langle\ ,\ \rangle$ is *nonsingular* if α and β are isomorphisms. ◊

Definition B.1.4. The bilinear form $\langle\ ,\ \rangle$ is *symmetric* if

$$\langle m_1, m_2\rangle = \langle m_2, m_1\rangle$$

for all $m_1, m_2 \in M$, and is *skew-symmetric* if

$$\langle m_1, m_2\rangle = -\langle m_2, m_1\rangle$$

for all $m_1, m_2 \in M$. ◊

Example B.1.5. Let $M = R^n$ and let A be any $n \times n$ matrix with entries in R. Then $[A]$ is the bilinear form

$$[A] = \langle\ ,\ \rangle : M \times M \longrightarrow R \quad \text{by } \langle x, y\rangle = x^t A y$$

is a bilinear form. It is symmetric if $A = A^t$ and skew-symmetric if $A = -A^t$. It is nonsingular if A is nonsingular (i.e., invertible). ◊

Remark B.1.6. In the situation of Example B.1.5, either α an isomorphism or β an isomorphism implies A nonsingular, so in the finite rank case it is only necessary to check one of these conditions. ◊

Remark B.1.7. Upon choosing a basis of M, a module of finite rank, every bilinear form arises in this way. ◊

Here is a simple but basic construction.

Definition B.1.8. Let $\langle\ ,\ \rangle$ be a bilinear form on M and let $\langle\ ,\ \rangle'$ be a bilinear form on M'. Their direct sum $\langle\ ,\ \rangle'' = \langle\ ,\ \rangle \oplus \langle\ ,\ \rangle'$ is the bilinear form on $M'' = M \oplus M'$ given by

$$\langle m_1 + m_1', m_2 + m_2'\rangle'' = \langle m_1, m_2\rangle + \langle m_1', m_2'\rangle'$$

for all $m_1, m_2 \in M$ and $m_1', m_2' \in M'$. Also, $k\langle\ ,\ \rangle$ denotes $\langle\ ,\ \rangle \oplus \cdots \oplus \langle\ ,\ \rangle$, where there are k summands. ◊

Remark B.1.9. If, in the notation of Example B.1.5, $\langle\ ,\ \rangle = [A]$ and $\langle\ ,\ \rangle' = [A']$, then $\langle , \rangle'' = \langle , \rangle \oplus \langle , \rangle' = [A'']$ with A'' the block diagonal matrix $A'' = \left[\begin{smallmatrix} A & 0 \\ 0 & A' \end{smallmatrix}\right]$. ◊

B.2 Classification Theorems

We now give some of the basic classification theorems for nonsingular bilinear forms. First we consider skew-symmetric forms.

Theorem B.2.1. *Let $R = \mathbb{Z}$ or a field of characteristic not equal to two. Then any nonsingular skew-symmetric bilinear form on a free R-module M of finite rank is isometric to*

$$k\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for some integer k. In particular, if M admits a nonsingular skew-symmetric bilinear form, then $\operatorname{rank}(M)$ *is even.*

Next we consider symmetric forms over the real numbers $\mathbb{R}$.

Definition B.2.2. A symmetric bilinear form $\langle\ ,\ \rangle$ on an $\mathbb{R}$-vector space V is positive definite if $\langle v,v\rangle > 0$ for every $v \in V$, $v \neq 0$, and is negative definite if $\langle v,v\rangle < 0$ for every $v \in V$, $v \neq 0$. ◊

Lemma B.2.3. *Let $\langle\ ,\ \rangle$ be a nonsingular symmetric bilinear form on a real vector space V of dimension n. Let V_+ be a subspace of V of largest possible dimension with $\langle\ ,\ \rangle$ restricted to V_+ positive definite and let V_- be a subspace of V of largest possible dimension with $\langle\ ,\ \rangle$ restricted to V_- negative definite. Then $V = V_+ \oplus V_-$.*

Remark B.2.4. The spaces V_+ and V_- are in general not unique. ◊

Theorem B.2.5 (Sylvester's Law of Inertia). *Let $\langle\ ,\ \rangle$ be a nonsingular symmetric bilinear form on a real vector space V of dimension t. Let V_+ and V_- be as in Lemma B.2.3, and let $r =$ $\dim(V_+)$ and $s =$ $\dim(V_-)$. Then $\langle\ ,\ \rangle$ is isometric to*

$$r[1] + s[-1].$$

Definition B.2.6. In the situation of Theorem B.2.5, the *signature* $\sigma(\langle\ ,\ \rangle) = r - s$. ◊

Remark B.2.7. Since $r + s = t$, a nonsingular symmetric bilinear form $\langle\ ,\ \rangle$ on a real vector space V is determined up to isometry by $\dim(V)$ and $\sigma(\langle\ ,\ \rangle)$. ◊

Lemma B.2.8. *Let $\langle\ ,\ \rangle$ be a nonsingular symmetric bilinear form on a real vector space V of even dimension t. Suppose that V has a subspace V_0 of dimension $t/2$ such that the restriction of $\langle\ ,\ \rangle$ to V_0 is identically 0. Then $\sigma(\langle\ ,\ \rangle) = 0$.*

Proof. If $v \in V_+ \cap V_0$, and $v \neq 0$, then $\langle v,v\rangle > 0$ as $v \in V_+$, but $\langle v,v\rangle = 0$ as $v \in V_0$, so this is impossible. Hence $V_+ \cap V_0 = \{0\}$, and so $r + t/2 \leq t$. Similarly $V_- \cap V_0 = \{0\}$ so $s + t/2 \leq t$. But $r + s = t$, so we must have $r = s = t/2$, and hence $\sigma(\langle\ ,\ \rangle) = 0$. □

Appendix C
Categories and Functors

Category theory provides a very convenient, and for some purposes essential, formulation for algebraic topology. We have minimized its use in this book, but we give the basics here.

C.1 Categories

Definition C.1.1. A *category* $\mathfrak{C}$ consists of a class of *objects* $\mathrm{Obj}(\mathfrak{C}) = \{A,B,C,\ldots\}$ and for any ordered pair (A,B) of objects a class of *morphisms* $\mathrm{Mor}(A,B)$, with the following properties:

(1) Given any $f \in \mathrm{Mor}(A,B)$ and $g \in \mathrm{Mor}(B,C)$ there is their *composition* $gf \in \mathrm{Mor}(A,C)$, and furthermore composition is *associative*, i.e., given $f \in \mathrm{Mor}(A,B)$, $g \in \mathrm{Mor}(B,C)$, and $h \in \mathrm{Mor}(C,D)$, then $h(gf) = (hg)f$.
(2) Given any object A of $\mathfrak{C}$ there is the *identity* morphism $\mathrm{id}_A \in \mathrm{Mor}(A,A)$, and, given any pair of objects A and B, if $f \in \mathrm{Mor}(A,B)$, then $f\mathrm{id}_A = f = \mathrm{id}_B f$. $\Diamond$

Example C.1.2. (1) $\mathrm{Obj}(\mathfrak{C}) = \{\text{sets}\}$ and for $X,Y \in \mathfrak{C}$, $\mathrm{Mor}(X,Y) = \{\text{functions} f : X \to Y\}$.
(2) $\mathrm{Obj}(\mathfrak{C}) = \{\text{topological spaces}\}$ and for $X,Y \in \mathfrak{C}$, $\mathrm{Mor}(X,Y) = \{\text{continuous maps } f : X \to Y\}$.
(3) $\mathrm{Obj}(\mathfrak{C}) = \{(X,A) \mid X \text{ is a topological space and } A \text{ is a subspace of } X\}$ and $\mathrm{Mor}((X,A),(Y,B)) = \{\text{continuous maps } f : X \to Y \text{ with } f(A) \subseteq B\}$.
(4) $\mathrm{Obj}(\mathfrak{C}) = \{\text{abelian groups}\}$ and for $G,H \in \mathfrak{C}$, $\mathrm{Mor}(G,H) = \{\text{group homomorphisms } f : G \to H\}$.
(5) $\mathrm{Obj}(\mathfrak{C}) = \{\text{graded abelian groups } \{G_i\}_{i\in\mathbb{Z}}\}$ and for $\{G_i\},\{H_i\} \in \mathfrak{C}$, $\mathrm{Mor}(G,H) = \{\{f_i\} \mid f_i : G_i \to H_i \text{ is a group homomorphism}\}$.

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(6) $\mathrm{Obj}\,(\mathfrak{C}) = \{\text{chain complexes } \{G_i,\ \partial_i^G : G_i \to G_{i-1}\}\}$ and $\mathrm{Mor}(\{G_i, \partial_i^G\}, \{H_i, \partial_i^H\}) = \{\{f_i\} \mid f_i : G_i \to H_i$ is a group homomorphism with $\partial_{i-1}^H f_i = f_{i-1}\partial_i^G\}$.

(7) $\mathrm{Obj}\,(\mathfrak{C}) = \{\text{graded rings } \{G^i\}\}$. A graded ring has the additive structure of a graded abelian group and a multiplicative structure $u_G : G^i \otimes G^j \to G^{i+j}$ satisfying:

  (a) There is an identity element $1^G \in G^0$.
  (b) Multiplication is *commutative* in the sense that if $\alpha \in G^i$ and $\beta \in G^j$, then $\alpha\beta = (-1)^{ij}\beta\alpha$.

  $\mathrm{Mor}\,(\{G^i\}, \{H^i\}) = \{\{f^i\}\}$ where $\{f^i\}$ is a homomorphism of graded rings, i.e., a morphism of graded abelian groups that is also a homomorphism of rings with identity.

(8) $\mathrm{Obj}\,(\mathfrak{C}) = \{\text{CW-complexes}\}$ and $\mathrm{Mor}\,(X,Y) = \{\text{cellular maps } f : X \to Y\}$, where $f : X \to Y$ is cellular if $f(X^n) \subseteq Y^n$ for every n.

(9) $\mathrm{Obj}\,(\mathfrak{C}) = \{\text{pointed spaces}\} = \{(X, x_0)\}$, i.e., a nonempty topological space X and a point $x_0 \in X$, and $\mathrm{Mor}\,((X,x_0),(Y,y_0)) = \{f : X \to Y \mid f(x_0) = y_0\}$.

(10) $\mathrm{Obj}\,(\mathfrak{C}) = \{\text{groups}\}$ and $\mathrm{Mor}\,(G,H) = \{\text{group homomorphisms } f : G \to H\}$.

(11) $\mathrm{Obj}\,(\mathfrak{C}) = \{\text{cochain complexes } \{G^i,\ \delta_G^i : G^i \to G^{i+1}\}\}$ and $\mathrm{Mor}\,(\{G^i, \int_G^i\}, \{H^i, \delta_H^i\}) = \{\{f^i\} \mid f^i : G^i \to H^i$ is a group homomorphism with $\delta_H^{i+1} f_i = f_{i+1}\delta_G^i\}$. ◊

C.2 Functors

Given two categories $\mathfrak{C}$ and $\mathfrak{D}$, we may regard them each as objects and ask for the appropriate notion of a function between them. This notion is that of a functor, and comes in two varieties.

Definition C.2.1. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A *covariant functor* $T : \mathfrak{C} \to \mathfrak{D}$ consists of:

(1) A function $T : \mathrm{Obj}\,(\mathfrak{C}) \to \mathrm{Obj}\,(\mathfrak{D})$.
(2) A function $T : \mathrm{Mor}\,(C_1, C_2) \to \mathrm{Mor}\,(D_1, D_2)$, where C_1 and C_2 are objects of $\mathfrak{C}$, and $D_1 = T(C_1)$, $D_2 = T(C_2)$, objects of $\mathfrak{D}$, with the properties:

  (a) $T(\mathrm{id}_C) = \mathrm{id}_{T(C)}$ for any object C of $\mathfrak{C}$.
  (b) If C_1, C_2, and C_3 are objects of $\mathfrak{C}$, $f \in \mathrm{Mor}\,(C_1, C_2)$, and $g \in \mathrm{Mor}\,(C_2, C_3)$, then $T(gf) = T(g)T(f)$.

A *contravariant functor* $T : \mathfrak{C} \to \mathfrak{D}$ consists of:

(1) A function $T : \mathrm{Obj}\,(\mathfrak{C}) \to \mathrm{Obj}\,(\mathfrak{D})$.
(2) A function $T : \mathrm{Mor}\,(C_1, C_2) \to \mathrm{Mor}\,(D_2, D_1)$, where C_1 and C_2 are objects of $\mathfrak{C}$, and $D_1 = T(C_1)$, $D_2 = T(C_2)$, objects of $\mathfrak{D}$, with the properties:

(a) $T(\mathrm{id}_C) = \mathrm{id}_{T(C)}$ for any object C of $\mathfrak{C}$.
(b) If C_1, C_2, and C_3 are objects of $\mathfrak{C}$, $f \in \mathrm{Mor}(C_1, C_2)$, and $g \in \mathrm{Mor}(C_2, C_3)$, then $T(gf) = T(f)T(g)$. ◊

In the following example, we just give the objects of the categories involved. The morphisms should be clear.

Example C.2.2. (1) Let $\mathfrak{C} = \{\text{pointed spaces}\}$ and $\mathfrak{D} = \{\text{groups}\}$. We have the covariant functor $T : \mathfrak{C} \to \mathfrak{D}$ given by $T(X, x_0) = \pi_1(X, x_0)$. If $f : (X, x_0) \to (Y, y_0)$ then $T(f) = f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

(2) Let $\mathfrak{C} = \{(X,A)\}$ pairs of topological spaces and let $\mathfrak{D} = \{\text{graded abelian groups}\}$. We have the covariant functor $T : \mathfrak{C} \to \mathfrak{D}$ given by $T(X,A) = \{H_i(X,A)\}$, and if $f : (X,A) \to (Y,B)$, $T(f) = \{f_i : H_i(X,A) \to H_i(Y,B)\}$, for any fixed homology theory. Similarly, we have the contravariant functor $T : \mathfrak{C} \to \mathfrak{D}$ given by $T(X,A) = \{H^i(X,A)\}$, and if $f : (X,A) \to (Y,B)$, $T(f) = \{f^i : H^i(Y,B) \to H^i(X,A)\}$, for any fixed cohomology theory.

(3) In this and the remaining examples, we simply describe what T does on objects; its effect on morphisms should then be clear.

Let $\mathfrak{C} = \{\text{chain complexes}\}$ and $\mathfrak{D} = \{\text{graded abelian groups}\}$. Then $T : \mathfrak{C} \to \mathfrak{D}$ by $T(\{C_i\}) = \{H_i\}$ where $\{H_i\}$ are the homology groups of $\{C_i\}$. Similarly, if $\mathfrak{C} = \{\text{cochain complexes}\}$ we have $T : \mathfrak{C} \to \mathfrak{D}$ by $T(\{C^i\}) = \{H^i\}$ where $\{H^i\}$ are the cohomology groups of $\{C^i\}$. Note in both cases T is covariant.

(4) Let $\mathfrak{C} = \{\text{chain complexes}\}$ and $\mathfrak{D} = \{\text{cochain complexes}\}$. Let $T : \mathfrak{C} \to \mathfrak{D}$ be the contravariant functor given by $T(\{C_i\}) =$ the dual cochain complex $\{C^i = \mathrm{Hom}(C_i, \mathbb{Z})\}$.

(5) Let $\mathfrak{C} = \{\text{topological pairs } (X,A)\}$ and let $\mathfrak{D} = \{\text{chain complexes}\}$. Let $T : \mathfrak{C} \to \mathfrak{D}$ be the covariant functor given by $T(X,A) = \{C_i(X,A)\}$, the singular chain complex of the pair (X,A).

(6) Let $\mathfrak{C} = \{(X,A)\}$ pairs of topological spaces and let $\mathfrak{D} = \{\text{graded rings}\}$. We have the contravariant functor $T : \mathfrak{C} \to \mathfrak{D}$ given by $T(X,A) = \{H^i(X,A)\}$ where $H^i(X,A)$ denotes singular cohomology. (We have constructed a ring structure on singular cohomology in Sect. 5.6.)

(7) Let $\mathfrak{C} = \{\text{CW-complexes}\}$ and $\mathfrak{D} = \{\text{graded abelian groups}\}$. Then $T : \mathfrak{C} \to \mathfrak{D}$ by $T(X) = \{H_i^{\mathrm{cell}}(X)\}$ is a covariant functor and $T : \mathfrak{C} \to \mathfrak{D}$ by $T(X) = \{H^i_{\mathrm{cell}}(X)\}$ is a contravariant functor. Note that $f \in \mathrm{Mor}(X,Y)$ induces maps on cellular homology or cohomology as by the definition of $\mathfrak{C}$, $\mathrm{Mor}(X,Y)$ consists of cellular maps. ◊

Remark C.2.3. Suppose that $H_i(X,A)$ is singular homology, and that $H^i(X,A)$ is singular cohomology. Then the covariant functor in Example C.2.2(2) is in an obvious way the composition of the covariant functor in (5) with the covariant functor in (3), and the contravariant functor in (2) is in an obvious way the composition of the covariant functor in (5), the contravariant functor in (4), and the covariant functor in (3). Note that the functors in (3) and (4) are purely algebraic. It is the functor in (5) that makes the connection between topology and algebra. ◊

Bibliography

Here is a short annotated guide to various other books on algebraic topology.

1. J. F. Davis and P. Kirk, *Lecture Notes in Algebraic Topology*, American Mathematical Society, 2001.
   Roughly speaking, this book picks up more or less where our book leaves off.
2. S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, 1952.
   This monograph introduced the Eilenberg-Steenrod axioms, which codified homology and cohomology theory and laid the foundation for future developments. It is very clearly written and has stood the test of time well.
3. B. Gray, *Homotopy Theory: An Introduction to Algebraic Topology*, Academic Press, 1975.
   As its title indicates, an exposition of algebraic topology from the viewpoint of homotopy (rather than homology) theory.
4. M. J. Greenberg and J. R. Harper, *Algebraic Topology: A First Course*, Westview Press, 1981.
   The first edition of this book (written by Greenberg alone and published in 1967) was this author's introduction to the subject.
5. A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
   A wide-ranging modern text, written from a geometric viewpoint. It can also be downloaded, free and legally, from the author's website.
6. S. MacLane, *Homology*, Springer, 1963.
   A clear and thorough introduction to homological algebra, written by one of the giants in the field.
7. W. S. Massey, *A Basic Course in Algebraic Topology*. Springer, 1991.
   An excellent introductory text written by a leading algebraic topologist. While Massey's book does not adopt an axiomatic standpoint, its spirit is in many ways similar to ours.
8. E. H. Spanier, *Algebraic Topology* McGraw-Hill, 1966.
   This book was and remains the bible of the subject, as it existed at the time of its writing. Precisely because it is so thorough, and written in the greatest generality, it is not easy to read, but is an invaluable reference.

S.H. Weintraub, *Fundamentals of Algebraic Topology*, Graduate Texts in Mathematics 270, DOI 10.1007/978-1-4939-1844-7

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