Appendix

We shall give brief reviews of the proofs of the algebraic theorems which have been quoted in this chapter.

We first discuss "formal linear combinations." Let S be a set. We wish to define what we mean by expressions

$$c_1 s_1 + \cdots + c_n s_n$$

where $\{c_i\}$ are numbers, and $\{s_i\}$ are distinct elements of S. What do we wish such a "sum" to be like? Well, we wish it to be entirely determined by the "coefficients" c_i, and each "coefficient" c_i should be associated with the element s_i of the set S. But an association is nothing but a function. This suggests to us how to define "sums" as above.

For each $s \in S$ and each number c we define the symbol

$$cs$$

to be the function which associates c to s and 0 to z for any element $z \in S$, $z \neq s$. If b, c are numbers, then clearly

$$b(cs) = (bc)s \qquad \text{and} \qquad (b + c)s = bs + cs.$$

We let T be the set of all functions defined on S which can be written in the form

$$c_1 s_1 + \cdots + c_n s_n$$

where c_i are numbers, and s_i are distinct elements of S. Note that we have no problem now about addition, since we know how to add functions.

We contend that if $s_1, \ldots, s_n$ are distinct elements of S, then

$$1s_1, \ldots, 1s_n$$

are linearly independent. To prove this, suppose $c_1, \ldots, c_n$ are numbers such that

$$c_1 s_1 + \cdots + c_n s_n = 0 \qquad \text{(the zero function).}$$

Then by definition, the left-hand side takes on the value c_i at s_i and hence $c_i = 0$. This proves the desired linear independence.

In practice, it is convenient to abbreviate the notation, and to write simply s_i instead of $1s_i$. The elements of T, which are called **formal linear combinations of elements of** S, can be expressed in the form

$$c_1 s_1 + \cdots + c_n s_n,$$

and any given element has a *unique* such expression, because of the linear independence of $s_1, \ldots, s_n$. This justifies our terminology.

We now come to the statements concerning multilinear alternating products. Let E, F be vector spaces over $\mathbf{R}$. As before, let

$$E^{(r)} = E \times \cdots \times E,$$

taken r times. Let

$$f\colon E^{(r)} \to F$$

be an r-multilinear alternating map. Let $v_1, \ldots, v_n$ be linearly independent elements of E. Let $A = (a_{ij})$ be an $r \times n$ matrix and let

$$\begin{aligned} u_1 &= a_{11} v_1 + \cdots + a_{1n} v_n, \\ &\vdots \\ u_r &= a_{r1} v_1 + \cdots + a_{rn} v_n. \end{aligned}$$

Then

$$\begin{aligned} f(u_1, \ldots, u_r) &= f(a_{11} v_1 + \cdots + a_{1n} v_n, \ldots, a_{r1} v_1 + \cdots + a_{rn} v_n) \\ &= \sum_\sigma f(a_{1,\sigma(1)} v_{\sigma(1)}, \ldots, a_{r,\sigma(r)} v_{\sigma(r)}) \\ &= \sum_\sigma a_{1,\sigma(1)} \cdots a_{r,\sigma(r)} f(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \end{aligned}$$

where the sum is taken over all maps $\sigma\colon \{1, \ldots, r\} \to \{1, \ldots, n\}$.

In this sum, all terms will be 0 whenever σ is not an injective mapping, that is whenever there is some pair i, j with $i \neq j$ such that $\sigma(i) = \sigma(j)$, because of the alternating property of f. From now on, we consider only injective maps σ. Then $\{\sigma(1), \ldots, \sigma(r)\}$ is simply a permutation of some r-tuple $(i_1, \ldots, i_r)$ with $i_1 < \cdots < i_r$.

We wish to rewrite this sum in terms of a determinant.

For each subset S of $\{1, \ldots, n\}$ consisting of precisely r elements, we can take the $r \times r$ submatrix of A consisting of those elements a_{ij} such that $j \in S$. We denote by

$$\operatorname{Det}_S(A)$$

the determinant of this submatrix. We also call it the subdeterminant of A corresponding to the set S. We denote by $P(S)$ the set of maps

$$\sigma: \{1, \ldots, r\} \to \{1, \ldots, n\}$$

whose image is precisely the set S. Then

$$\operatorname{Det}_S(A) = \sum_{\sigma \in P(S)} \epsilon_S(\sigma) a_{1,\sigma(1)} \cdots a_{r,\sigma(r)},$$

and in terms of this notation, we can write our expression for $f(u_1, \ldots, u_r)$ in the form

$$(1) \qquad \boxed{f(u_1, \ldots, u_r) = \sum_S \operatorname{Det}_S(A) f(v_S)}$$

where v_S denotes $(v_{i_1}, \ldots, v_{i_r})$ if $i_1 < \cdots < i_r$ are the elements of the set S. The first sum over S is taken over all subsets of of $1, \ldots, n$ having precisely r elements.

Theorem A. *Let E be a vector space over* $\mathbf{R}$, *of dimension n. Let r be an integer $1 \leqq r \leqq n$. There exists a finite dimensional space $\bigwedge^r E$ and an r-multilinear alternating map $E^{(r)} \to \bigwedge^r E$ denoted by*

$$(u_1, \ldots, u_r) \mapsto u_1 \wedge \cdots \wedge u_r$$

satisfying the following properties:

AP 1. *If F is a vector space over* $\mathbf{R}$ *and $g: E^{(r)} \to F$ is an r-multilinear alternating map, then there exists a unique linear map*

$$g_*: \bigwedge^r E \to F$$

such that for all $u_1, \ldots, u_r \in E$ we have

$$g(u_1, \ldots, u_r) = g_*(u_1 \wedge \cdots \wedge u_r).$$

AP 2. *If $\{v_1, \dots, v_n\}$ is a basis of E, then the set of elements*

$$v_{i_1} \wedge \cdots \wedge v_{i_r}, \qquad 1 \leqq i_1 < \cdots < i_r \leqq n,$$

is a basis of $\bigwedge^r E$.

Proof. For each subset S of $\{1, \dots, n\}$ consisting of precisely r elements, we select a letter t_S. As explained at the beginning of the section, these letters t_S form a basis of a vector space whose dimension is equal to the binomial coefficient $\binom{n}{r}$. It is the space of formal linear combinations of these letters. Instead of t_S, we could also write $t_{(i)} = t_{i_1 \cdots i_r}$ with $i_1 < \cdots < i_r$. Let $\{v_1, \dots, v_n\}$ be a basis of E and let $u_1, \dots, u_r$ be elements of E. Let $A = (a_{ij})$ be the matrix of numbers such that

$$\begin{aligned} u_1 &= a_{11}v_1 + \cdots + a_{1n}v_n, \\ &\vdots \\ u_r &= a_{r1}v_1 + \cdots + a_{rn}v_n. \end{aligned}$$

Define

$$u_1 \wedge \cdots \wedge u_r = \sum_S \mathrm{Det}_S(A) t_S.$$

We contend that this product has the required properties.

The fact that it is multilinear and alternating simply follows from the corresponding property of the determinant.

We note that if $S = \{i_1, \dots, i_r\}$ with $i_1 < \cdots < i_r$, then

$$t_S = v_{i_1} \wedge \cdots \wedge v_{i_r}.$$

A standard theorem on linear maps asserts that there always exists a unique linear map having prescribed values on basis elements. In particular, if $g\colon E^{(r)} \to F$ is a multilinear alternating map, then there exists a unique linear map

$$g_*\colon \bigwedge^r E \to F$$

such that for each set S, we have

$$g_*(t_S) = g(v_S) = g(v_{i_1}, \dots, v_{i_r})$$

if $i_1, \dots, i_r$ are as above. By formula (1), it follows that

$$g(u_1, \dots, u_r) = g_*(u_1 \wedge \cdots \wedge u_r)$$

for all elements $u_1, \dots, u_r$ of E. This proves **AP 1**.

As for **AP 2**, let $\{w_1, \ldots, w_n\}$ be a basis of E. From the expansion of (1), it follows that the elements $\{w_S\}$, i.e. the elements $\{w_{i_1} \wedge \cdots \wedge w_{i_r}\}$ with all possible choices of r-tuples $(i_1, \ldots, i_r)$ satisfying $i_1 < \cdots < i_r$ are generators of $\bigwedge^r E$. The number of such elements is precisely $\binom{n}{r}$. Hence they must be linearly independent, and form a basis of $\bigwedge^r E$, as was to be shown.

Theorem B. *For each pair of positive integers* (r, s) *there exists a unique bilinear map*

$$\bigwedge^r E \times \bigwedge^s E \to \bigwedge^{r+s} E$$

such that if $u_1, \ldots, u_r, w_1, \ldots, w_s \in E$ *then*

$$(u_1 \wedge \cdots \wedge u_r) \times (w_1 \wedge \cdots \wedge w_s) \mapsto u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s.$$

This product is associative.

Proof. For each r-tuple $(u_1, \ldots, u_r)$ consider the map of $E^{(s)}$ into $\bigwedge^{r+s} E$ given by

$$(w_1, \ldots, w_s) \mapsto u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s.$$

This map is obviously s-multilinear and alternating. Consequently, by **AP 1** of Theorem A, there exists a unique linear map

$$g_{(u)} = g_{u_1, \ldots, u_r} : \bigwedge^s E \to \bigwedge^{r+s} E$$

such that for any elements $w_1, \ldots, w_s \in E$ we have

$$g_{(u)}(w_1 \wedge \cdots \wedge w_s) = u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s.$$

Now the association $(u) \mapsto g_{(u)}$ is clearly an r-multilinear alternating map of $E^{(r)}$ into $L(\bigwedge^s E, \bigwedge^{r+s} E)$, and again by **AP 1** of Theorem A, there exists a unique linear map

$$g_* : \bigwedge^r E \to L(\bigwedge^s E, \bigwedge^{r+s} E)$$

such that for all elements $u_1, \ldots, u_r \in E$ we have

$$g_{u_1, \ldots, u_r} = g_*(u_1 \wedge \cdots \wedge u_r).$$

To obtain the desired product $\bigwedge^r E \times \bigwedge^s E \to \bigwedge^{r+s} E$, we simply take the association

$$(\omega, \psi) \mapsto g_*(\omega)(\psi).$$

It is bilinear, and is uniquely determined since elements of the form $u_1 \wedge \cdots \wedge u_r$ generate $\bigwedge^r E$, and elements of the form $w_1 \wedge \cdots \wedge w_s$ generate $\bigwedge^s E$. This product is associative, as one sees at once on decomposable elements, and then on all elements by linearity. This proves Theorem B.

Let E, F be vector spaces, finite dimensional over $\mathbf{R}$, and let $\lambda\colon E \to F$ be a linear map. If $\mu\colon F \to \mathbf{R}$ is an element of the dual space F^*, i.e. a linear map of F into $\mathbf{R}$, then we may form the composite linear map

$$\mu \circ \lambda\colon E \to \mathbf{R}$$

which we visualize as

$$E \xrightarrow{\lambda} F \xrightarrow{\mu} \mathbf{R}.$$

We denote this composite $\mu \circ \lambda$ by $\lambda^*(\mu)$. It is an element of E^*.

Theorem C. *Let $\lambda\colon E \to F$ be a linear map. For each r there exists a unique linear map*

$$\lambda^*\colon \bigwedge^r F^* \to \bigwedge^r E^*$$

having the following properties:

(i) $\lambda^*(\omega \wedge \psi) = \lambda^*(\omega) \wedge \lambda^*(\psi)$ *for* $\omega \in \bigwedge^r F^*$, $\psi \in \bigwedge^s F^*$.

(ii) *If* $\mu \in F^*$ *then* $\lambda^*(\mu) = \mu \circ \lambda$, *and* λ^* *is the identity on* $\bigwedge^0 F^* = \mathbf{R}$.

Proof. The composition of mappings

$$F^* \times \cdots \times F^* = F^{*(r)} \to E^* \times \cdots \times E^* = E^{*(r)} \to \bigwedge^r E^*$$

given by

$$(\mu, \ldots, \mu_r) \mapsto (\mu_1 \circ \lambda, \ldots, \mu_r \circ \lambda) \mapsto (\mu_1 \circ \lambda) \wedge \cdots \wedge (\mu_r \circ \lambda)$$

is obviously multilinear and alternating. Hence there exists a unique linear map $\bigwedge^r F^* \to \bigwedge^r E^*$ such that

$$\mu_1 \wedge \cdots \wedge \mu_r \mapsto \lambda^*(\mu_1) \wedge \cdots \wedge \lambda^*(\mu_r).$$

Property (i) now follows by linearity and the fact that decomposable elements $\mu_1 \wedge \cdots \wedge \mu_r$ generate $\bigwedge^r F^*$. Property (ii) comes from the definition. This proves Theorem C.

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