Appendix

In Exercise 8 of Chapter VI the concept of <u>limit point</u> is defined and it is proved that C is closed $\Leftrightarrow$ (x lp $C \Rightarrow x \in C$) ; i.e., a set is closed if and only if it contains all of its limit points.

<u>Lemma</u>: <u>If</u> $C \subset R^n$ <u>is bounded and infinite then</u> C <u>has a limit point</u>.

<u>Proof</u>: We must find $x \in R^n$ such that each $B(x,\varepsilon) \cap C$ is an infinite set.

Since C is bounded we can find an n - dimensional cube K_1 containing C. Let λ be the length of the sides of K_1. Divide each edge into two equal parts and cut K_1 into 2^n equal cubes of side length $\frac{\lambda}{2}$. Call those cubes $K_{11}, K_{12}, \ldots, K_{12^n}$. At least one of those must contain infinitely-many points of C. Choose one such and call it K_2. Subdivide K_2 into 2^n cubes of side $\frac{\lambda}{4}$ and choose one of these, K_3, containing infinitely-many points of C. Continue this process to get $K_1 \supset K_2 \supset K_3 \supset \cdots$. Clearly $K_1 \cap K_2 \cap K_3 \cap \cdots$ is a single point $x \in R^n$.

Then x lp C. For, consider any $B(x,\varepsilon)$. Take m such that $\frac{\lambda}{2^m} < \frac{\varepsilon}{2}$. Then

$$K_m \subset B(x,\varepsilon)$$

and K_m contains infinitely-many points of C. q.e.d.

<u>Theorem</u>: (Proposition 5 of Chapter VI) <u>If</u> $C \subset R^n$ <u>is closed and bounded and</u>

$$f\colon C \to R^m$$

<u>is continuous, then</u> $f(C)$ <u>is closed and bounded.</u>

<u>Proof</u>: Suppose $f(C)$ is not bounded. Choose $x_1 \in C$ such that $y_1 = f(x_1) \notin B(o,1)$.

Choose $x_2 \in C$ such that $y_2 = f(x_2) \notin B(o,2)$.

⋮

Choose $x_k \in C$ such that $y_k = f(x_k) \notin B(o,k)$. It is easy to prove that $Y = \{y_1,y_2,...\}$ has no limit point and that $X = \{x_1,x_2,...\}$ is an ifinite set. Also $X \subset C$ is bounded. So by the lemma X has some limit point x . Then also x lp C . Since C is closed $x \in C$. But then $y = f(x)$ must be a limit point of $Y = f(X)$ by continuity. Thus $f(C)$ is bounded.

Suppose $f(C)$ is not closed. Then there exists $y \in R^m$ such that y lp $f(C)$ but $y \notin f(C)$. Choose $x_1 \in C$ such that $y_1 = f(x_1) \in B(y,1)$.

Choose $x_2 \in C$ such that $y_2 = f(x_2) \in B(y,\frac{1}{2})$.

⋮

Choose $x_k \in C$ such that $y_k = f(x_k) \in B(y,\frac{1}{k})$. Clearly y is the only limit point of $Y = \{y_1,...,y_k,...\}$. Just as before $X = \{x_1,...,x_k,...\}$ is infinite and bounded. Let x lp X . Then x lp C so $x \in C$ and $f(x)$ lp $Y = f(X)$. This implies $f(x) = y$, but $f(x) \in f(C)$. So $f(C)$ is closed. q.e.d.

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