Appendix A
A Green's Function *Mathematica* Package

This appendix is directed to the construction of a *Mathematica* Package valid to calculate the explicit expression of Green's function related to the two-point boundary value problem (1.4.1), where the nth-order linear operator L_n defined on (1.4.3) has constant coefficients. This algorithm has been published in [19] and it can be downloaded from the web page

http://webspersoais.usc.es/persoais/alberto.cabada/index.html

A.1 The Algorithm

By assuming the uniqueness of solutions of problem (1.4.1) in [19] an algorithm is developed to obtain the expression of Green's function when the operator L_n defined on (1.4.3) has constant coefficients. The method is applied to any two-point boundary conditions by means of the expression of Green's function related to suitable initial problems. By using (1.4.7) for a general initial boundary conditions, the algorithm finds the values of the solution and the successive derivatives up to order $n-1$, for which the two-point boundary conditions hold. Special mention has been made for the periodic case. In this situation the algorithm calculates function (1.4.9).

In the sequel we present the arguments developed in [19, Sect. 2].

First, it is not difficult to verify [42] that Green's function related to the initial value problem

$$L_n\, y(t) = 0,\ t \in I, \quad y^{(i)}(a) = 0,\ i = 0, \ldots, n-1, \tag{A.1.1}$$

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is given by

$$\tilde{K}(t,s) = \begin{cases} K(t,s), & \text{if} \quad a \le s \le t, \\ 0, & \text{if} \quad t < s \le b, \end{cases}$$

where

$$K(t,s) := \frac{\begin{vmatrix} y_1(s) & \dots & y_n(s) \\ y_1'(s) & \dots & y_n'(s) \\ \vdots & \ddots & \vdots \\ y_1^{(n-2)}(s) & \dots & y_n^{(n-2)}(s) \\ y_1(t) & \dots & y_n(t) \end{vmatrix}}{W(y_1,\dots,y_n)(s)},$$

being $(y_1,\dots,y_n)$ a fundamental set of solutions of equation $L_n y = 0$ and

$$W(y_1,\dots,y_n)(s) = \begin{vmatrix} y_1(s) & \dots & y_n(s) \\ y_1'(s) & \dots & y_n'(s) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(s) & \dots & y_n^{(n-1)}(s) \end{vmatrix}$$

its corresponding Wronskian.

To see this, it is enough to check that function $\tilde{K}$ satisfies properties $(g1)$–$(g6)$ in Definition 1.4.1 (See [19, Theorem 2.4] for details).

To obtain the expression of Green's function related to problem (1.4.1), let us consider n continuous functions on J, $c_1,\dots,c_n$. Now we are going to look for a Green's function of the form

$$g(t,s) = \tilde{K}(t,s) + c_1(s)\, y_1(t) + \cdots + c_n(s)\, y_n(t).$$

It is easy to verify that function g satisfies conditions $(g1)$–$(g5)$ in Definition 1.4.1. Now we must obtain the unique functions $c_1,\dots,c_n$ for which $(g6)$ is fulfilled, i.e., for each $s \in (a,b)$ we need to verify that

$$U_i(G(\cdot,s)) = 0, \quad \forall\, i = 1,\dots,n, \qquad \forall\, s \in I.$$

By linearity, we have that

$$U_i(G(\cdot,s)) = U_i(\tilde{K}(\cdot,s)) + \sum_{j=1}^{n} c_j(s)\, U_i(y_j), \quad i = 1,\dots,n,$$

that is, $(c_1(s),\dots,c_n(s))$ should be a solution of the linear system

$$\begin{pmatrix} U_1(y_1) & \dots & U_1(y_n) \\ \vdots & \ddots & \vdots \\ U_n(y_1) & \dots & U_n(y_n) \end{pmatrix} \begin{pmatrix} c_1(s) \\ \vdots \\ c_n(s) \end{pmatrix} = - \begin{pmatrix} U_1(\tilde{K}(\cdot,s)) \\ \vdots \\ U_n(\tilde{K}(\cdot,s)) \end{pmatrix}.$$

Now, since (1.4.1) is uniquely solvable, we have that the system has a unique solution given by

$$\begin{pmatrix} c_1(s) \\ \vdots \\ c_n(s) \end{pmatrix} = - \begin{pmatrix} U_1(y_1) & \dots & U_1(y_n) \\ \vdots & \ddots & \vdots \\ U_n(y_1) & \dots & U_n(y_n) \end{pmatrix}^{-1} \begin{pmatrix} U_1(\tilde{K}(\cdot, s)) \\ \vdots \\ U_n(\tilde{K}(\cdot, s)) \end{pmatrix}.$$

Moreover, from this expression, we know that functions $c_1, \dots, c_n$ are continuous and, therefore, g is Green's function that we are looking for.

From the previous considerations, the problem of deducing the expression of Green's function is reduced to the one of finding a set of $(y_1, \dots, y_n)$ fundamental solutions of equation $L_n y = 0$. To this end we use the following result for initial value problems, which is proved in [9].

Theorem A.1. *Let r be the unique solution of the initial value problem*

$$\begin{aligned} u^{(n)}(t) + \sum_{i=0}^{n-1} a_{n-i}\, u^{(i)}(t) = 0, &\quad t \in \mathbb{R}, \\ u^{(i)}(0) = 0, &\quad i = 0, \dots, n-2, \\ u^{(n-1)}(0) = 1. & \end{aligned} \tag{A.1.2}$$

Then, the unique solution of the initial value problem

$$\begin{aligned} y^{(n)}(t) + \sum_{i=0}^{n-1} a_{n-i}\, y^{(i)}(t) &= \sigma(t), \quad t \in J, \\ y^{(i)}(a) &= \lambda_i, \quad i = 0, \dots, n-1, \end{aligned} \tag{A.1.3}$$

with $\sigma \in \mathscr{L}(J\, \mathbb{R})$ and $\lambda_i \in \mathbb{R}$, $i = 0, \dots, n-1$, is given by

$$y(t) = \int_a^t r(t-s)\, \sigma(s)\, ds + \sum_{k=0}^{n-1} y_k(t)\, \lambda_k, \tag{A.1.4}$$

where

$$y_k(t) = r^{(n-1-k)}(t-a) + \sum_{j=k+1}^{n-1} a_{n-j}\, r^{(j-k-1)}(t-a), \quad t \in \mathbb{R}, \quad k = 0, \dots, n-1. \tag{A.1.5}$$

We note that in [9] the proof has been done for a continuous function σ. To extend the formula to $\mathscr{L}^1(J, \mathbb{R})$ is immediate.

Now, when we consider the boundary value problem (1.4.1), we will search for a Green's function of the form

$$g(t,s) = \begin{cases} r(t-s) + \sum_{k=0}^{n-1} y_k(t)\, d_k(s), & \text{if} \quad a \le s \le t \le b, \\ \sum_{k=0}^{n-1} y_k(t)\, d_k(s), & \text{if} \quad a \le t < s \le b, \end{cases} \tag{A.1.6}$$

where the continuous real functions d_k are the unknowns.

The expression of d_k came from the verification of the boundary conditions.

$$\begin{aligned} 0 &= \sum_{j=0}^{n-1} \Big(\alpha_j^i\, y^{(j)}(a) + \beta_j^i\, y^{(j)}(b) \Big) \\ &= \sum_{j=0}^{n-1} \Big[\alpha_j^i \int_a^b \sum_{k=0}^{n-1} y_k^{(j)}(a)\, d_k(s)\, \sigma(s)\, ds \\ &\quad + \beta_j^i \int_a^b r^{(j)}(b-s)\, \sigma(s)\, ds + \beta_j^i \int_a^b \sum_{k=0}^{n-1} y_k^{(j)}(b)\, d_k(s)\, \sigma(s)\, ds \Big] \\ &= \sum_{j=0}^{n-1} \Big[\beta_j^i \int_a^b r^{(j)}(b-s)\, \sigma(s)\, ds \Big] \\ &\quad + \sum_{j=0}^{n-1} \int_a^b d_k(s) \Big[\alpha_j^i \sum_{k=0}^{n-1} y_k^{(j)}(a) + \beta_j^i \sum_{k=0}^{n-1} y_k^{(j)}(b) \Big] \sigma(s)\, ds \\ &= \int_a^b \left[\sum_{j=0}^{n-1} \beta_j^i\, r^{(j)}(b-s) + \sum_{k=0}^{n-1} d_k(s)\, U_i(y_k) \right] \sigma(s)\, ds. \end{aligned}$$

Since y_k, r and U_i have been previously obtained, by solving the linear system

$$\sum_{k=0}^{n-1} d_k(s)\, U_i(y_k) = -\sum_{j=0}^{n-1} \beta_j^i\, r^{(j)}(b-s), \qquad i = 1, \dots, n, \tag{A.1.7}$$

we obtain the expression of $d_k(s)$ and, therefore, we have the formula for $g(t,s)$.

Notice that system (A.1.7) is equivalent to

$$\begin{pmatrix} U_1(y_0) & \cdots & U_1(y_{n-1}) \\ \vdots & \ddots & \vdots \\ U_n(y_0) & \cdots & U_n(y_{n-1}) \end{pmatrix} \begin{pmatrix} d_0(s) \\ \vdots \\ d_{n-1}(s) \end{pmatrix} = - \begin{pmatrix} \sum_{j=0}^{n-1} \beta_j^1\, r^{(j)}(b-s) \\ \vdots \\ \sum_{j=0}^{n-1} \beta_j^n\, r^{(j)}(b-s) \end{pmatrix}, \tag{A.1.8}$$

which is uniquely solvable.

A.1.1 The **Module** *Environment to Calculate Green's Function*

The purpose of this section, which is essentially [19, Sect. 5], is to construct an algorithm for computing Green's function of problem (1.4.1). Such construction is based on the expression (A.1.6). To arrive at such expression, we must previously find the functions r, y_k, and d_k.

Due to the fact that the function r is the unique solution of the initial value problem (A.1.2), the first step consists on solving such problem.

Once we have this expression, we obtain the expression of the y_k's as the unique solutions of the related problems:

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_{n-i}\, y^{(i)}(t) = 0, \quad t \in \mathbb{R}, \tag{A.1.9}$$

$$y^{(i)}(a) = 0, \quad i = 0, \ldots, n-1,\ i \neq k, \tag{A.1.10}$$

$$y^{(k}(a) = 1. \tag{A.1.11}$$

The next step of the algorithm consists of solving the system (A.1.7). In consequence, to ensure the existence and uniqueness of Green's function, we must verify, first, that the matrix of system (A.1.7) is invertible. Otherwise, there is not Green's function and this ends the process.

When the system (A.1.7) is uniquely solvable, once we have obtained its unique solution, d_k, we arrive at the expression of Green's function, $g(t,s)$, by means of the expression (A.1.6) defined in the two triangles $a \leq s < t \leq b$ and $a \leq t < s \leq b$.

As we have noticed in the previous section, the calculations involved in this process are very complicated, so, for higher order equations and several boundary conditions, the resolution may be very slow and the simplifications unavailable. On the other hand, for the particular case of periodic boundary conditions, we only must obtain the function r, defined in (1.4.9), in order to have the expression of Green's function from (1.4.12). The outline of the described algorithm can be seen in the flow diagram of Fig. A.1.

By using the scientific software *Mathematica 8.0.1.0* and the previously described algorithm, we implemented a program in which, by supplying the order equation, the coefficients of the linear operator and the two-point boundary conditions on the interval J, the related Green's function is calculated.

Using the *Manipulate* environment, and taking advantage of the *Module* that we are going to describe, we will design a "friendly" environment that allows the user to enter data and interact with the program in a simple manner. Moreover, we can see and manipulate both the analytical outputs as well as the graphical results.

In this section we focus our attention on the technical aspects of the *Module* and we leave the aspects of the *Manipulate* to the next subsection.

The *Mathematica Module* that we are going to describe runs once we previously know the following values:

1. The order of the equation (**n**)
2. The coefficients (**vector c**)

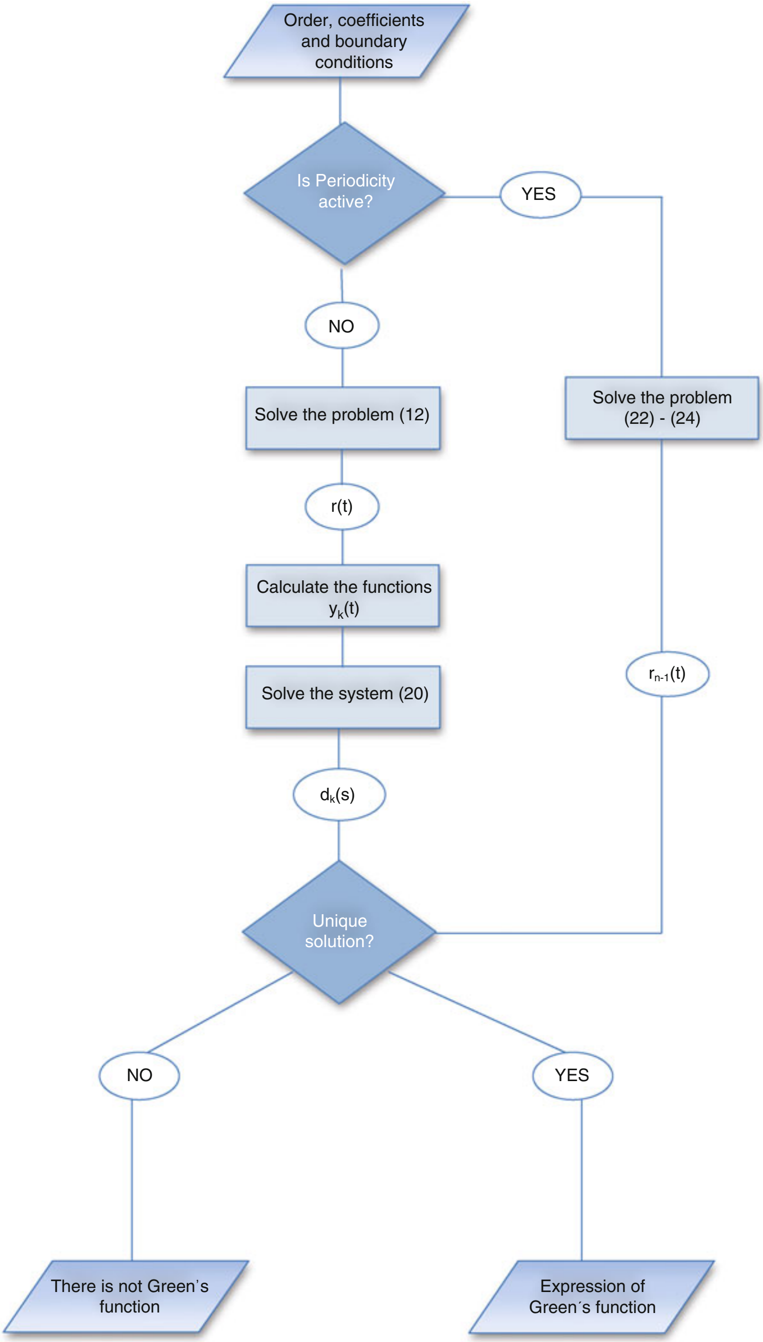

Fig. A.1 Flow diagram of the algorithm

3. The extremes of the interval (**exta** and **extb**)
4. The two-point boundary conditions depending on a and b (**vector cc**)

Moreover we have the option **Periodicity** (False/True).

As a consequence, we can enter the values for the aforementioned variables and simply pick up at the *Module* to run. As we have noticed, in the next section we will explain how to insert this *Module* into the *Manipulate* to enhance the user interaction.

Our *Module* is divided in two algorithms, depending on whether the boundary conditions are, or not, the periodic ones. In this last case, we can choose a more specific algorithm that calculates the function r defined in (1.4.9) Let us start to describe the generic algorithm presented at the beginning of this Appendix, which is the one to be used if we keep the **Periodicity** option as False.

At the beginning, the program checks that both the number of coefficients and of boundary conditions equal the order of the considered equation. Due to the fact that such conditions should be evaluated in the *Module*, in order to find the coefficients of the system described in (A.1.7), these boundary conditions must be a real function and not a vector, as they have been introduced by the keyboard. We have decided to ask for a n-dimensional vector because in this case it is not necessary to enter all the coefficients α^i_j, β^i_j, in functions U_i. It is the *Module* which calculates these coefficients directly from the introduced vector.

To carry out the transformation of each boundary condition into a real function, it is defined as an auxiliary function at the beginning of the *Module* (**aux**). This function saves the vector that contains the boundary conditions as a function that depends on u, ext$a$ and ext$b$. After transforming it into a system, the coefficients that multiply $u^{(j)}(\text{ext}a)$ and $u^{(j)}(\text{ext}b)$ are extracted as α^i_j and β^i_j, respectively. So the function is well defined.

The part of the code that makes this step is showed below. Here **aux** is the auxiliary function that has been previously defined from the vector that contains the boundary conditions:

$$\texttt{Do}\left[\texttt{alfa}[i,j] = \texttt{Coefficient}\left[\text{aux}[u][\text{ext}a,\text{ext}b][[i]], u^{(j)}[\text{ext}a]\right], \{j,0,\texttt{Length}[c]-1\}, \{i,1,\texttt{Length}[c]\}\right];$$

$$\texttt{Do}\left[\texttt{beta}[i,j] = \texttt{Coefficient}\left[\text{aux}[u][\text{ext}a,\text{ext}b][[i]], u^{(j)}[\text{ext}b]\right], \{j,0,\texttt{Length}[c]-1\}, \{i,1,\texttt{Length}[c]\}\right];$$

Once the coefficients α^i_j and β^i_j have been extracted, the functionals are defined U_i, which depends on u, ext$a$, and ext$b$:

$$\texttt{Do}\left[U_i[\text{u\_}][\text{exta\_},\text{extb\_}] = \sum_{j=0}^{\texttt{Length}[c]-1} \left(\texttt{alfa}[i,j] * u^{(j)}[\text{ext}a] + \texttt{beta}[i,j] * u^{(j)}[\text{ext}b]\right), \{i,1,\texttt{Length}[c]\}\right].$$

The first step of the algorithm mentioned above is to find the unique solution of the initial value problem (A.1.2). This problem is solved by using the *Mathematica DSolve* command. The result is saved as a function r that depends on t.

Denoting by n the order of the equation and by c the vector where the coefficients are saved, (A.1.2) is introduced in the program as

$$\texttt{ec:=}y^{(\texttt{Length}[c])}[t]+\sum_{i=1}^{\texttt{Length}[c]} c[[i]]\, y^{(\texttt{Length}[c]-i)}[t].$$

The initial value problem (A.1.2) is solved in the sentence

$$\texttt{DSolve}\left[\texttt{Join}\left[\{\texttt{ec}==0\},\texttt{Table}\left[y^{(i)}[0]==0,\{i,0,n-2\}\right],\left\{y^{(n-1)}[0]==1\right\}\right],y,t\right];$$

Note that this output is a list, so we need to extract the corresponding part of the function with $y[t]/.\text{DSolve}[\cdots][[1]]$. The input *DSolve* returns the simplified solution, and so on numerous occasions this result shows an expression in which it appears complex numbers when this expression of the solution is the shortest one. We notice that, for higher order equations, it is very usual that *Mathematica* solves the previous equation with dependence on the roots of the characteristic polynomial. In this case the expression of the solution appears as a function of the (unknown for *Mathematica*) corresponding roots. This makes the expression of Green's function impossible to process in practical situations. For this reason the program checks if the word *Root* appears in the expression and, if it is the case, it makes the transformation $c = N[c]$ over the coefficients. This fact implies that *Mathematica* considers such coefficients as a numerical approximation of the corresponding exact numbers, and it makes the next calculations for functions r, y_k and, as consequence, for Green's function G, as numerical approximations too. Our experiments show us that the numerical error is around 10^{-15}.

The second step of the algorithm consists on solving the associated problem (A.1.9)–(A.1.11). The solution is obtained either directly by using *DSolve* or by means of the expression (A.1.5), depending if either all the introduced data is real numbers or there is some parameter. The implementation in *Mathematica* is as follows:

```
If[(c ∈ Reals ∧ extb ∈ Reals ∧ exta ∈ Reals)===True,
Do [soluci[k] = DSolve [Join [{y^(Length[c])[t] + ∑_{i=1}^{Length[c]} c[[i]]y^(Length[c]-i)[t] == 0},
Table [y^(i)[0] == 0, {i, 0, k - 2}], {y^(k-1)[0] == 1},
 Table [y^(i)[0] == 0, {i, k, Length[c] - 1}]], y, t];
yk[k][t_] = FullSimplify[ComplexExpand[y[t]/.Extract[soluci[k], {1, 1}]]];,
{k, 1, Length[c]}],
Do [yk[k][t_] = r^(Length[c]-k)[t] + ∑_{j=k}^{Length[c]-1} c[[Length[c] - j]]r^(j-k)[t];,
{k, 1, Length[c]}]]
```

This different choice follows from our experimental experience with *Mathematica*. We have noticed that when all the variables are real constants, the use of formula (A.1.5) gives us bigger numerical errors than the direct resolution of (A.1.9)–(A.1.11), but (A.1.5) is more adequate if some real parameter is involved in the equations.

The coefficients of the system (A.1.7) are given by the boundary conditions U_i evaluated at y_k. To solve it, we use the *Mathematica* command *Solve*, and the unknown variables are saved on the vector d:

$$\texttt{Solve}\left[\texttt{Table}\left[0 == \sum_{j=0}^{\texttt{Length}[c]-1} \texttt{beta}[i,j] * r^{(j)}[\text{ext}b - s] \right.\right.$$
$$\left.\left. + \sum_{j=1}^{\texttt{Length}[c]} d_j[s] U_i[y[j]][\text{ext}a, \text{ext}b], \{i, 1, \texttt{Length}[c]\}\right], \texttt{Table}\,[d_i[s], \{i, 1, \texttt{Length}[c]\}]\right]$$

We notice that the variables used to solve this system must to be local ones. This is due to the fact that any overlap in the value of the vector d implies that if the dimension of the equation is changed, then the system is incorrectly solved.

Finally, by means of the performed calculations, Green's function is defined as in expression (A.1.6). To make it, we need to extract the coefficients of the solution of the system (A.1.7) returned by *Mathematica*, which is given in a list form. In the following lines of code we present the extraction of such coefficients and the definition of the function h that corresponds to the summation part on both sides of the expression (A.1.6)

$$\texttt{coef} := \texttt{Sort}[\texttt{Extract}[\texttt{ecuacion}, \{1\}]];$$

$$\texttt{Do}\left[e[i][\texttt{s\_}] := d_i[s] \,/\,.\,\texttt{Extract}[\texttt{coef}, \{i\}];, \{i, 1, n\}\right];$$

$$h[\texttt{t\_}, \texttt{s\_}] := \sum_{i=1}^{\texttt{Length}[c]} \texttt{Simplify}\left[e[i][s]\right] y[i][t];$$

The so-called function $h(t,s)$ is the most complicated part of the expression of Green's function and the simplification becomes harder to do.

We recall that this *Module* has two parts: the first one is the generic algorithm, described above, and the second part consists on a specific algorithm to solve equations with periodic boundary conditions. The option **Periodicity** makes the program to run in one way or in another.

In the specific algorithm for periodic boundary conditions it is only needed to solve a boundary value problem and to define Green's function as in (1.4.12).

A.1.2 An Environment Based on Manipulate

Once the program has been implemented, the next step is to make a simple environment for the input of the data. To this end, we have programed a *Manipulate* package, which is an interactive environment where users must enter the data through boxes or menus.

While running the program an environment will appear as in Fig. A.2.

In this environment the user must enter the order of the equation, n, that should be a natural number. Then the vector of coefficients must be introduced, which must be written in *Mathematica* format, i.e., between keys and separated by commas. The vector $\{c_1, \ldots, c_n\}$ must have length n and it contains the coefficients accompanying $u^{n-1)}, \ldots, u$.

The coefficients could be real numbers as well as parameters. These parameters can be any lowercase letter that has not been previously used in the calculations (for instance: $j, k, l, m, \ldots$). Notice that the parameter is not considered as a variable by Green's function, so the Graph option does not run in this case.

In the next two boxes the user must insert the endpoints of the interval I. The second one must be strictly bigger than the first one. It is allowed to insert at the endpoints the values a and/or b in a generic way, but in this case the option Graph does not run again, because they are considered as parameters and not as variables.

In the last box the boundary conditions are introduced. They must be inserted as a n-dimensional vector and must take its values at the previously given endpoints a and b. If we choose the option **Periodicity**, the considered boundary conditions are the periodic ones and the program calculates them with the alternative algorithm explained at the beginning of this section. Of course, the periodic boundary conditions can also be introduced in the corresponding box and the program will make the calculations in a generic way.

The **Enter** button controls the execution of the *Manipulate*. While the program is making calculations, it appears pressed.

A.1.3 Final Remarks

Because some of the calculations can exceed the maximum execution time of the machine outputs, we can have an unsatisfactory output that does not return *Manipulate* to the initial situation. In such a case it will be necessary to abort the execution and relaunch the program evaluating again the code.

By making suitable simplifications the output given by the program can be substantially improved. For instance, some commands as **ComplexExpand**, **ExpToTrig**, or **Simplify** may help to get the expression of Green's function without complex numbers. We remark that $G[t, s]$, where Green's function is saved, is a global variable, so once the execution is completed the user can simplify it outside the *Module* by applying the commands that are more suitable in each case.

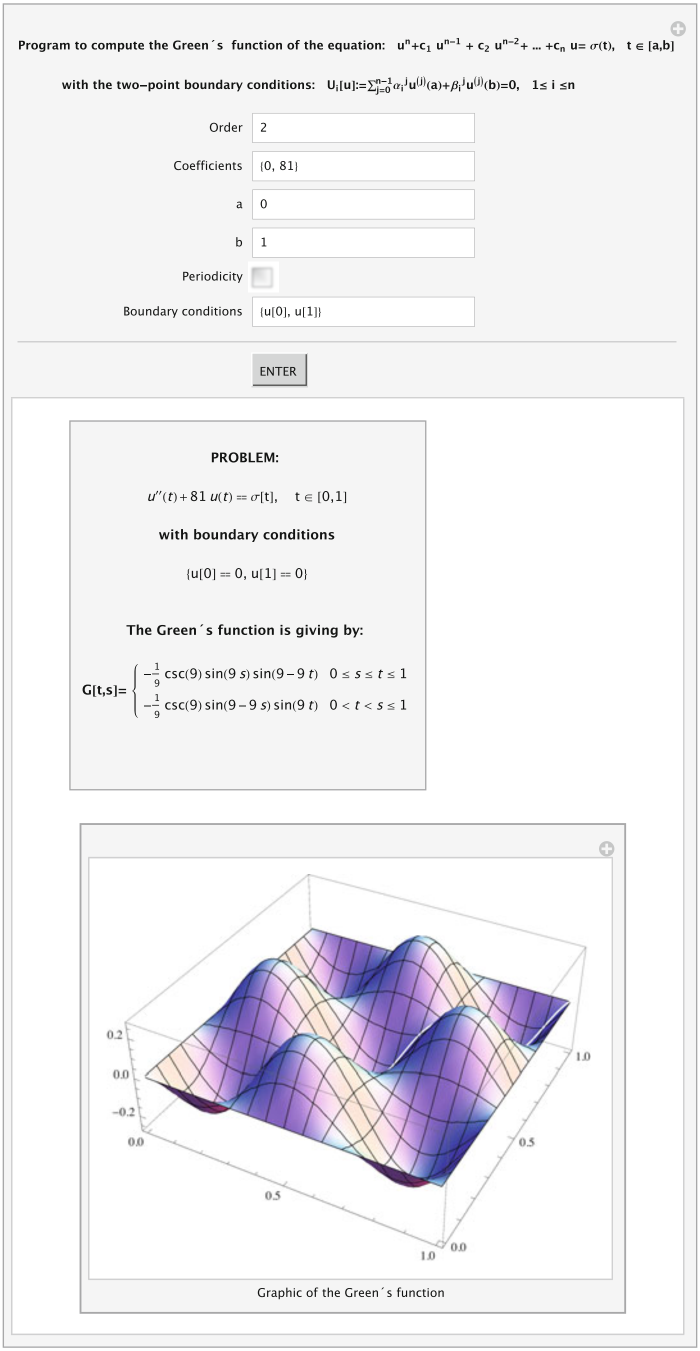

Fig. A.2 Environment for the calculation of Green's function

The complete code of the program is included at the end of the paper as an appendix. Alternatively it can be downloaded from the web page

http://webspersoais.usc.es/persoais/alberto.cabada/index.html

Notice that *Mathematica 8.0.1.0* is recommended in order to run the program. Some trouble could arise if the program is executed in other versions of *Mathematica*.

Appendix B
Expressions of Some Particular Green's Functions

At the Wolfram site a restricted version of the program that has been shown in previous appendix is posted. It may be used without *Mathematica*. In this appendix, the expressions of Green's functions obtained by the *Mathematica* package on that demonstration are given.

B.1 First-Order Problems

Periodic Problem: $u'(t) + m\,u(t) = \sigma(t), \quad t \in [a,b], \qquad u(a) = u(b).$

$$g(t,s,a,b,m) = \frac{1}{e^{bm} - e^{am}} \begin{cases} e^{m(b+s-t)}, & a \le s < t \le b, \\ e^{m(a+s-t)}, & a < t < s \le b. \end{cases}$$

Initial Problem: $u'(t) + m\,u(t) = \sigma(t), \quad t \in [a,b], \qquad u(a) = 0.$

$$g(t,s,a,b,m) = \begin{cases} e^{m(s-t)} & a \le s < t \le b, \\ 0 & a < t < s \le b. \end{cases}$$

Terminal Problem: $u'(t) + m\,u(t) = \sigma(t), \quad t \in [a,b], \qquad u(b) = 0.$

$$g(t,s,a,b,m) = \begin{cases} 0 & a \le s < t \le b \\ -e^{m(s-t)} & a < t < s \le b. \end{cases}$$

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B.2 Second-Order Problems

Periodic Problem: $u''(t) + m^2 u(t) = \sigma(t),\ t \in [a,b], \quad u(a) = u(b),\ u'(a) = u'(b).$

$$g(t,s,a,b,m) = \begin{cases} r(a+t-s,a,b,m), & a \le s \le t \le b \\ r(b+t-s,a,b,m), & a < t \le s \le b. \end{cases}$$

with

$$r(t,a,b,m) = -\frac{\csc\left(\frac{1}{2}m(a-b)\right)\cos\left(\frac{1}{2}m(-a+b-2t)\right)}{2m}.$$

Periodic Problem: $u''(t) - m^2 u(t) = \sigma(t),\ t \in [a,b], \quad u(a) = u(b),\ u'(a) = u'(b).$

$$g(t,s,a,b,m) = \begin{cases} r(a+t-s,a,b,m), & a \le s \le t \le b \\ r(b+t-s,a,b,m), & a < t \le s \le b. \end{cases}$$

with

$$r(t,a,b,m) = \frac{e^{m(a+t)} + e^{m(b-t)}}{2m\left(e^{am} - e^{bm}\right)}.$$

Dirichlet Problem: $u''(t) + m^2 u(t) = \sigma(t), \quad t \in [a,b], \qquad u(a) = u(b) = 0.$

$$g(t,s,a,b,m) = \begin{cases} h(s,t,a,b,m), & a \le s \le t \le b, \\ h(t,s,a,b,m), & a < t \le s \le b. \end{cases}$$

with

$$h(t,s,a,b,m) = -\frac{\sin(m(a-t))\sin(m(b-s))}{m\ \sin(m(a-b))}.$$

Dirichlet Problem: $u''(t) - m^2 u(t) = \sigma(t), \quad t \in [a,b], \qquad u(a) = u(b) = 0.$

$$g(t,s,a,b,m) = \begin{cases} h(s,t,a,b,m), & a \le s \le t \le b, \\ h(t,s,a,b,m), & a < t \le s \le b. \end{cases}$$

with

$$h(t,s,a,b,m) = \frac{\left(e^{2mt} - e^{2am}\right)\left(e^{2m(b-s)} - 1\right)e^{m(s-t)}}{2m\left(e^{2am} - e^{2bm}\right)}.$$

Neumann Problem: $u''(t) + m^2\, u(t) = \sigma(t), \quad t \in [a,b], \qquad u'(a) = u'(b) = 0.$

$$g(t,s,a,b,m) = \begin{cases} h(s,t,a,b,m), & a \le s \le t \le b, \\ h(t,s,a,b,m), & a < t \le s \le b. \end{cases}$$

with

$$h(t,s,a,b,m) = -\frac{\cos(m(a-t))\cos(m(b-s))}{m\ \sin(m(a-b))}.$$

Neumann Problem: $u''(t) - m^2\, u(t) = \sigma(t), \quad t \in [a,b], \qquad u'(a) = u'(b) = 0.$

$$g(t,s,a,b,m) = \begin{cases} h(s,t,a,b,m), & a \le s \le t \le b, \\ h(t,s,a,b,m), & a < t \le s \le b. \end{cases}$$

with

$$h(t,s,a,b,m) = \frac{\left(e^{2am} + e^{2mt}\right)\left(e^{2m(b-s)} + 1\right)e^{m(s-t)}}{2m\left(e^{2am} - e^{2bm}\right)}.$$

B.3 Third-Order Problems

Periodic Problem: $u'''(t) + m^3\, u(t) = \sigma(t),\ t \in [a,b], \quad u^{(i)}(a) = u^{(i)}(b);\ i = 0,\ 1,\ 2.$

$$g(t,s,a,b,m) = \begin{cases} r(a+t-s,a,b,m)\,, & a \le s \le t \le b \\ r(b+t-s,a,b,m)\,, & a < t \le s \le b. \end{cases}$$

Where

$$r(t,a,b,m) = -\frac{e^{-m(a+t)}}{3m^2\left(e^{am} - e^{bm}\right)\left(-2e^{\frac{1}{2}m(a+b)}\cos\left(\frac{1}{2}\sqrt{3}m(a-b)\right) + e^{am} + e^{bm}\right)}$$

$$\left[\sqrt{3}e^{\frac{3}{2}m(a+b+t)}\sin\left(\frac{1}{2}\sqrt{3}m(-a+b-t)\right) - \sqrt{3}e^{\frac{1}{2}m(5a+b+3t)}\sin\left(\frac{1}{2}\sqrt{3}m(-a+b-t)\right)\right.$$

$$+\sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}mt\right)e^{\frac{1}{2}m(4a+2b+3t)}+e^{\frac{3}{2}m(a+b+t)}\cos\left(\frac{1}{2}\sqrt{3}m(-a+b-t)\right)$$

$$-e^{\frac{1}{2}m(5a+b+3t)}\cos\left(\frac{1}{2}\sqrt{3}m(-a+b-t)\right)+\left(e^{am}-e^{bm}\right)e^{\frac{1}{2}m(4a+3t)}\cos\left(\frac{1}{2}\sqrt{3}mt\right)$$

$$+e^{m(2a+b)}+e^{m(a+2b)}-2e^{\frac{3}{2}m(a+b)}\cos\left(\frac{1}{2}\sqrt{3}m(a-b)\right)-\sqrt{3}e^{\frac{3}{2}m(2a+t)}\sin\left(\frac{1}{2}\sqrt{3}mt\right)\Bigg].$$

B.4 Fourth-Order Problems

Periodic Problem: $u^{(4)}(t)+m^4\,u(t)=\sigma(t),\ t\in[0,1],\quad u^{(i)}(0)=u^{(i)}(1);\ i=0,\ 1,\ 2,\ 3.$

$$g(t,s,m)=\begin{cases} r(t-s,m), & 0\le s\le t\le 1\\ r(1+t-s,m), & 0\le t\le s\le 1.\end{cases}$$

with

$$r(t,m)=\frac{e^{\frac{mt}{\sqrt{2}}}}{2\sqrt{2}m^3\left(e^{\sqrt{2}m}-2e^{\frac{m}{\sqrt{2}}}\cos\left(\frac{m}{\sqrt{2}}\right)+1\right)}\left[e^{\frac{m}{\sqrt{2}}}\sin\left(\frac{m(1-t)}{\sqrt{2}}\right)\right.$$

$$+e^{\frac{m(1-2t)}{\sqrt{2}}}\sin\left(\frac{m(1-t)}{\sqrt{2}}\right)+e^{\sqrt{2}m(1-t)}\sin\left(\frac{mt}{\sqrt{2}}\right)+\sin\left(\frac{mt}{\sqrt{2}}\right)$$

$$\left.+\left(e^{\frac{m}{\sqrt{2}}}-e^{\frac{m(1-2t)}{\sqrt{2}}}\right)\cos\left(\frac{m(1-t)}{\sqrt{2}}\right)+\left(e^{\sqrt{2}m(1-t)}-1\right)\cos\left(\frac{mt}{\sqrt{2}}\right)\right].$$

Periodic Problem: $u^{(4)}(t)-m^4\,u(t)=\sigma(t),\ t\in[0,1],\quad u^{(i)}(0)=u^{(i)}(1);\ i=0,\ 1,\ 2,\ 3.$

$$g(t,s,m)=\begin{cases} r(t-s,m), & 0\le s\le t\le 1\\ r(1+t-s,m), & 0\le t\le s\le 1.\end{cases}$$

with

$$r(t,m) = \frac{e^{-mt}\left(-e^{2mt} + (e^m - 1)\left(-e^{mt}\right)\csc\left(\frac{m}{2}\right)\cos\left(\frac{1}{2}m(1-2t)\right) - e^m\right)}{4\left(e^m - 1\right)m^3}.$$

Simply Supported Conditions:

$$u^{(4)}(t) + m^4\, u(t) = \sigma(t),\ t \in [0,1],\ u(0) = u(1) = u''(0) = u''(1) = 0.$$

$$g(t,s,m) = \begin{cases} h(s,t,m), & a \le s \le t \le b, \\ h(t,s,m), & a \le t \le s \le b. \end{cases}$$

with

$$\begin{aligned}
h(t,s,m) = &\frac{1}{2\sqrt{2}m^3\left(e^{2\sqrt{2}m} - 2e^{\sqrt{2}m}\cos\left(\sqrt{2}m\right) + 1\right)} e^{-\frac{m(3s+t-6)}{\sqrt{2}}} \\
&\left(-e^{2\sqrt{2}m(s-1)}\sin\left(\frac{m(s-t-2)}{\sqrt{2}}\right) - e^{\sqrt{2}m(s+t-2)}\sin\left(\frac{m(s-t-2)}{\sqrt{2}}\right)\right. \\
&+ e^{\sqrt{2}m(2s-3)}\sin\left(\frac{m(s-t)}{\sqrt{2}}\right) + e^{\sqrt{2}m(s+t-1)}\sin\left(\frac{m(s-t)}{\sqrt{2}}\right) \\
&+ e^{\sqrt{2}m(s-2)}\sin\left(\frac{m(s+t-2)}{\sqrt{2}}\right) + e^{\sqrt{2}m(2s+t-2)}\sin\left(\frac{m(s+t-2)}{\sqrt{2}}\right) \\
&- e^{\sqrt{2}m(s-1)}\sin\left(\frac{m(s+t)}{\sqrt{2}}\right) - e^{\sqrt{2}m(2s+t-3)}\sin\left(\frac{m(s+t)}{\sqrt{2}}\right) \\
&+ \left(e^{2\sqrt{2}m(s-1)} - e^{\sqrt{2}m(s+t-2)}\right)\cos\left(\frac{m(s-t-2)}{\sqrt{2}}\right) \\
&+ \left(e^{\sqrt{2}m(s+t-1)} - e^{\sqrt{2}m(2s-3)}\right)\cos\left(\frac{m(s-t)}{\sqrt{2}}\right) + e^{\sqrt{2}m(s-2)}\cos\left(\frac{m(s+t-2)}{\sqrt{2}}\right) \\
&- e^{\sqrt{2}m(2s+t-2)}\cos\left(\frac{m(s+t-2)}{\sqrt{2}}\right) - e^{\sqrt{2}m(s-1)}\cos\left(\frac{m(s+t)}{\sqrt{2}}\right) \\
&\left. + e^{\sqrt{2}m(2s+t-3)}\cos\left(\frac{m(s+t)}{\sqrt{2}}\right)\right).
\end{aligned}$$

Simply Supported Conditions:

$$u^{(4)}(t) - m^4\, u(t) = \sigma(t),\ t \in [0,1],\ u(0) = u(1) = u''(0) = u''(1) = 0.$$

$$g(t,s,m) = \begin{cases} h(s,t,m), & 0 \le s \le t \le 1, \\ h(t,s,m), & 0 \le t \le s \le 1. \end{cases}$$

with

$$
\begin{aligned}
&h(t,s,m)\\
&=\frac{1}{4\left(e^{2m}-1\right)m^3}\left(e^{-m(s+t-2)}\left(e^{2m(s+t-1)}\right.\right.\\
&\left.\left.-2\csc(m)\left(e^{m(s+t-2)}-e^{m(s+t)}\right)\sin(m-ms)\sin(mt)-e^{2m(s-1)}-e^{2mt}+1\right)\right).
\end{aligned}
$$

Clamped Beam Conditions:

$$u^{(4)}(t)+m^4\,u(t)=\sigma(t),\ t\in[0,1],\ u(0)=u(1)=u'(0)=u'(1)=0.$$

$$g(t,s,m)=\begin{cases}h(s,t,m), & 0\le s\le t\le 1,\\ h(t,s,m), & 0\le t\le s\le 1.\end{cases}$$

with

$$
\begin{aligned}
&h(t,s,m)\\
&=\frac{e^{-\frac{m(3s+t-6)}{\sqrt{2}}}}{2\sqrt{2}m^3\left(-4e^{\sqrt{2}m}+e^{2\sqrt{2}m}+2e^{\sqrt{2}m}\cos\left(\sqrt{2}m\right)+1\right)}\\
&\left[\left(-e^{\sqrt{2}m(s-2)}\sin\left(\frac{m(s-2)}{\sqrt{2}}\right)+e^{2\sqrt{2}m(s-1)}\sin\left(\frac{m(s-2)}{\sqrt{2}}\right)\right.\right.\\
&+e^{\sqrt{2}m(s-1)}\sin\left(\frac{ms}{\sqrt{2}}\right)-e^{\sqrt{2}m(2s-3)}\sin\left(\frac{ms}{\sqrt{2}}\right)\\
&+\left(e^{\sqrt{2}m(s-2)}+e^{2\sqrt{2}m(s-1)}\right)\cos\left(\frac{m(s-2)}{\sqrt{2}}\right)\\
&\left.+\left(-2e^{\sqrt{2}m(s-2)}+e^{\sqrt{2}m(s-1)}-2e^{2\sqrt{2}m(s-1)}+e^{\sqrt{2}m(2s-3)}\right)\cos\left(\frac{ms}{\sqrt{2}}\right)\right)\\
&\left(\left(e^{\sqrt{2}mt}-1\right)\cos\left(\frac{mt}{\sqrt{2}}\right)-\left(e^{\sqrt{2}mt}+1\right)\sin\left(\frac{mt}{\sqrt{2}}\right)\right)\\
&-2\left(e^{\sqrt{2}mt}-1\right)\sin\left(\frac{mt}{\sqrt{2}}\right)\\
&\left(\left(e^{\sqrt{2}m(s-2)}-e^{\sqrt{2}m(s-1)}+e^{2\sqrt{2}m(s-1)}-e^{\sqrt{2}m(2s-3)}\right)\sin\left(\frac{ms}{\sqrt{2}}\right)\right.\\
&\left.\left.+\left(e^{\sqrt{2}m(s-2)}-e^{2\sqrt{2}m(s-1)}\right)\cos\left(\frac{m(s-2)}{\sqrt{2}}\right)+\left(e^{2\sqrt{2}m(s-1)}-e^{\sqrt{2}m(s-2)}\right)\cos\left(\frac{ms}{\sqrt{2}}\right)\right)\right].
\end{aligned}
$$

Clamped Beam Conditions:

$$u^{(4)}(t) - m^4\, u(t) = \sigma(t),\ t \in [0,1],\ u(0) = u(1) = u'(0) = u'(1) = 0.$$

$$g(t,s,m) = \begin{cases} h(s,t,m), & 0 \le s \le t \le 1, \\ h(t,s,m), & 0 \le t \le s \le 1. \end{cases}$$

with

$$\begin{aligned}
&h(t,s,m) \\
&= \frac{e^{m(s-1)}}{8m^3\left(\left(e^{2m}+1\right)\cos(m)-2e^m\right)}\Big[\big(e^{-mt}+e^{mt}-2\cos(mt)\big) \\
&\quad \big(e^{-2m(s-1)}+e^{m(3-2s)}\sin(m)-2e^{-m(s-2)}\sin(ms)-e^{-m(s-3)}\sin(m-ms) \\
&\quad - e^{m-ms}\sin(m-ms)+\big(e^{m}-e^{m(3-2s)}\big)\cos(m) \\
&\quad + \big(e^{2m}-1\big)e^{m-ms}\cos(m-ms)-e^{2m}+e^{m}\sin(m)\big) \\
&\quad - \big(e^{-mt}-e^{mt}+2\sin(mt)\big)\big(e^{-2m(s-1)}-e^{m(3-2s)}\sin(m)+e^{-m(s-3)}\sin(m-ms) \\
&\quad - e^{m-ms}\sin(m-ms)-\big(e^{m(3-2s)}+e^{m}\big)\cos(m)+2e^{-m(s-2)}\cos(ms) \\
&\quad - e^{-m(s-3)}\cos(m-ms)-e^{m-ms}\cos(m-ms)+e^{2m}+e^{m}\sin(m)\big)\Big].
\end{aligned}$$

Glossary

$J = [a, b] = \{x \in \mathbb{R},\ a \leq x \leq b\}$. In some particular situations J corresponds to the intervals $[0, 1]$, $[0, \pi]$ or $[0, R]$.

I_n: Identity matrix of dimension n.

$\|f\|_\infty = \sup_{t \in J} \{|f(t)|\}$.

$\|f\|_p = \left(\int_a^b |f(t)|^p\, dt\right)^{1/p}$.

$\mathscr{M}_{n \times n}$: Set of square matrices of dimension equals to n.

$\ker(L)$: Kernel of operator L, i.e., the set of x such that $L\,x = 0$.

$\det(M)$: Determinant of the matrix M.

rank (M): Rank of the matrix M.

M^{-1}: Inverse matrix of the matrix M.

M^T: Transposed matrix of the matrix M.

T^*: Adjoint of operator T.

$\mathscr{L}^p(J, \mathbb{R}) = \{f : J \to \mathbb{R},\ f$ is measurable on J and $\int_a^b |f(t)|^p\, dt < \infty\}$.

$\mathscr{C}^m(J, \mathbb{R}) = \{f : J \to \mathbb{R},\ f^{(j)}$ is continuous on J for all $j \in \{0, \ldots, m\}\}$.

$\mathscr{AC}(J, \mathbb{R}) = \{f : J \to \mathbb{R},\ f \in \mathscr{C}(J, \mathbb{R}),\ f' \in \mathscr{L}^1(J, \mathbb{R})$ and $f(t) - f(s) = \int_s^t f'(r)\, dr$ for all $t, s \in J\}$.

$\mathscr{W}^{m,p}(J, \mathbb{R}) = \{f : J \to \mathbb{R},\ f \in \mathscr{C}^{m-1}(J, \mathbb{R}),\ f^{(m-1)} \in \mathscr{AC}(J, \mathbb{R})$ and $f^{(m)} \in \mathscr{L}^p(J, \mathbb{R})\}$.

$\mathscr{L}^p(J, \mathbb{R}^n) = \{f \equiv (f_1, \ldots, f_n) : J \to \mathbb{R}^n,\ f_j \in \mathscr{L}^p(J, \mathbb{R})$ for all $j \in \{0, \ldots, n\}\}$.

$\mathscr{C}^m(J, \mathbb{R}^n) = \{f \equiv (f_1, \ldots, f_n) : J \to \mathbb{R}^n,\ f_j \in \mathscr{C}^m(J, \mathbb{R})$ for all $j \in \{0, \ldots, n\}\}$.

$\mathscr{AC}(J, \mathbb{R}^n) = \{f \equiv (f_1, \ldots, f_n) : J \to \mathbb{R}^n,\ f_j \in \mathscr{AC}(J, \mathbb{R})$ for all $j \in \{0, \ldots, n\}\}$.

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$$\mathscr{W}^{m,p}(J,\mathbb{R}^n) = \{f \equiv (f_1,\ldots,f_n) : J \to \mathbb{R}^n,\ f_j \in \mathscr{W}^{m,p}(J,\mathbb{R}) \text{ for all } j \in \{0,\ldots,n\}\}.$$

$$\mathscr{L}^p(J,\mathscr{M}_{n\times n}) = \{f \equiv (f_{i,j})_{i,\,j\in\{1,\ldots,n\}} : J \to \mathscr{M}_{n\times n},\ f_{i,j} \in \mathscr{L}^p(J,\mathbb{R}) \text{ for all } i,\ j \in \{0,\ldots,n\}\}.$$

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