Appendix A: Math Programming

In spite of the name, a mathematical program is a concise statement of a mathematical optimization problem, not a computer program. It specifies a vector of real variables **u**, together with a scalar objective function *f*(**u**) that is to be optimized (either maximized or minimized), and also possibly some constraints on **u**. The most compact statement of the constraints would be to say $\mathbf{u} \in U$; that is, only members of the set *U* are eligible. If the problem is to maximize *f*(**u**), the notation that we will use for specifying the associated mathematical program is

$$\begin{aligned}&\max_{\mathbf{u}} f(\mathbf{u})\\&\text{subject to } \mathbf{u}\in U\end{aligned}$$

Note that the variables are named in the subscript on max or min; if a symbol is not named there, then it is not a variable. If the set *U* is empty, the program is said to be "infeasible". If the objective function can be made arbitrarily large, the program is said to be "unbounded". An example of an unbounded problem is to maximize u subject to $u \geq 0$. In practice most mathematical programs are feasible and bounded.

For brevity we will also permit two elaborations of the subscript. One is to state the constraining set in the subscript, so that the above program is simply $\max_{\mathbf{u} \in U} f(\mathbf{u})$. The other is to state that all components of **u** must be nonnegative. Consider the following example

$$\begin{aligned}&\min_{v,\mathbf{u}\geq 0} v^2 + u_1 + w{u_2}^4\\&\text{subject to } u_1 + \sin(v) \leq 5\\&u_1 + v = -3\end{aligned}$$

There are three variables in this mathematical program, the two components of **u** and the scalar v. The symbol w is not a variable, since it is not among the subscripts, so its value must somehow be determined before the problem is solved. The subscript on min requires both components of **u** to be nonnegative. The first explicit constraint is that a particular function of the variables cannot exceed 5,

A. Washburn, *Two-Person Zero-Sum Games*, International Series in Operations Research & Management Science 201, DOI 10.1007/978-1-4614-9050-0,

and the second is that a different function must equal −3. The relationship "≥" is permitted in constraints in addition to "≤" and "=", but only those three. Mathematical programming packages are prepared to find the solutions of such problems, and some of them are also prepared to accept constraints to the effect that certain variables must be integer-valued.

To distinguish variables that are optimal, we will simply affix an asterisk to the variable name. Thus, if the mathematical program is $\min_x x^2$, we say $x^* = 0$ because the quadratic function is minimized when x is set to 0.

The most important special case of a mathematical program is a linear program, since software exists that will solve large instances of linear programs quickly and reliably. A linear program is the problem of optimizing a linear function subject to constraints that are also linear. An example is LP1 of Chap. 3, which finds the optimal mixed strategy $\mathbf{x}$ for player 1 in a game whose payoff matrix is $\mathbf{a} = (a_{ij})$:

$$\begin{aligned}&\max_{v,\mathbf{x}\geq 0} v\\ &\text{subject to } \sum_{i=1}^{m} x_i a_{ij} - v \geq 0;\ \ j = 1, \ldots, n\\ &\sum_{i=1}^{m} x_i = 1\end{aligned}$$

LP1 has $m + 1$ variables, only one of which (v, the game value) is involved in the objective function. In stating LP1 we have observed the convention that the right-hand-side of every constraint is a constant, and that there are no constant terms on the left-hand-side. Linear program LP2 of Chap. 2 similarly expresses player 2's problem of minimizing the payoff, instead of maximizing it, using a linear program with $n + 1$ variables and $m + 1$ constraints. LP1 and LP2 are duals of each other.

Every linear program has a dual linear program with the opposite objective. If the original "primal" program LP is:

$$\begin{aligned}&\max_{\mathbf{x}\geq 0} \sum_{i=1}^{m} c_i x_i\\ &\text{subject to } \sum_{i=1}^{m} a_{ij} x_i \leq b_j;\ \ j = 1, \ldots, n,\end{aligned}$$

then the dual linear program is LP′:

$$\begin{aligned}&\min_{\mathbf{y}\geq 0} \sum_{j=1}^{n} b_j y_j\\ &\text{subject to } \sum_{j=1}^{n} a_{ij} y_j \geq c_i;\ \ i = 1, \ldots, m.\end{aligned}$$

LP is stated in "canonical form",[1] meaning that it is a maximization with nonnegative variables and upperbounding constraints, but any linear program can be put into canonical form. Consider LP1, for example. LP1 has an unconstrained variable v, but v can be expressed as $v_1 - v_2$, the difference of two nonnegative variables. LP1 also has an equality constraint, but any constraint of the form $z = b$ is equivalent to two upperbounding constraints $z \leq b$ and $(-1)z \leq -b$. Thus LP1 could be put into canonical form using $m + 2$ variables and $n + 2$ constraints.

The two important facts about duality are

1. $(\text{LP}')'$ is LP; that is, the dual of the dual is the primal
2. If LP and LP′ have solutions $\mathbf{x}^*$ and $\mathbf{y}^*$, then $\sum_{i=1}^{m} c_i x_i^* = \sum_{j=1}^{n} b_j y_j^*$; that is, the dual and the primal both have the same optimized objective.

The first fact is not difficult to prove. If the reader wishes to try his hand at it, remember that any minimization problem can be turned into a maximization problem by changing the sign of the objective function, so even LP′ can be put into canonical form. The second fact implies Theorem 3.2-1, the fundamental theorem that underlies most of the results in TPZS game theory.

Many linear programming packages are available for download, or the reader may have access to one already. Two possible package sources are Solver in Microsoft's Excel ™ and the linprog() function in MATLAB™. These packages return dual variables that have a useful sensitivity interpretation, as well as primal variables. INFORMS periodically conducts a survey of available linear programming packages (Fourer 2013).

To solve LP1 using Solver, input the matrix **a** in rows and columns, designate $m + 1$ cells for the variables, do the math to form the required linear expressions, start Solver and tell it about the variables (Solver's term is "adjustable cells") and constraints, and then maximize. The constraints should say that all components of **x** must be nonnegative, but not (since games are allowed to have negative values) v. Solver will adjust the variables to be maximizing. If you prefer, solve LP2 instead, as in Sheet "MorraLP" of the TPZS workbook. The other player's optimal strategy is available through the dual variables (Solver's term is "shadow prices") on the "sensitivity report", a new sheet that Solver offers after solving the problem.

In MATLAB, the following LP2-based script will take an input matrix **A** (note upper case, and that MATLAB is case sensitive) and produce the game value and optimal strategies for both players. The script uses the linprog() function that is available (as of this writing in 2013) in the Optimization toolbox. The script can be copied from sheet "MATLAB" in the TPZS workbook.

[1] Linear programming computer packages do not require that inputs be expressed in canonical form, but use of the form makes it easy to express the dual.

```
% gamesolver.mat
% Solves TPZS (2-person, zero sum) games using LINPROG command
% Payoff matrix must be called A

[m,n] = size(A);
% Create the various inputs for the LINPROG function
B = [P -1*ones(m,1)];      % creates the B matrix
b = zeros(m,1);            % creates the goal
f = [zeros(n,1); 1];       % want the value of the game to be minimized
Beq = [ones(1,n) 0];       % Beq and beq set up criterion that the sum of
the solution probabilities = 1
beq = 1;                   %
lb = [zeros(n,1); -Inf];   % lower bound on solution
% Using the LINPROG command to solve this
[y,fval,exitflag,output,lambda] = linprog(f,B,b,Beq,beq,lb);
% Output
v = fval;
y = y(1:n);
x = lambda.ineqlin;
% Formatted, output

Answer = char('For the payoff matrix', int2str(P),'','The value of the
game is ', num2str(v),'', 'Player I''s strategy is', num2str(x),'',
'Player II''s strategy is', num2str(y'))
```

As mentioned above, solving LP1 or LP2 is a matter of convenience, since the optimal strategy for the other player is available through dual variables. The call to linprog() in the above script puts the dual variables in the output array lambda, from which player 1's optimal mixed strategy **x** is retrieved. Different LP packages have different conventions about whether variables are assumed by default to be non-negative. They are not so assumed in MATLAB, so the statement defining array **lb** in the above script states explicitly that the first n variables (the first n components of **y**) must not be negative because they all have a lower bound of 0, whereas the last variable (v is named y_{n+1}) is effectively unconstrained.

Appendix B: Convexity

Convexity is a property of both functions and sets, and plays a heavy role in game theory. This appendix is an introduction to terms and fundamental theorems.

B.1 Convex Sets and Convex Hulls

Figure B.1 shows two subsets of 2-dimensional Euclidean space. Set A on the left is bounded, but not convex because the illustrated line segment does not lie entirely within A. The definition of a convex set in any number of dimensions is that every line segment connecting two points of the set must lie entirely within the set, and A fails that test. Set B on the right is convex, as would be a circle. Set B is a convex polygon with five extreme points (corners).

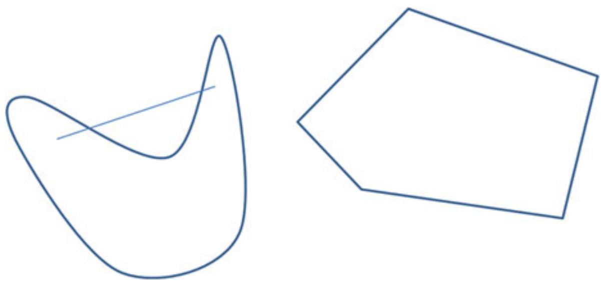

Fig. B.1 A set A that is not convex on the left and a convex polygon B on the right

Every bounded set such as A has a convex hull that is by definition the intersection of all convex sets that contain it, or more simply the smallest convex set that contains it. In two dimensions, put a fence on the edge of A, surround the fence with a rubber band and let go. The rubber band will settle at the convex hull. A convex set is its own convex hull, so the convex hull of B is already pictured. Set B is the convex hull of just five extreme points, so the extreme points themselves are a compact way of describing it.

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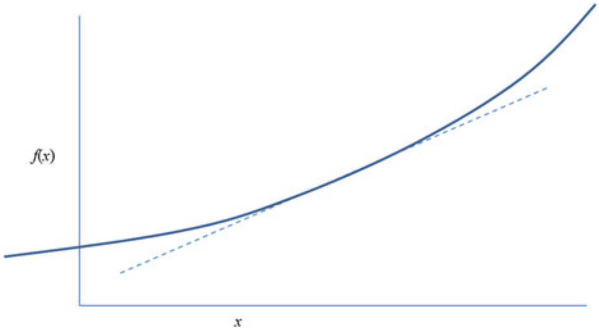
B.2 Convex Functions and Jensen's Inequality

Figure B.2 is a graph of a convex function, together with a tangent line. The function lies entirely above the tangent line no matter where the tangent is drawn, a property that characterizes convex functions.

Fig. B.2 A convex function (solid) lying above one of its tangents (dashed)



More generally, suppose that $f(x)$ maps some convex set S in n-dimensional Euclidean space onto the real line. We say that the function is convex if and only if, at every point $y \in S$, there exists a gradient vector **s** such that

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \sum_{i=1}^{n} s_i(x_i - y_i) \quad \text{for all} \quad x \in S$$

The linear expression on the right-hand-side is a hyperplane that is tangent to the function at point **y**. The gradient **s** of that hyperplane need not be unique. For example $|x|$ is a convex function on the whole real line even though it is not differentiable at 0. At point 0 the function lies above many different tangent lines.

A function is concave if its negation is convex. Linear functions are both concave and convex.

One of the important theorems about convex functions is Jensen's inequality, which states that, if $f(x)$ is a convex function defined on convex set S and if $\mathbf{X}$ is a vector of random variables that takes values in S, then $E(f(\mathbf{X})) \geq f(E(\mathbf{X}))$. To prove this, let $\mathbf{y} \equiv E(\mathbf{X})$. Point **y** is in S because S is convex, so there is a gradient **s** such that always

$$f(\mathbf{X}) \geq f(\mathbf{y}) + \sum_{i=1}^{n} s_i(X_i - y_i)$$

Upon taking expected values of both sides, the sum on the right vanishes and we are left with Jensen's inequality.

The difference between the expected value of a function and the function of the expected value is actually intuitive, and is even the source of a certain kind of perverted humor. For example, if half of your tennis balls go too far and the other half go into the net, you opponent might say “You’re doing great, on the average”, thinking himself clever. Even though all your shots are losers, a shot with the average length, if only you could hit one, would be a winner.

Appendix C: Exercise Solutions

Chapter 1

There are no exercises in chapter 1.

Chapter 2

1. $v_1 = 3$ and $v_2 = 4$
2. S should first include $\{1, 2\}$, then $\{3, 4, 5\}$ in response to player 1's choice of row 2, and then any subset of two columns in response to player 1's choice of row 3. Since $\{1, 2\}$ is among those subsets and already in S, the final S includes only two subsets.

Chapter 3

1.

	x*	**y***	v
a	(0, 0.5, 0.5)	(0, 0, 0.75, 0.25)	3.5
b	(0.25, 0.75)	(0.5, 0.5)	2.5
c	(1/3, 2/3)	(1/6, 5/6, 0)	4/3
d	(0, 1 ,0)	(0, 1, 0)	2
e	(0, 0, 1, 0)	(1, 0, 0)	1
f	(0, 9/16, 0, 0, 7/16)	(0.5, 0.5, 0)	4.5
g	(1/3, 2/3)	(1/3, 0, 2/3, 0)	8/3
h	(0, 1/4, 0, 0, 0, 3/4)	(0, 7/8, 0, 1/8)	15/4
i	(0.5, 0.5, 0)	(0, 0, 0, 0.5, 0.5)	2.5

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6. maximize $qx_1{*}y_1{*}$ subject to $v = 0$ to find that the maximum is $1.5 - \sqrt{2} = 0.0858$, slightly larger than 1/12, the probability of a called bluff in Basic Poker.
8. Each side has four strategies in the normal form.
10. a 4 × 5 matrix with value 0.5
11. The modification of 1c is: $\begin{bmatrix} 4 & 0 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

    For example, the third column is the strategy "choose column 3 of the original game if there is no leak, else choose the best response to player 1's choice".
12b. $\mathbf{y}^* = (0.5, 0, 0.5)$ is also optimal
13. 18 rows and 2 columns
14. The payoff matrix is $\begin{bmatrix} 7 & 0 \\ 3 & 4 \end{bmatrix}$, with the first strategy for both players being "up". The solution is $\mathbf{x}^* = (1/8, 7/8)$, $\mathbf{y}^* = (1/2, 1/2)$, and $v = 3.5$. I say go with the mixed strategy, but Aumann and Machsler (1972) would like a chance to talk you out of it.
15. The solution of the game is that $v = \frac{6}{\pi^2}$, and that $x_j = y_j = v/j^2$ for all positive integers j.
16. Correct to two decimal places, you should find that $v = 2.57$, $\mathbf{x^*} = (0.43, 0.57)$, and $\mathbf{y^*} = (0.86, 0, 0.14)$
17. (a) The upper bound is 5/9 and the lower bound is −5/9.
    (b) $\mathbf{x}^* = \mathbf{y}^* = (0, 0, 0.433, 0, 0.324, 0, 0.243, 0, 0)$, and $v = 0$
18. (a) $\mathbf{x}^* = \mathbf{y}^* = (10/66, 26/66, 13/66, 16/66, 1/66)$
    (b) $v = -0.28$, so player 2 is favored
20. $v = 1.8$. (1/3, 1/3, 1/3) is not optimal for player 1.
21. The payoff matrix will depend on the order in which you list player 2's strategies, but the game value should not. One version of the matrix is $\begin{bmatrix} 0 & 0 & 3 & 9 & 5 & 7 \\ 3 & 9 & 0 & 0 & 8 & 4 \\ 7 & 5 & 8 & 4 & 0 & 0 \end{bmatrix}$, where the last column corresponds to searching cells in the order CBA. There is no dominance. The value of the game is 7/2. Player 2 is equally likely to use ABC or CBA, the first and last columns. Player 1's optimal probabilities are (14,7,15)/36.
23. $\mathbf{x}^* = \mathbf{y}^* = (1,1,1,1,1,0,0, \ldots)/5$, $v^* = 1/5$

Chapter 4

2. You should find $\mathbf{x}^* = (0.238, 0.191, 0.190, 0.381)$, $\mathbf{y}^* = (0.619, 0.238, 0.095, 0.048)$, $v = 0.381$
3. $v(2,1,2,1) = 0.5$, with player 1 entering a woman with probability 0.5 and player 2 entering a mouse with probability 0.75.

5. In part a, the answer is the same as whatever the value of 4 & 10 is when punts are short, which is about 67 yards. In part b, the answer is that more additional yards can be gained from 2 & 2 than from 1 & 10.
7. $v(3) = (0.200, 0.333, 0.600)$
8. $u_1 = 0, u_2 = -0.5, g = v_1 = v_2 = 0.5.$
9. (a) The last row is (2, 2, 2) and the last column is (1.5, 2, 2, 2).
   (b) The first row is (2, 3.625, 3.75) and the first column is (2, 0.75, 1, 2), so there is a saddle point where $v(3, 2, 3) = 2$ and both sides use pure CA.
   (c) $\lim_{n\to\infty} V(2,1,n) = 3$.
10. Player 1 has a strategy that will always trap the nickel, but it is hard to find and very unforgiving of mistakes. As a result, actual play of the game often leads to the incorrect conclusion that an expert nickel is impossible to trap.
11. You should find that $v(2, 7) = v(7, -2) = v(0, -5) = 11$, with all other (finite) values being smaller.
12. There are many solutions of (4.2-8). $(u_1, u_2) = (3, 4)$ is the smallest.
13. If the submarine survives the current encounter, the payoff for all future encounters is vq, so $v = \min_r \max\{P_1(r) + vq, P_2(r)(1 + vq)\}$. The two expressions within { } must be equal, so there are two equations to solve for the two unknowns r and v.
15. $\mathbf{x}^* = (k/n, 1 - k/n)$ and $\mathbf{y}^* = (1/n, 1 - 1/n)$ in game element G_{nk}.
17c. Follow the NimSum strategy with two exceptions. If there is only one pile remaining, with that pile having at least two stones, remove all stones but one. If there are two piles remaining, with one of them having a single stone, remove the other pile.
18. Player 2 can win by always leaving player 1 with a sum that is a multiple of 11. If player 1 adds x, player 2 adds $11 - x$.

Chapter 5

2. By equating $(d/dx)A(x,y)$ and $(d/dy)A(x,y)$ to 0 and solving the resulting pair of equations, one obtains $y^* = b/(b + 2a), x^* = (y^*)^2$, and $v = A(x^*,y^*) = -ax^*$. Since $x^* < y^*$, the assumption that the saddle point occurs in the region $x < y$ is confirmed.
3. The curve should go through the points $(v, c) = (1, 3)$ and $(2, 0.5)$.
4. $\mathbf{x}^* = (15/26, 10/26, 1/26)$.
10. The optimal CDF for player 1 is 0 for $x \leq 1/3$. For $x > 1/3$ it has derivative (probability density) $F'(x) = 1/(4x)^3$.
11. $\mathbf{y}^* = (0, 1, 2)$
12. $v = -0.2$
14. $v = 0.25$. Player 1's strategy is mixed, but player 2's is not.
15. $\mathbf{y}^* = (0, 0.5, 0.75, 0.9)$, and $c = 2.15$.
16a. $\mathbf{x}^* = \mathbf{y}^* = (0.358, 0.642), v = 0.328.$
17. In the revised game $\mathbf{x}^* = (0, 16/21, 5/21), \mathbf{y}^* = (6/21, 15/21)$, and $v = 0.305$.

18. (a) no (b) 0.2 (c) 0.4 (d) The value of the game is 1/3.

    The edge effect is central to the game. In the equivalent game played on a circle (imagine connecting the 0 and 1 ends of the unit interval), the optimal mixed strategies are both uniform.
20. Player 2 makes $y = \pi$, in which case player 1 sees a payoff of $-\cos(x)$ and can do no better than make $x = \pm 0.2$. Player 1 flips a coin between 0.2 and -0.2, in which case player 2 sees $0.5(\cos(y - 0.2) + \cos(y + 0.2)) = \cos(y)\cos(0.2)$ and can do no better than make $y = \pi$.
21. When two constraints are added for each player, LP of Sect. 5.5 becomes

$$\max_{\mathbf{x}\geq 0,\,\mathbf{u}\geq 0} u_1 - u_2$$
$$\text{subject to } \sum_i x_i \leq 1,$$
$$-\sum_i x_i \leq -1,$$
$$u_1 - u_2 - \sum_i x_i a_{ij} \leq 0;\;\; j = 1, \ldots, n$$

    The difference $u_1 - u_2$ is simply an unconstrained variable, so this is equivalent to LP1 of Sect. 3.10.
23. The **b** matrix is $\begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \\ -0.5 & 0 \\ -1 & 2 \end{bmatrix}$ and the value of the game is $-1/8$. Player 1 uses mixed strategy (1,3,0,0)/4 in game **b**, and player 2 uses mixed strategy (3,1)/4.
25. In part a, the most sweepers should go on segment 4, not segment 5. In part b, you would need 76 sweepers.

Chapter 6

3. In part b, choose one of the three permutations of (0, 3, 3) with probability 1/6, or one of the six permutations of (1, 2, 3) with probability 1/12. In part c after two steps, X_{12} and X_{34} are both equally likely to be 3, 4, or 5. Combining the two random variables antithetically produces a random variable that is always 8.
4. In part a, you should find that $f^* = 2.825$, $g^* = 1.528$, and $\Pi = 0.565$. In part b, since doubling the target value is tactically equivalent to halving the two costs, the average number of units consumed per target is as in part a. Considering both classes of target, the total consumptions are therefore $100(0.250 + 1.528) = 177.8$ interceptors and $100(0.750 + 2.825) = 357.5$ missiles. Since type 2 targets have value 2 each, the average value captured is $100(0.3 + 2(0.565)) = 143.0$.
5. $f^* = 2.1$, $g^* = 1.2$, and $\Pi = u^* + cf^* = 0 + (0.2)(2.1) = 0.42$. With continuous allocations, using Fig. 6.5-2 with $\beta = 0.693$, the same average resources

result in a kill probability of about 0.44. The attacker benefits most from the continuity assumption because he loses ties in the discrete model.

6. Consider the generic target with parameters α, β, and v. The two cost-effectiveness ratios α/λ and β/μ are crucial to finding the saddle point. If $v \leq \alpha/\lambda$, player 1 will not attack and of course player 2 will not defend. If $\lambda/\alpha \leq v \leq \lambda/\alpha + \mu/\beta$, player 1 attacks, but player 2 does not defend. It is only when $\lambda/\alpha + \mu/\beta \leq v$ that both players have positive allocations. The first, second and third cases apply to targets 1, 2 and 3, respectively The solution is $\mathbf{x} = (0, 0.3466, 0.3054)$ and $\mathbf{y} = (0, 0, 0.1823)$.
7. Use formula 6.5-3 with $\beta = 0.223$, $A = 1.785$, and $D = 1.116$. Π is 0.45, so 55 % of the targets should survive.

Chapter 7

1. Use Solver to solve a succession of linear programs with varying budgets. The graph of shortest path length versus budget starts at 4 when the budget is 0 and ends at 17.5 when the budget is 20. It is concave.

2b. You can reduce the maximum flow to 1 by removing arcs c and e, which is within the budget of 2, and setting the cut to $\{a, c, e\}$. If you give Solver that starting point, it will not go back to spending all of Breaker's budget on arc d. This answer is for Solver in Excel 2010—who knows what future versions of Excel may bring?

3. The average number of detections increases to 1.5 when you increase the number of asset type 2 from 1 to 2. In part b, Breaker assigns something to all arcs except for arc c. Arcs b and d will have a zero survival probability, so the only hope for User is to travel arcs a, c, and e. The product of the three survival probabilities on those arcs is $(0.5)(1)(0.2) = 0.1$, so User would at best have one chance in ten of surviving.
4. $(1.66, 2.34) = 0.66(2, 2) + 0.34(1, 3)$, where (m, n) means m units of type 1 to arc 1 and n units of type 1 to arc 3. Therefore we can play (2, 2) with probability 0.66 and (1, 3) with probability 0.34 to achieve the desired result for units of type 1. The other two types can be treated similarly and independently, since there is no prohibition on multiple assets on an arc. Each play requires three random numbers.
5. $v = 0.05$. The min cut has five arcs in it.
6. The optimized objective function in part a) is 0.12, with arcs (3, 5) and (4, 6) receiving attention from Breaker. It increases to 0.7 in part b), with 12 arcs receiving attention. In part c), the solution is ten times the solution of part a).

Chapter 8

1. $v = 1/17$, $\mathbf{x}^* = \mathbf{y}^* = (10/17, 5/17, 2/17)$.
2. You should find $J = 3$, $A = 3$, $D = 9$.
3. If Evader uses column j, the payoff is $\sum_{i \neq j} x_i^* q_i + x_j^* p_j = \sum_{i=1}^{J} x_i^* q_i - x_j^* (q_j - p_j)$, which is equivalent to the expression given in the text.
4. Evader hides at a random point, and Searcher starts at a random point and proceeds through the cells until detection. The value is n, the number of cells. See Ruckle (1983).
5. $\mathbf{x}^* = (0, 0.2, 0.8, 0, 0, \ldots)$ $\mathbf{y}^* = (0.6, 0.4)$.
6. Searcher's active strategies are s_3 and s_4, and his optimal mixed strategy is $\mathbf{x}^* = (0, 0, 4/31, 27/31, \ldots)$. The value of the game is $105/31 = 3.387$, and $\mathbf{y}^* = (19/31, 12/31)$.
8. $x^* = (1,2,3,2)/8$. You can get this result either by linear programming, since you know v and therefore the payoff matrix, or by using the equation that, in game element 1, we must have $x_i v = d_{1i}$.
9. $\mathbf{x}^* = (1,2,2)/5$, $\mathbf{y}^* = (1,1,1)/3$, and $v = 5$.
10. $\sum_{i=1}^{4} \alpha_i^{-1} = 2$, so $\mathbf{y} = (4,2,1,1)$ and $v = \exp(-4) = 0.0183$. Detection is almost certain.
11. The game is covered in Sect. 8.2.1 because the network has an Eulerian path from 1 to 3 that includes all five arcs exactly once, namely 123413. That path has a length of 10. The value of the game is therefore 5.
12. If Searcher looks within 0.5 of the center, he will cover the center and at most one point on the circumference. Since all points on the circumference have probability 0 of including Evader, such looks will detect Evader with probability 1/7. If Searcher looks farther away from the center he can cover at most 1/6 of the circumference, but not the center, so the detection probability cannot exceed (1/6) (6/7).
13. Strategies A and B are not in equilibrium. If Monster follows A, it is true that Princess can do no better than B. But if Princess follows B, Monster can start at the origin and detect Princess immediately. The value of this game is unknown, although Alpern et al. (2008) establish bounds for it.

Chapter 9

1. Assume that column 2 is not active and solve the resulting 2×2 game. If the value is to be 0, then we must have $v^2 + v - 2.5 = 0$. Solve for $v = 1.158$, verify that column 2 is indeed not active, and then solve for $\mathbf{x}^* = (0.302, 0.698)$. If instead you mistakenly assume that column 3 is not active, you will find that $v = 1.5$. However, that game value makes column 3 become active, a contradiction.

2. Changing player 2's first vector reduces the game value to 1.19, and the saddle point changes to one where both players use their first vector.
3. If **p** is changed to (1, 0), Solver's solution of the Lopside game has $\mathbf{x}^* = (2,7)/9$ and $v = 4/3$, which agrees with the solution of the first game element.
6. Make the last row of the 3×2 element all 0's, or any negative number. That will keep player 1 from using it. Also make the last two columns of the 3×4 element all 9's, or any large number, since doing so will make columns 3 and 4 unattractive to player 2.
7. Both players know which game element is operative, but player 2 has to move first. Let m_k be the minmax value of game element k. Then the value of the game is $\sum_k p_k m_k$.
8. If the initial guess is $\mathbf{x} = (0.5, 0.5)$ and $v = 1$, Solver gets the correct solution. In fact, Solver in Excel 2010 always seems to get the correct solution regardless of the intial guess.

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