Appendix A
Preliminaries

Abstract In this appendix we include some classical inequalities that will be needed in the book, and a short review of the basic theory of p-Laplace operators and some properties of their eigenvalues.

A.1 Some Basic Inequalities

We include here some inequalities for the sake of completeness. The proofs can be found in the classic book of Hardy et al. [60].

A.1.1 The Arithmetic–Geometric–Harmonic Mean Inequality

In several places we need the well-known arithmetic–geometric inequality

$$\sqrt{st} \leq \frac{s+t}{2}, \tag{1.1}$$

and we will use it mainly with $s = (b-c)$ and $t = (c-a)$.

This is only a small part of the arithmetic–geometric–harmonic mean inequality: given n positive numbers $\{a_n\}_{i=1}^n$, we have

$$n \cdot \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right)^{-1} \leq (a_1 \cdot \cdots \cdot a_n)^{\frac{1}{n}} \leq \frac{a_1 + \cdots + a_n}{n},$$

and equality holds only when all of the terms are equal, $a_1 = \cdots = a_n$.

Sometimes we will use the following variant of the arithmetic–geometric–harmonic mean inequality:

$$\frac{n^2}{a_1 + \cdots + a_n} \leq \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

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A.1.2 Minkowski's Inequality

There are different inequalities related to Minkowski. Let us mention here the following ones:

Minkowsi's inequality: given $f, g \in L^p(\Omega)$, $\Omega \subset R^N$, one has

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

A companion to Minkowski's inequality, Theorem 27 in [60]: given n positive numbers $\{a_i\}_{i=1}^n$, we have

$$(a_1+\cdots+a_n)^r > a_1^r+\cdots+a_n^r \qquad \text{for } r>1,$$

$$(a_1+\cdots+a_n)^r < a_1^r+\cdots+a_n^r \qquad \text{for } 0<r<1.$$

This inequality implies Theorem 199 in [60]: for positive functions f, g and $0<r<1$, we have

$$\int (f+g)^r \mathrm{d}x < \int f^r \mathrm{d}x + \int g^r \mathrm{d}x.$$

Minkowski's integral inequality, Theorem 202 in [60]: for $p \geq 1$,

$$\left(\int_A \left|\int_B F(x,t)\mathrm{d}t\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq \int_B \left(\int_A |F(x,t)\mathrm{d}t|^p \mathrm{d}x\right)^{\frac{1}{p}} \mathrm{d}t.$$

A.1.3 A Useful Lemma

The arithmetic–geometric–harmonic mean inequality together with Minkowski's inequalities implies the following result:

Lemma A.1. *Given n positive numbers* $\{a_i\}_{i=1}^n$, *we have*

$$\min\left\{\frac{1}{a_1^{p-1}}+\cdots+\frac{1}{a_n^{p-1}} \;:\; p\geq 1\right\} = n\cdot\left(\frac{n}{\sum_{i=1}^n a_i}\right)^{p-1}.$$

A.1.4 Hölder's Inequality

Given $f \in L^p(\Omega)$, $g \in L^{\frac{p}{p-1}}(\Omega)$, $\Omega \subset R^N$, we have

$$\int_\Omega fg\mathrm{d}x \leq \left(\int_\Omega |f|^p \mathrm{d}x\right)^{\frac{1}{p}} \cdot \left(\int_\Omega |g|^{\frac{p}{p-1}} \mathrm{d}x\right)^{\frac{p-1}{p}}.$$

A.1.5 Young's Inequality

Given $f \in L^p(\Omega)$, $g \in L^{\frac{p}{p-1}}(\Omega)$, $\Omega \subset R^N$, we have

$$\int_\Omega fg\mathrm{d}x \leq \frac{1}{p}\int_\Omega |f|^p \mathrm{d}x + \frac{p-1}{p}\int_\Omega |g|^{\frac{p}{p-1}}\mathrm{d}x.$$

A.1.6 Jensen's Inequality

Given $f \in L^1(\Omega)$, $\Omega \subset R^N$, and $\varphi : R \to R$ a convex function, we have

$$\varphi\left(\frac{1}{|\Omega|}\int_\Omega f(x)\mathrm{d}x\right) \leq \frac{1}{|\Omega|}\int_\Omega \varphi(f(x))\mathrm{d}x.$$

A.2 Sobolev Spaces and Related Inequalities

We recall the definition of the Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$, for $1 \leq p < \infty$ and $m \geq 1$. We refer the interested reader to [47] for details.

We say that $v \in L^1_{\mathrm{loc}}(\Omega)$ is the kth weak partial derivative of $u \in L^1_{\mathrm{loc}}(\Omega)$, $D^k u = v$, if

$$\int_\Omega u D^k \varphi \mathrm{d}x = (-1)^k \int_\Omega v\varphi \mathrm{d}x$$

for all test functions $\varphi \in C_0^\infty(\Omega)$, the space of C^∞ functions with compact support in Ω.

The Sobolev space $W^{m,p}(\Omega)$ is the set of functions $u \in L^1_{\mathrm{loc}}(\Omega)$ such that for each multi-index $k = (k_1, \ldots, k_N)$ with $0 \leq |k| \leq m$, $D^k u$ exists and belongs to $L^p(\Omega)$.

The norm of $u \in W^{m,p}(\Omega)$ for $1 \leq p < \infty$ is

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{0 \leq |k| \leq m} \int_\Omega |D^k u|^p \mathrm{d}x.$$

We denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_0^{m,p}(\Omega)$.

We will need the following classical inequalities for functions in Sobolev spaces.

Theorem A.1 (Sobolev's inequality). *Let $\Omega \subset R^N$ be a bounded open set, and $u \in W_0^{1,p}$ with $1 \leq p < N$. Then*

$$\int_\Omega |u|^q \mathrm{d}x \leq C_{q,p}\left(\int_\Omega |\nabla u|^p\right)^{\frac{q}{p}} \tag{2.2}$$

with $1 \leq q \leq p^* = pN/(N-p)$.

Theorem A.2 (Morrey's lemma). *If $p > n$, there exists a constant C such that*

$$\|u\|_{C^{0,\alpha}(R^N)} \leq C\|u\|_{W^{1,p}(R^N)}.$$

In particular, there exists a constant $C(N,p)$ such that for all $u \in W_0^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C(n,p)|x-y|^{\alpha} \, \|\nabla u\|_{L^p} \tag{2.3}$$

for all x, $y \in \overline{\Omega}$ and $\alpha = 1 - \frac{N}{p}$.

The proofs can be found, for example, in [47].

Theorem A.3 (Hardy's inequality). *Let $1 < p < N$, $u \in C_0^{\infty}(\Omega)$, and suppose that there exists a constant $\gamma > 0$ such that*

$$C_p((R^N \setminus \Omega) \cap \bar{B}(x,r), B(x,2r)) \geq \gamma C_p(\bar{B}(x,r), B(x,2r))$$

for every $x \in (R^N \setminus \Omega)$ and $r > 0$, where $C_p(K, B(x,2r))$ is the variational p-capacity

$$C_p(K,U) = \inf\left\{ \int_U |\nabla u|^p \mathrm{d}x : u \in C_0^{\infty}(U), u(x) \geq 1 \text{ for } x \in K \right\}.$$

Then there exists a constant C_h such that

$$\int_{\Omega} \frac{|u|^p}{d(x,\partial\Omega)^p} \mathrm{d}x \leq C_h \int_{\Omega} |\nabla u|^p \mathrm{d}x, \tag{2.4}$$

where $d(x,\partial\Omega)$ is the distance from $x \in \Omega$ to the boundary,

$$d(x,\partial\Omega) = \inf_{y \in \partial\Omega} |x-y|.$$

This theorem is due to Lewis [77]. Let us note that $C_{q,s}$ in Sobolev's inequality is a universal constant depending only on p, q, and N. However, C_h depends on the p-capacity of $R^N \setminus \Omega$, although for convex domains, we have $C_h = \left(\frac{p}{N-p}\right)^p$.

A.3 The *p*-Laplace Operator

In this section we review briefly the main results for the eigenvalue problem associated to the p-Laplace operator

$$-\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u),$$

where $1 < p < \infty$.

This is a quasilinear operator (also called a half-linear operator due to its homogeneity), which is singular if $p < 2$ and degenerate if $p > 2$. The eigenvalue problem is

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda w(x)|u|^{p-2}u, \qquad x \in \Omega \subset R^N,$$

and it was considered first by Browder [11–13], who proved the existence of a sequence of eigenvalues when $p \geq 2$. Later, several authors studied the eigenvalue problem, among them Amman and Fučík, (see [1, 52]), and then there was an explosion of research in the 1980s by Anane, de Thelin, Elbert, Lieb, Otani, Velin, to name only a few. For a more detailed survey, see [50]

A.3.1 The One-Dimensional Eigenvalue Problem

Let us consider the following eigenvalue problem for the p-Laplacian equation on (a,b):

$$-(|u'|^{p-2}u')' = \lambda w(x)|u|^{p-2}u, \tag{3.5}$$

with zero Dirichlet boundary conditions

$$u(a) = u(b) = 0. \tag{3.6}$$

Here, the weight $w(x) \in L^1(a,b)$ is a positive function, and $\lambda \in R$ is the eigenvalue parameter.

Although this problem can be analyzed in different ways (see [39]), we will use the fact that it has a variational structure, and Eq. (3.5) is the Euler–Lagrange equation of the functional

$$\phi(u) = \frac{1}{p}\int_a^b |u|^p \mathrm{d}x + \frac{1}{p}\int_a^b \lambda w(x)|u|^p \mathrm{d}x. \tag{3.7}$$

We will say that $\lambda \in R$ is an eigenvalue if there exists a nontrivial weak solution of problem (3.5), where by a weak solution we mean a critical point of the functional Eq. (3.7), a function $u \in W_0^{1,p}(a,b)$, and $\lambda \in R$ such that

$$\int_a^b |u'|^{p-2}u' \cdot v' \mathrm{d}x = \lambda \int_s^b w(x)|u|^{p-2}uv$$

for every test function $v \in W_0^{1,p}(a,b)$.

A.3.2 Constant-Coefficient Case

When w is a constant function, the eigenvalues and eigenfunctions can be computed explicitly (see [38, 57]). Let us denote by $\sin_p(x)$ the solution of the initial value problem

$$-(|u'|^{p-2}u')' = (p-1)|u|^{p-2}u,$$
$$u(0)=0, \qquad u'(0)=1,$$

and by direct integration, we can check that $\sin_p(\cdot)$ is defined implicitly as

$$x = \int_0^{\sin_p(x)} \frac{\mathrm{d}t}{\sqrt[p]{1-t^p}} = 2\frac{\pi/p}{\sin(\pi/p)}.$$

Clearly, when $p=2$, this is the definition of $\arcsin(\cdot)$.

Let us denote by $\hat{\pi}_p$ the first zero of $\sin_p(x)$, given by

$$\hat{\pi}_p = 2\int_0^1 \frac{\mathrm{d}t}{\sqrt[p]{1-t^p}}.$$

Remark A.1. Let us observe that there are alternative definitions of $\sin_p$ and π_p, depending on the presence of the factor $p-1$ in the equation; see [41], for example. It is convenient to introduce

$$\pi_p = \sqrt[p]{p-1}\,\hat{\pi}_p$$

to recover an expression similar to the one that holds for linear problems.

With these definitions, $\sin_p(x)$ and $\sin_p'(x)$ satisfy

$$|\sin_p(x)| \le 1, \quad |\sin_p'(x)| \le 1.$$

Moreover, they satisfy a Pythagorean-like identity

$$|\sin_p(x)|^p + |\sin_p'(x)|^p = 1. \tag{3.8}$$

We have the following characterization of the spectrum:

Theorem A.4 (Del Pino et al. [38]). *The Dirichlet eigenvalues $\lambda_k(p)$ and eigenfunctions $u_{p,k}$ of problems (3.5)–(3.6) are given by*

$$\lambda_k(p) = \left(\frac{\pi_p k}{b-a}\right)^p, \qquad u_{p,k}(x) = \sin_p\left(\frac{\hat{\pi}_p k x}{b-a}\right).$$

Let us observe that the kth eigenvalue is simple, and the associated eigenfunction $u_{p,k}$ has k nodal domains, that is, $u_{p,k}$ has $k+1$ simple zeros in $[a,b]$.

Remark A.2. Similar computations show that the Neumann eigenvalues $\{\mu_k(p)\}_{k\ge 0}$ corresponding to the boundary condition $u'(a)=u'(b)=0$ and the eigenvalues

$\{\nu_k(p)\}_{k\geq 1}$ corresponding to the mixed boundary condition $u(a) = u'(b) = 0$ or $u'(a) = u(b) = 0$ are given by

$$\mu_k = \left(\frac{\pi_p k}{b-a}\right)^p, \qquad \nu_k = \left(\frac{\pi_p k}{2(b-a)}\right)^p.$$

A.3.3 General Case

The eigenvalue problems (3.5)–(3.6) for general w need some heavy machinery. However, the following results also hold for-higher dimensional problems:

$$-\Delta_p u = \lambda w(x)|u|^{p-2}u, \qquad x \in \Omega \subset R^N.$$

For brevity we will focus only on one-dimensional problems, although the extension of the following results to the N-dimensional case is straightforward.

A.3.3.1 The Functional Setting

Let us define the functionals $\mathscr{A}, \mathscr{B} : W_0^{1,p}(a,b) \to R$:

$$\mathscr{A}(u) = \frac{1}{p}\int_a^b |u'|^p, \quad \mathscr{B}(u) = \frac{1}{p}\int_a^b w(x)|u|^p.$$

Let $A : W_0^{1,p}(a,b) \to W^{-1,p'}(a,b)$ be the Gâteaux derivative of $\mathscr{A}$,

$$A(u) = -(|u'|^{p-2}u')',$$

which is an odd potential operator, uniformly continuous on bounded sets, satisfying the following condition:

(S) If $u_j \rightharpoonup u$ weakly in $W_0^{1,p}(a,b)$ and $A(u_j) \to u^*$, then $u_j \to u$ strongly in $W_0^{1,p}(a,b)$

Let $B : W_0^{1,p}(a,b) \to W^{-1,p'}(a,b)$ be the odd potential operator

$$B(u) = w(x)|u|^{p-2}u,$$

which is strongly sequentially continuous, and let us observe that $\mathscr{B}(u) \neq 0$ implies that $B(u) \neq 0$.

A.3.3.2 Lyusternik–Schnirelmann Theory

Let us recall the definition of the Krasnoselskii genus gen(C), where C is a closed subset of $W_0^{1,p}(a,b)\setminus\{0\}$, and C is symmetric, that is, $C=-C$.

We say that $\mathrm{gen}(C)=n$ if n is the minimum integer such that there exists an odd continuous mapping $\varphi : C\to R^n\setminus\{0\}$.

Lyusternik and Schnirelmann proved that an even function $I : S^{N-1}\to R$ has at least N pairs of critical points. For homogeneous operators, their arguments can be applied to eigenvalue problems if we consider them as Lagrange multipliers. There are several generalizations of the Lyusternik.Schnirelmann result to the infinite-dimensional setting; see [26, 27, 68, 99].

Let us fix $m>0$, and define the level set of $\mathscr{A}$,

$$M_m=\{u\in W_0^{1,p}(a,b) : \mathscr{A}(u)=m\}.$$

Since M_m is a bounded set, the coercivity of A implies that there exists a positive constant ρ_m such that

$$\langle A(u),u\rangle \geq \rho_m$$

for every $u\in M_m$, and $\langle\cdot,\cdot\rangle$ is the duality product between $W_0^{1,p}(a,b)$ and $W^{-1,p'}(a,b)$.

Following Amann [1], under the previous hypotheses on the functionals and their associated operators, there exists a sequence of eigenvalues:

Theorem A.5. *Let $\mathscr{A}$, A, $\mathscr{B}$, B be defined as before, and let us define*

$$\beta_k=\sup_{C\in\mathscr{C}_k, C\subset M_m}\inf_{u\in C}\mathscr{B}(u),$$

where $\mathscr{C}_k$ is the class of compact symmetric subsets of the space $W_0^{1,p}(a,b)$ of genus greater than or equal to k.

If $\beta_k>0$, there exists an eigenfunction $u_k\in M_m$ with

$$\mathscr{B}(u)=\beta_k=m/\lambda_k.$$

Moreover, if

$$\gamma(\{w\in M_m : \mathscr{B}(u)\neq 0\})=\infty,$$

then there exist infinitely many eigenfunctions.

We will say that $\{\lambda_k\}_k$ is the set of *variational* eigenvalues. For problems (3.5)–(3.6), it is easy to show that they exhaust the spectrum; see [48]. For the Neumann problem the same result holds, and it was proved in [95]. However, for periodic problems there exist nonvariational eigenvalues; see [5]. For higher dimensions, it is an open problem to fully characterize the spectrum. This is the main difference between one-dimensional and N-dimensional problems, and perhaps the most important open problem in this area.

It is possible to work with an equivalent characterization of the eigenvalues by introducing a Rayleigh-type quotient as in the linear case,

$$\lambda_k = \inf_{C \in \mathscr{C}_k} \sup_{u \in C} \frac{\int_a^b |u'|^p}{\int_a^b w(x)|u|^p}. \tag{3.9}$$

The equivalence follows from the homogeneity of $\mathscr{B}(u)$. Indeed, for a given $C \subset W_0^{1,p}(a,b) \setminus \{0\}$, we have a set $\tilde{C}$ in M_m by taking the retraction

$$u \mapsto \frac{u}{\mathscr{A}^{1/p}(u)}.$$

Both sets have the same genus, and

$$\inf_{u \in C} \frac{\mathscr{B}(u)}{\mathscr{A}(u)} = \inf_{u \in \tilde{C}} \mathscr{B}(u).$$

See also [53, 103] for more details.

Finally, let us observe that w can be allowed to change sign. Assuming that

$$|\Omega^+(w)| = |\{x \in (a,b) : w(x) > 0\}|_1 > 0,$$

$$|\Omega^-(w)| = |\{x \in (a,b) : w(x) < 0\}|_1 > 0,$$

where $|A|$ denotes the measure of A, there exist a sequence of positive variational eigenvalues and another sequence of negative eigenvalues.

A.3.4 Resonant Systems

We will consider the system

$$\begin{aligned} -(|u'|^{p-2}u')' &= \lambda w(x)\alpha|u|^{\alpha-2}u|v|^\beta \qquad x \in (a,b), \\ -(|v'|^{q-2}v')' &= \lambda w(x)\beta|u|^\alpha|v|^{\beta-2}v \qquad x \in (a,b), \end{aligned} \tag{3.10}$$

with zero Dirichlet boundary conditions

$$u(a) = u(b) = v(a) = v(b) = 0,$$

where the weight $w \in L^1(a,b)$ is allowed to change sign. We assume that

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1 \quad \text{and} \quad 1 < q \le p < \infty, \tag{3.11}$$

a condition that gives some homogeneity, since the solutions are invariant under the rescaling $(u,v) \to (t^{\frac{1}{p}}u, t^{\frac{1}{q}}v)$. See Boccardo and De Figueiredo [8], and the book [117] for a recent survey of different aspects of the theory of resonant systems.

The system (3.10) corresponds to the Euler–Lagrange equations of the functional

$$\Phi(u,v) = \frac{1}{p}\int_a^b |u'|^p \mathrm{d}x + \frac{1}{q}\int_a^b |v'|^q \mathrm{d}x - \lambda \int_a^b w(x)|u|^\alpha |v|^\beta \mathrm{d}x, \tag{3.12}$$

and we will say that $(u,v) \in W_0^{1,p}(a,b) \times W_0^{1,q}(a,b)$ is a solution (in the weak sense) if it is a critical point of the functional Eq. (3.12) satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \mathrm{d}x + \int_\Omega |\nabla v|^{q-2}\nabla v \cdot \nabla\psi \mathrm{d}x$$
$$= \lambda \int_\Omega r(x)\alpha|u|^{\alpha-2}u\phi|v|^\beta \mathrm{d}x + \lambda \int_\Omega r(x)\beta|u|^\alpha|v|^{\beta-2}v\psi \mathrm{d}x$$

for every test pair $(\phi,\psi) \in W_0^{1,p}(a,b) \times W_0^{1,q}(a,b)$.

We will say that $\lambda \in R$ is an eigenvalue if there exists a nontrivial solution (u,v), and the eigenvalues are obtained as before:

$$\lambda_k = \inf_{C\in\mathscr{C}_k} \sup_{(u,v)\in C} \frac{\frac{1}{p}\int_a^b |u'|^p + \frac{1}{q}\int_a^b |v'|^q}{\int_a^b r(x)|u|^\alpha|v|^\beta},$$

with $C \subset W_0^{1,p}(a,b) \times W_0^{1,q}(a,b)$.

A.4 Some Properties of the Eigenvalues and Eigenfunctions

For the weighted one-dimensional p-Laplacian with Dirichlet boundary conditions we have the following result:

Theorem A.6. *Let $\{\lambda_k\}_k$ be the eigenvalues of problems (3.5)–(3.6). Then:*

1. *Every eigenfunction corresponding to the kth eigenvalue λ_k has exactly $k+1$ zeros in $[a,b]$.*
2. *For every k, λ_k is simple.*
3. *If $\lambda_k < \lambda < \lambda_{k+1}$, the only solution of problems (3.5)–(3.6) is $u \equiv 0$.*

Now we can state the classical Sturmian theorems for the p-Laplacian equation. For a proof, see [39].

Theorem A.7. *Let $w(x)$ be a continuous and positive function, and let u be a solution of*

$$-(|u'|^{p-2}u')' = w(x)|u|^{p-2}u.$$

Then the zeros of u and u' alternate.

Moreover, if v is another solution, the zeros of u and v alternate.

Theorem A.8. *Let us consider the following problems for the p-Laplacian:*

$$-(|u'|^{p-2}u')' = w(x)|u|^{p-2}u, \tag{4.13}$$

$$-(|v'|^{p-2}v')' = W(x)|v|^{p-2}v, \tag{4.14}$$

where $w(x) \le W(x)$ are positive functions. Then every solution v of Eq. (4.14) has at least one zero between two zeros of a solution u of Eq. (4.13).

Moreover, the eigenvalues of the problems

$$-(|u'|^{p-2}u')' = \lambda w(x)|u|^{p-2}u, \qquad u(a) = u(b) = 0, \tag{4.15}$$

$$-(|v'|^{p-2}v')' = \Lambda W(x)|v|^{p-2}v, \qquad v(a) = v(b) = 0, \tag{4.16}$$

satisfy

$$\Lambda_k(W) \le \lambda_k(w).$$

Let us note that the last part of Theorem 4.15 follows immediately from the variational characterization of eigenvalues given by Eq. (3.9). As a corollary, we have the following result.

Theorem A.9. *Let us assume that $0 \le w \le M$ in $(0,L)$, and let λ_1 be the first eigenvalue of*

$$-(|u'|^{p-2}u')' = \lambda w(x)|u|^{p-2}u, \qquad u(0) = u(L) = 0.$$

Then

$$\frac{\pi_p^p}{ML^p} \le \lambda_1.$$

The result follows from Theorem 4.15, with $W(x) \equiv M$, and the explicit formula for the constant coefficient case in Theorem A.4

Finally, as k goes to infinity, the asymptotic behavior of the eigenvalues is given by

$$\lambda_k \sim \frac{\pi_p^p k^p}{\left(\int_a^b w^{1/p}(x)\mathrm{d}x\right)^p};$$

see [48] for a proof. This expression agrees with the corresponding one for the linear case; it is enough to replace $p = 2$ to get the classical one. We will use this fact at the end of Chap. 4, and it provides a motivation for Theorem C.

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