Answers to Problems

Chapter 2

2.9. Yes, it can. Consider, for example, probability space $\Omega = [0,1]$ with Borel sigma-algebra and Lebesgue measure, and let $\xi(\omega) = \frac{1}{\omega}\mathbb{I}\{\omega < 1/2\}$ and $\eta(\omega) = \xi(1-\omega)$. Then $\min(\xi,\eta) = 0$.

2.14. Exponential distribution, for example.

2.18. (i) $\alpha > 2$.

2.19. $n \ge 3$. Yes, the product is long-tailed.

2.20. $\beta < n$.

2.21. (ii) The same result holds for any distribution with negative mean.

2.26. (i) *Solution*. Denote by τ the first return time to the state 1, that is, $\tau = \min(n \ge 1 : X_n = 1)$ given $X_0 = 1$. Then $\tau = n$ if and only if $X_1 = n$, so that $\mathbb{P}\{\tau = n\} = F\{n-1\}$ and $\mathbb{E}\tau = \sum_{n\ge1} nF\{n-1\}$. Therefore, the Markov chain is positive recurrent if and only if F has finite mean.

Let $\{\pi_i\}_{i\ge1}$ be the stationary distribution. For any $i \ge 1$,

$$\begin{aligned}\pi_i &= \pi_{i+1} + \pi_1 F\{i-1\}\\ &= \pi_{i+2} + \pi_1 F\{i\} + \pi_1 F\{i-1\}\\ &= \dots\\ &= \pi_1 F[i-1,\infty).\end{aligned}$$

Then

$$\begin{aligned}\sum_{i\ge1}\pi_i &= \pi_1 \sum_{i\ge1}\sum_{j\ge i-1} F\{j\}\\ &= \pi_1 \sum_{j\ge0}(j+1)F\{j\},\end{aligned}$$

which implies that the invariant distribution is given by the residual distribution F_r.

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Chapter 3

3.8. Let the ξ's have exponential distribution with parameter λ.

(i) $n_1 e^{-\lambda x^{1/\beta_1}}$ where $\beta_1 := \max \alpha_i$ and n_1 is the number of α_i that are equal to β_1;
(ii) $n_2 \lambda x^{1/\beta_2}$ where $\beta_2 := \min \alpha_i$ and n_2 is the number of α_i that are equal to β_2;
(iii) $n_1 e^{-\lambda x^{1/\beta_1}}$ if $\beta_2 > 1$ and $n_2 \lambda x^{1/\beta_2}$ if $\beta_2 < 0$.

3.9. $\alpha < 1 - \beta$.
3.10. For all positive values.
3.12. $\mathbb{P}\{S_n > x\} \le n\mathbb{P}\{\xi_1 > x - 1\}$.
3.13. $\lambda t \overline{F}(x)$ where λ is the intensity and F is the jump distribution.
3.15. Proportional to $\overline{F}(x)$ with the following coefficients:

(i) $\mathbb{P}\{X_0 = 1\}(2p_{11} + p_{12}(1+c)) + \mathbb{P}\{X_0 = 2\}(p_{22}2c + p_{21}(1+c))$;
(ii) $\mathbb{P}\{X_0 = 1\}\Big[1 + p_{11} + \sum_{j=0}^{\infty} p_{12}p_{21}p_{22}^{j}((j+1)c+1)\Big] + \mathbb{P}\{X_0 = 2\}\Big[c + p_{22}c + \sum_{j=0}^{\infty} p_{21}p_{11}^{j}p_{12}(j+1+c)\Big]$;
(iii) $\mathbb{P}\{X_0 = 1\}\Big[1 + kp_{11} + k\sum_{j=0}^{\infty} p_{12}p_{22}^{j}p_{21}((j+1)c+1)\Big] + \mathbb{P}\{X_0 = 2\}\Big[c + kp_{22}c + k\sum_{j=0}^{\infty} p_{21}p_{11}^{j}p_{12}(j+1+c)\Big]$.

3.16. (i) $H(x) = x$; (ii) $H(x) = x/\log x$; (iii) $H(x) = x^{1-\beta}$; (iv) $H(x) = \log x$.
3.19. (i) $\frac{1}{x\log x}$ and $\frac{2}{x\log x}$; (ii) $\frac{e^{-\sqrt{x}}}{\sqrt{x}\log x}$ and $\frac{2e^{-\sqrt{x}}}{\sqrt{x}\log x}$; (iii) $\frac{c_1}{\log x}$ and $\frac{c_2}{\log x}$ where $c_2 < 2c_1$.
Solution. The tail distribution function of ξ_i is equal to

$$\overline{F}(x) = \int_0^1 e^{-x^y} dy = \frac{1}{\log x}\int_1^x \frac{e^{-u}}{u} du \sim \frac{c_1}{\log x} \quad \text{as } x \to \infty,$$

where

$$c_1 := \int_1^\infty \frac{e^{-u}}{u} du.$$

Further,

$$\mathbb{P}\{\xi_1 + \xi_2 > x\} = \mathbb{P}\{\xi_1 > x\} + \mathbb{P}\{\xi_2 > x\} - \mathbb{P}\{\xi_1 > x, \xi_2 > x\} + \mathbb{P}\{\xi_1 \le x, \xi_2 \le x, \xi_1 + \xi_2 > x\}.$$

By the conditional independence,

$$\mathbb{P}\{\xi_1 > x, \xi_2 > x\} = \mathbb{E}\mathbb{P}\{\xi_1 > x, \xi_2 > x \mid \eta\} = \int_0^1 e^{-2x^t} dt \sim \frac{1}{\log x}\int_2^\infty \frac{e^{-u}}{u} du \quad \text{as } x \to \infty.$$

It is left to prove that

$$\mathbb{P}\{\xi_1 \le x, \xi_2 \le x, \xi_1 + \xi_2 > x\} = o(\overline{F}(x)) \quad \text{as } x \to \infty. \tag{5.60}$$

Indeed, this probability is equal to

$$\begin{aligned}
&\int_0^1 dt \int_0^x tu^{t-1} e^{-u^t} du \int_{x-u}^x tv^{t-1} e^{-v^t} dv \\
&= \int_0^1 x^{2t} dt \int_0^1 \int_{1-z}^1 t^2 y^{t-1} z^{t-1} e^{-x^t(y^t+z^t)} dydz.
\end{aligned}$$

Since $y^t + z^t \geq 1$, the latter term is not greater than

$$\int_0^1 x^{2t} e^{-x^t} dt \int_0^1 \int_{1-z}^1 t^2 y^{t-1} z^{t-1} dydz = \int_0^1 x^{2t} e^{-x^t} J(t) dt,$$

where $J(t) = \mathbb{P}\{Y_1^{1/t} + Y_2^{1/t} > 1\}$ and Y_1, Y_2 are independent random variables both uniformly distributed in the interval $[0,1]$. As $t \downarrow 0$, $J(t) \downarrow 0$. Hence, (5.60) follows from the estimates, for any $\varepsilon > 0$,

$$\begin{aligned}
\int_0^1 x^{2t} e^{-x^t} J(t) dt &= \Big(\int_0^\varepsilon + \int_\varepsilon^1\Big) x^{2t} e^{-x^t} J(t) dt \\
&\leq J(\varepsilon) \int_0^\varepsilon x^{2t} e^{-x^t} dt + \int_\varepsilon^1 x^{2t} e^{-x^t} dt \\
&\leq \frac{J(\varepsilon)}{\log x} \int_1^\infty t e^{-t} dt + x^2 e^{-x^\varepsilon}.
\end{aligned}$$

Thus, the answer is:

$$\mathbb{P}\{\xi_1 + \xi_2 > x\} \sim \frac{c_2}{\log x} \quad \text{as } x \to \infty,$$

where

$$c_2 := \Big(2\int_1^\infty - \int_2^\infty\Big) \frac{e^{-u}}{u} du.$$

3.20. Proportional to $\overline{F}(x)$ with the following coefficient:

(i) $2\mathbb{E}e^{-\eta_1}$; (ii) $(1 + \mathbb{E}e^{-\eta_1})\mathbb{E}e^{-\eta_2}$.

3.22. All the integrated tail distributions $F_{i,I}$ are tail-proportional to a reference subexponential distribution.

Chapter 4

4.6. Yes, it is.

4.11. Let the ξ's have exponential distribution with parameter λ.

(i) $n_1 \lambda x^{1/\beta_1 - 1} e^{-\lambda x^{1/\beta_1}}$ where $\beta_1 := \max \alpha_i$ and n_1 is the number of α_i that are equal to β_1;

(ii) $n_2\lambda x^{1/\beta_2}/|\beta_2|$ where $\beta_2 := \min \alpha_i$ and n_2 is the number of α_i that are equal to β_2;

(iii) $n_1\lambda x^{1/\beta_1 - 1}e^{-\lambda x^{1/\beta_1}}$ if $\beta_2 > 1$ and $n_2\lambda x^{1/\beta_2}/|\beta_2|$ if $\beta_2 < 0$.

4.12. $\sum_n \frac{n\mathbb{P}\{\tau=n\}}{\pi(n+x^2)}$.

4.14. $\lambda t f(x)$.

4.19.

(i) $p_{i,i+1} = p - c/i + o(1/i)$ and $p_{i,i-1} = p + c/i + o(1/i)$ as $i \to \infty$ where $p < 1/2$ and $c > p/2$;

(ii) $p_{i,i+1} = p - c/i^\beta + o(1/i^\beta)$ and $p_{i,i-1} = p + c/i^\beta + o(1/i^\beta)$ as $i \to \infty$ where $p < 1/2$, $c > 0$ and $0 < \beta < 1$.

4.20. *Solution.* It follows from the solution to Problem 2.26 that the invariant distribution for the Markov chain coincides with the residual distribution F_r. Then the result follows from Theorem 4.32 with $T = 1$.

4.21. All the distributions F_i are tail equivalent to some distribution from $\mathcal{S}^*$.

4.22. $\frac{(1-p)^2}{(2\pi)^{d/2}\sqrt{\det B}}$.

Chapter 5

5.3. Let the ξ's have exponential distribution with parameter λ.

(i) $ne^{-\lambda x^{1/\alpha}}$; (ii) $n\lambda x^{1/\alpha}$.

5.4. $\frac{\alpha}{\varepsilon(1-\alpha)}x^{1-1/\alpha}$.

5.5. It is proportional to the tail of the jump distribution with coefficient λT where λ is the intensity of the jumps.

5.6. It is proportional to the tail $\overline{F_I}$ with coefficient λ/a where λ is the intensity of the jumps.

5.8. $\overline{F_I}(x)/a$. If the left tail of the distribution of η is much heavier than the right tail, then the asymptotic tail behaviour of M will be determined by the left tail of η.

5.9. $\mathbb{P}\{M > x\} \sim c\overline{F_I}(x/c)/a$ where $c := c_0 + \ldots + c_k$.

5.12. *Solution.* We observe that

$$\mathbb{P}\{X_n > x \mid X_0 = 1\} = \sum_{k=1}^{n} \mathbb{P}\{X_{n-k} = 1, X_{n-k+1} \ge 2, \ldots, X_{n-1} \ge 2, X_n > x \mid X_0 = 1\}.$$

Then, by the Markov property,

$$\begin{aligned}\mathbb{P}\{X_n > x \mid X_0 = 1\} &= \sum_{k=1}^{n} \mathbb{P}\{X_{n-k} = 1 \mid X_0 = 1\} \\ &\qquad \times \mathbb{P}\{X_k > x, X_{k-1} \ge 2, \ldots, X_1 \ge 2 \mid X_0 = 1\} \\ &= \sum_{k=1}^{n} \mathbb{P}\{X_{n-k} = 1 \mid X_0 = 1\} F[x+k, \infty).\end{aligned}$$

By the ergodic theorem for Markov chains,

$$\mathbb{P}\{X_n = 1 \mid X_0 = 1\} \to 1/\mathbb{E}\{\tau \mid X_0 = 1\} \quad \text{as } n \to \infty,$$

where $\tau = \min\{n \geq 1 : X_n = 1\}$. Taking also into account that the distribution F is long-tailed, we deduce that, as $n, x \to \infty$,

$$\begin{aligned}\mathbb{P}\{X_n > x \mid X_0 = 1\} &\sim \frac{1}{\mathbb{E}\{\tau \mid X_0 = 1\}} \sum_{k=1}^{n} F[x+k, \infty) \\ &\sim \frac{1}{\mathbb{E}\{\tau \mid X_0 = 1\}} \int_x^{x+n} \overline{F}(y)dy.\end{aligned}$$

Since $\mathbb{E}\{\tau \mid X_0 = 1\} = \mathbb{E}\{X_1 \mid X_0 = 1\}$, the result follows.

5.13. The answers differ for even and odd valued of n. If n is even number, $n = 2k$, then

$$\mathbb{P}\{\max_{0 \leq i \leq n} S_i > x\} \sim \frac{(c^{(1)} + c^{(2)})}{|a^{(1)} + a^{(2)}|} \int_0^{k|a^{(1)} + a^{(2)}|} \overline{F}(x+y)dy$$

as $x \to \infty$ uniformly in $k \geq 1$. If n is odd, $n = 2k+1$, then

$$\mathbb{P}\{\max_{0 \leq i \leq n} S_i > x\} \sim \frac{(c^{(1)} + c^{(2)})}{|a^{(1)} + a^{(2)}|} \int_0^{k|a^{(1)} + a^{(2)}|} \overline{F}(x+y)dy + c^{(2)}\overline{F}(x).$$

5.14. $\frac{c}{a} \int_0^{n|a|} \overline{F}(x+y)dy$ as $n, x \to \infty$.

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