Appendix A
Taylor Series

A.1 Single Variable

(a) For x near a:

$$f(x) = f(a)+(x-a)f'(a)+\frac{1}{2}(x-a)^2f''(a)+\cdots+\frac{1}{n!}(x-a)^nf^{(n)}(a)+\cdots .$$

(b) For h near 0:

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \cdots + \frac{1}{n!}h^nf^{(n)}(x) + \cdots .$$

A.2 Two Variables

(a) For h and k near 0:

$$\begin{aligned} f(x+h,t+k) =& f(x,t) + hf_x(x,t) + kf_t(x,t) \\ &+ \frac{1}{2}h^2f_{xx}(x,t) + hkf_{xt}(x,t) + \frac{1}{2}k^2f_{tt}(x,t) + \cdots . \end{aligned}$$

(b) For x near a and t near b:

$$\begin{aligned} f(x,t) =& f(a,b) + (x-a)f_x(a,b) + (t-b)f_t(a,b) \\ &+ \frac{1}{2}(x-a)^2f_{xx}(a,b) + (x-a)(t-b)f_{xt}(a,b) + \frac{1}{2}(t-b)^2f_{tt}(a,b) \\ &+ \cdots . \end{aligned}$$

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(c) For h and k near 0:

$$f(x+h,t+k) = f(x,t) + Df(x,t) + \frac{1}{2}D^2 f(x,t) + \cdots + \frac{1}{n!}D^n f(x,t) + \cdots ,$$

where

$$D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t} .$$

A.3 Multivariable

(a) For $\mathbf{h}$ near $\mathbf{0}$:

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + Df(\mathbf{x}) + \frac{1}{2}D^2 f(\mathbf{x}) + \cdots + \frac{1}{n!}D^n f(\mathbf{x}) + \cdots ,$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{h} = (h_1, h_2, \ldots, h_k)$, and

$$\begin{aligned} D &= \mathbf{h} \cdot \nabla \\ &= h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \cdots + h_k \frac{\partial}{\partial x_k} . \end{aligned}$$

(b) For $\mathbf{x}$ near $\mathbf{a}$:

$$f(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x}-\mathbf{a}) \cdot \nabla f(\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^{\mathrm{T}} \mathbf{H}_a (\mathbf{x}-\mathbf{a}) + \cdots ,$$

where $\mathbf{H}_a = \mathbf{H}(\mathbf{a})$, and $\mathbf{H}(\mathbf{x})$ is the Hessian defined as

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_k} & \frac{\partial^2 f}{\partial x_2 \partial x_k} & \cdots & \frac{\partial^2 f}{\partial x_k^2} \end{pmatrix} .$$

A.4 Useful Examples for x Near Zero

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{6}x^3 f'''(0) + \cdots .$$

A.5 Power Functions

$$
\begin{aligned}
(a+x)^\gamma &= a^\gamma + \gamma x a^{\gamma-1} + \frac{1}{2}\gamma(\gamma-1)x^2 a^{\gamma-2} + \frac{1}{6}\gamma(\gamma-1)(\gamma-2)x^3 a^{\gamma-3} + \cdots\,,\\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots\,,\\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots\,,\\
\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots\,,\\
\frac{1}{\sqrt{1-x}} &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots\,.
\end{aligned}
$$

A.6 Trig Functions

$$
\begin{aligned}
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\,,\\
\arcsin(x) &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots\,,\\
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots\,,\\
\arccos(x) &= \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 + \cdots\,,\\
\tan(x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\,,\\
\arctan(x) &= x - \frac{1}{3}x^3 + \frac{1}{40}x^5 + \cdots\,,\\
\cot(x) &= \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 + \cdots\,,\\
\mathrm{arccot}(x) &= \frac{\pi}{2} - x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \cdots\,,\\
\sin(a+x) &= \sin(a) + x\cos(a) - \frac{1}{2}x^2\sin(a) + \cdots\,,\\
\cos(a+x) &= \cos(a) - x\sin(a) - \frac{1}{2}x^2\cos(a) + \cdots\,,\\
\tan(a+x) &= \tan(a) + x\sec^2(a) + x^2\tan(a)\sec^2(a).
\end{aligned}
$$

A.7 Exponential and Log Functions

$$\begin{aligned}
\mathrm{e}^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\,,\\
a^x &= \mathrm{e}^{x\ln(a)} = 1 + x\ln(a) + \frac{1}{2}[x\ln(a)]^2 + \frac{1}{6}[x\ln(a)]^3 + \cdots\,,\\
\ln(a+x) &= \ln(a) + \frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 + \cdots\,.
\end{aligned}$$

A.8 Hyperbolic Functions

$$\begin{aligned}
\sinh(x) &= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\,,\\
\operatorname{arcsinh}(x) &= x - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots\,,\\
\cosh(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\,,\\
\operatorname{arccosh}(x) &= \sqrt{2x}\left(1 - \frac{1}{12}x + \frac{3}{160}x^2 + \cdots\right),\\
\tanh(x) &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\,,\\
\operatorname{arctanh}(x) &= x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\,.
\end{aligned}$$

Appendix B
Solution and Properties of Transition Layer Equations

B.1 Airy Functions

This section concerns the solutions of Airy's equation. A more extensive presentation can be found in Vallée and Soares (2010), Abramowitz and Stegun (1972), and Olver et al. (2010).

B.1.1 Differential Equation

$$y'' = xy \quad \text{for} \quad -\infty < x < \infty.$$

B.1.2 General Solution

$$y(x) = \alpha_0 \mathrm{Ai}(x) + \beta_0 \mathrm{Bi}(x),$$

where

$$\begin{aligned}\mathrm{Ai}(x) &\equiv \frac{1}{3^{2/3}\pi}\sum_{k=0}^{\infty}\frac{1}{k!}\,\Gamma\left(\frac{k+1}{3}\right)\sin\left[\frac{2\pi}{3}(k+1)\right]\left(3^{1/3}x\right)^k \\ &= \mathrm{Ai}(0)\left(1+\frac{1}{6}x^3+\cdots\right)+\mathrm{Ai}'(0)\left(x+\frac{1}{12}x^4+\cdots\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Bi}(x) &\equiv \mathrm{e}^{\pi \mathrm{i}/6}\mathrm{Ai}\left(x\mathrm{e}^{2\pi\mathrm{i}/3}\right)+\mathrm{e}^{-\pi\mathrm{i}/6}\mathrm{Ai}\left(x\mathrm{e}^{-2\pi\mathrm{i}/3}\right) \\ &= \mathrm{Bi}(0)\left(1+\frac{1}{6}x^3+\cdots\right)+\mathrm{Bi}'(0)\left(x+\frac{1}{12}x^4+\cdots\right).\end{aligned}$$

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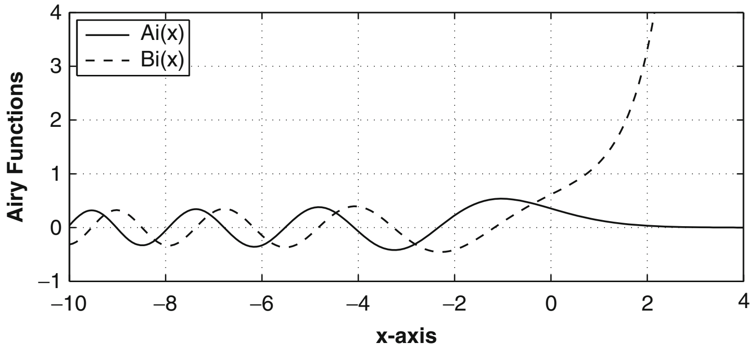

Figure B.1 Plot of the two Airy functions used in the development of the transition layer

B.1.3 Particular Values

$$\mathrm{Ai}(0) = \frac{\Gamma\left(\frac{1}{3}\right)}{2\pi 3^{1/6}}\,, \qquad \mathrm{Ai}'(0) = -\frac{3^{1/6}\Gamma\left(\frac{2}{3}\right)}{2\pi}\,,$$
$$\mathrm{Bi}(0) = \sqrt{3}\mathrm{Ai}(0)\,, \qquad \mathrm{Bi}'(0) = -\sqrt{3}\mathrm{Ai}'(0)\,,$$
$$\int_0^\infty \mathrm{Ai}(x)\mathrm{d}x = \frac{1}{3}\,, \quad \int_{-\infty}^0 \mathrm{Ai}(x)\mathrm{d}x = \frac{2}{3}\,, \quad \int_{-\infty}^0 \mathrm{Bi}(x)\mathrm{d}x = 0,$$
$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}\,.$$

B.1.4 Asymptotic Approximations

Setting $\zeta = \frac{2}{3}|x|^{3/2}$ and $\eta = \frac{5}{72\zeta}$, then

$$\mathrm{Ai}(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}|x|^{1/4}}\left[\cos\Big(\zeta - \dfrac{\pi}{4}\Big) + \eta(x)\sin\Big(\zeta - \dfrac{\pi}{4}\Big)\right] & \text{if } x \to -\infty, \\ \dfrac{1}{2\sqrt{\pi}x^{1/4}}\,\mathrm{e}^{-\zeta}\left[1 - \eta(x)\right] & \text{if } x \to +\infty, \end{cases}$$

and

$$\mathrm{Bi}(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}|x|^{1/4}}\left[\cos\Big(\zeta + \dfrac{\pi}{4}\Big) + \eta(x)\sin\Big(\zeta + \dfrac{\pi}{4}\Big)\right] & \text{if } x \to -\infty, \\ \dfrac{1}{\sqrt{\pi}x^{1/4}}\,\mathrm{e}^{\zeta}\left[1 + \eta(x)\right] & \text{if } x \to +\infty. \end{cases}$$

Also, setting $\nu = \frac{7}{72\zeta}$, then

$$\mathrm{Ai}'(x) \sim \begin{cases} -\frac{1}{\sqrt{\pi}}\,|x|^{1/4}\left[\cos\!\left(\zeta+\frac{\pi}{4}\right)-\nu(x)\sin\!\left(\zeta+\frac{\pi}{4}\right)\right] & \text{if } x\to-\infty,\\ -\frac{1}{2\sqrt{\pi}}\,x^{1/4}\mathrm{e}^{-\zeta}\left[1+\nu(x)\right] & \text{if } x\to+\infty,\end{cases}$$

and

$$\mathrm{Bi}'(x) \sim \begin{cases} -\frac{1}{\sqrt{\pi}}\,|x|^{1/4}\left[\cos\!\left(\zeta-\frac{\pi}{4}\right)-\nu(x)\sin\!\left(\zeta-\frac{\pi}{4}\right)\right] & \text{if } x\to-\infty,\\ -\frac{1}{2\sqrt{\pi}}\,x^{1/4}\mathrm{e}^{\zeta}\left[1-\nu(x)\right] & \text{if } x\to+\infty.\end{cases}$$

B.1.5 Connection with Bessel Functions

Setting $\zeta = \frac{2}{3}|x|^{3/2}$, then

$$\mathrm{Ai}(x) = \begin{cases} \sqrt{\frac{1}{3}|x|}\left[J_{\frac{1}{3}}(\zeta)+J_{-\frac{1}{3}}(\zeta)\right] & \text{if } x\le 0,\\ \sqrt{\frac{1}{3}x}\left[I_{-\frac{1}{3}}(\zeta)-I_{\frac{1}{3}}(\zeta)\right] & \text{if } x\ge 0,\end{cases}$$

and

$$\mathrm{Bi}(x) = \begin{cases} \sqrt{\frac{1}{3}|x|}\left[-J_{\frac{1}{3}}(\zeta)+J_{-\frac{1}{3}}(\zeta)\right] & \text{if } x\le 0,\\ \sqrt{\frac{1}{3}x}\left[I_{-\frac{1}{3}}(\zeta)+I_{\frac{1}{3}}(\zeta)\right] & \text{if } x\ge 0.\end{cases}$$

B.2 Kummer's Function

This section concerns the properties of the solutions of a differential equation that arises frequently when solving turning-point problems. It is related to the hypergeometric equation, and much of the material presented here is adopted from Slater (1960), Oldham et al. (2009), and Olver et al. (2010).

B.2.1 Differential Equation

$$y'' + \alpha x y' + \beta y = 0 \quad \text{for} \quad -\infty < x < \infty,$$

where α and β are nonzero constants.

B.2.2 General Solution

$$y(x) = \alpha_0 M\left(\frac{\beta}{2\alpha}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) + \beta_0 x M\left(\frac{\alpha+\beta}{2\alpha}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right),$$

where $M(a, b, z)$ is a confluent hypergeometric function and is known as Kummer's function [it is also denoted by ${}_1F_1(a, b, z)$]. Note $M(a, b, -\frac{1}{2}\alpha x^2)$ is an even function of x, so the foregoing solution consists of the sum of an even and an odd function. These functions are plotted for particular values of the coefficients in Fig. B.2.

The series definition of Kummer's function is

$$\begin{aligned} M(a, b, z) &= \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{1}{k!} z^k \\ &= 1 + \frac{a}{b}\, z + \frac{a(a+1)}{2b(b+1)}\, z^2 + \cdots , \end{aligned}$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ and $(a)_0 = 1$. This series is absolutely convergent for all values of a, b, and z except for $b = 0, -1, -2, -3, \ldots$. The latter values are assumed not to occur in the formulas that follow.

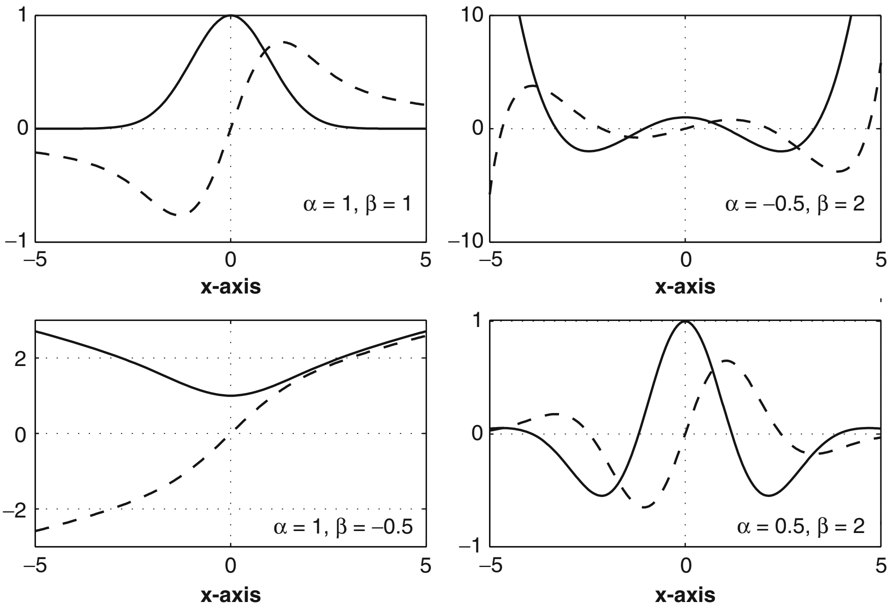

Figure B.2 Plots of $y_1 = M(\frac{\beta}{2\alpha}, \frac{1}{2}, -\frac{1}{2}\alpha x^2)$ (*solid curves*) and $y_2 = xM(\frac{\alpha+\beta}{2\alpha}, \frac{3}{2}, -\frac{1}{2}\alpha x^2)$ (*dashed curves*) for various values of α and β

B.2.3 Particular Values

$$M(a,b,0) = 1, \quad \partial_z M(a,b,0) = \frac{a}{b},$$
$$M(0,b,z) = 1, \quad M(a,a,z) = \mathrm{e}^z.$$

B.2.4 Useful Formulas

The following formulas are useful for deriving some of the basic properties, and special cases, of Kummer's function:

$$M(a,b,z) = \mathrm{e}^z M(b-a,b,-z),$$

$$M(a+1,b,z) = \frac{2a-b+z}{a} M(a,b,z) + \frac{b-a}{a} M(a-1,b,z),$$

$$M(a,b+1,z) = \frac{b(b-1+z)}{(b-a)z} M(a,b,z) - \frac{b(b-1)}{(b-a)z} M(a,b-1,z),$$

$$\frac{\mathrm{d}}{\mathrm{d}z} M(a,b,z) = \frac{a}{b} M(a+1,b+1,z).$$

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 \mathrm{e}^{zt} t^{a-1} (1-t)^{b-a-1} \mathrm{d}t, \text{ for } 0 < a < b$$

B.2.5 Special Cases

If $\alpha = \beta$, then

$$M\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = \mathrm{e}^{-\alpha x^2/2}$$

and

$$M\left(1, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = \frac{1}{x} \int_0^x \mathrm{e}^{\alpha(s^2-x^2)/2}\, \mathrm{d}s.$$

If $\alpha = -\beta$, then

$$M\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = \mathrm{e}^{-\alpha x^2/2} + \alpha x \int_0^x \mathrm{e}^{-\alpha r^2/2}\, dr$$

and

$$M\left(0, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = 1.$$

If $\beta = 2\alpha$, then

$$M\left(1, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = 1 - \alpha x \int_0^x e^{\alpha(s^2 - x^2)/2}\, ds$$

and

$$M\left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = e^{-\alpha x^2/2}.$$

If $\alpha = 2\beta$, then

$$M\left(\frac{1}{4}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = \Gamma\left(\frac{3}{4}\right)\left(-\frac{1}{8}\alpha x^2\right)^{1/4} e^{-\alpha x^2/4} I_{-1/4}\left(-\frac{1}{4}\alpha x^2\right)$$

and

$$M\left(\frac{3}{4}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = \Gamma\left(\frac{5}{4}\right)\left(-\frac{1}{8}\alpha x^2\right)^{-1/4} e^{-\alpha x^2/4} I_{1/4}\left(-\frac{1}{4}\alpha x^2\right).$$

B.2.6 Polynomials

When the first argument of $M(a, b, z)$ is a nonpositive integer, the function reduces to an expression involving a Laguerre polynomial. In particular, if $n = 0, 1, 2, 3, \ldots$, then

$$M\left(-n, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = \frac{\sqrt{\pi}\, n!}{2\Gamma\left(n + \frac{3}{2}\right)} \sum_{k=0}^{n} \binom{n + \frac{1}{2}}{n - k} \frac{1}{k!} \left(\frac{1}{2}\alpha x^2\right)^k$$

and

$$M\left(-n, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = \frac{\sqrt{\pi}\, n!}{\Gamma\left(n + \frac{1}{2}\right)} \sum_{k=0}^{n} \binom{n - \frac{1}{2}}{n - k} \frac{1}{k!} \left(\frac{1}{2}\alpha x^2\right)^k.$$

B.2.7 Asymptotic Approximations

The asymptotic expansions for large $|x|$ depend on the sign of α and also on whether or not the first argument of the function is a nonpositive integer (see section *Polynomials*).

1. For $x^2 \to \infty$,

$$M\left(\frac{\alpha+\beta}{2\alpha}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) \sim \begin{cases} \dfrac{\sqrt{\pi}}{2\Gamma\left(\frac{\alpha+\beta}{2\alpha}\right)}\left(-\dfrac{1}{2}\alpha x^2\right)^{\frac{\beta-2\alpha}{2\alpha}} \mathrm{e}^{-\frac{1}{2}\alpha x^2} & \text{if } \alpha < 0, \\ \dfrac{\sqrt{\pi}}{2\Gamma\left(\frac{2\alpha-\beta}{2\alpha}\right)}\left(\dfrac{1}{2}\alpha x^2\right)^{-\frac{\alpha+\beta}{2\alpha}} & \text{if } \alpha > 0. \end{cases}$$

   The approximation for $\alpha < 0$ does not hold when $\alpha + \beta = -2\alpha n$, where $n = 0, 1, 2, 3, \ldots$, and these cases are included in item 3 below. Also, the approximation for $\alpha > 0$ does not hold when $(2\alpha - \beta)/(2\alpha)$ is a nonpositive integer. The case where $\beta = 2\alpha$ is included in *Special Cases*, and Slater (1960) may be consulted for the others.

2. For $x^2 \to \infty$,

$$M\left(\frac{\beta}{2\alpha}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) \sim \begin{cases} \dfrac{\sqrt{\pi}}{\Gamma\left(\frac{\beta}{2\alpha}\right)}\left(-\dfrac{1}{2}\alpha x^2\right)^{\frac{-\alpha+\beta}{2\alpha}} \mathrm{e}^{-\frac{1}{2}\alpha x^2} & \text{if } \alpha < 0, \\ \dfrac{\sqrt{\pi}}{\Gamma\left(\frac{\alpha-\beta}{2\alpha}\right)}\left(\dfrac{1}{2}\alpha x^2\right)^{-\frac{\beta}{2\alpha}} & \text{if } \alpha > 0. \end{cases}$$

   The approximation for $\alpha < 0$ does not hold when $\beta = -2\alpha n$, where $n = 1, 2, 3, \ldots$, and these cases are included in item 3 below. Also, the approximation for $\alpha > 0$ does not hold when $(\alpha - \beta)/(2\alpha)$ is a nonpositive integer. The case where $\beta = \alpha$ is included in *Special Cases*, and Slater (1960) may be consulted for the others.

3. The nonpositive-integer cases are

$$M\left(-n, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) \sim \frac{\sqrt{\pi}}{2\Gamma\left(\frac{3}{2}+n\right)}\left(\frac{1}{2}\alpha x^2\right)^n$$

   and

$$M\left(-n, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) \sim \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+n\right)}\left(\frac{1}{2}\alpha x^2\right)^n.$$

B.2.8 Related Special Functions

Kummer's differential equation reduces to Hermite's equation in the special case where $\alpha = -1$ and $\beta = 2n$, or $\alpha - 1$ and $\beta = n$, where n is a positive integer. Similarly, it can be transformed into Hermite's differential equation when $\alpha < 0$ and $\beta = -\alpha n$, where n is a positive integer. It is also possible to express the solution of Kummer's equation in terms of parabolic cylinder functions (Oldham et al., 2009).

B.3 Higher-Order Turning Points

Analysis of higher order-turning points can be found in Willner and Rubenfeld (1976).

B.3.1 Differential Equation

$$y'' = x^\gamma y \quad \text{for} \quad -\infty < x < \infty \text{ and } \gamma \geq 0.$$

B.3.2 General Solution

$$y(x) = \alpha_0 y_1(x) + \beta_0 y_2(x),$$

where, setting $\nu = 1/(\gamma+2)$ and $\kappa = (\gamma+2)/2$,

$$y_1(x) = \begin{cases} \sqrt{|x|}\,[J_\nu(2\nu|x|^\kappa) + J_{-\nu}(2\nu|x|^\kappa)] & \text{if } x \leq 0, \\ \sqrt{x}\,[I_{-\nu}(2\nu x^\kappa) - I_\nu(2\nu x^\kappa)] & \text{if } x \geq 0, \end{cases}$$

and

$$y_2(x) = \begin{cases} \sqrt{|x|}\,[-J_\nu(2\nu|x|^\kappa) + J_{-\nu}(2\nu|x|^\kappa)] & \text{if } x \leq 0, \\ \sqrt{x}\,[I_{-\nu}(2\nu x^\kappa) + I_\nu(2\nu x^\kappa)] & \text{if } x \geq 0. \end{cases}$$

B.3.3 Asymptotic Approximations

Setting $\xi = \frac{2}{\gamma+2}|x|^{\frac{2+\gamma}{2}}$, then

$$y_1(x) \sim \begin{cases} \dfrac{2\cos(\nu\pi/2)}{\sqrt{\pi\nu}}\,|x|^{-\gamma/4}\cos\Big(\xi - \dfrac{\pi}{4}\Big) & \text{if } x \to -\infty, \\ \dfrac{\sin(\nu\pi)}{\sqrt{\pi\nu}}\,x^{-\gamma/4}\mathrm{e}^{-\xi} & \text{if } x \to +\infty, \end{cases}$$

and

$$y_2(x) \sim \begin{cases} \dfrac{2\sin(\nu\pi/2)}{\sqrt{\pi\nu}}\,|x|^{-\gamma/4}\cos\!\Big(\xi + \dfrac{\pi}{4}\Big) & \text{if } x \to -\infty, \\ \dfrac{1}{\sqrt{\pi\nu}}\,x^{-\gamma/4}\mathrm{e}^{\xi} & \text{if } x \to +\infty. \end{cases}$$

Appendix C
Asymptotic Approximations of Integrals

C.1 Introduction

This appendix summarizes some formulas for approximating integrals. Specifically, approximations of integrals of the form

$$\int_a^b f(t)\mathrm{e}^{-xg(t)}\mathrm{d}t \quad \text{and} \quad \int_a^b f(t)\mathrm{e}^{\mathrm{i}xg(t)}\mathrm{d}t$$

are given for the case of large x. Readers interested in a more extensive development of this material may consult Murray (1984) or Olver (1974).

In this appendix the following assumptions are made:

1. $-\infty \leq a < b \leq \infty$,
2. $f(t)$ and $g(t)$ are continuous and $g(t)$ is real-valued for $a < t < b$,
3. a, b, f, and g are independent of x.

The asymptotic behavior of the first integral depends on where $g(x)$ has a minimum value, while the second depends on where $g(x)$ has a stationary value.

C.2 Watson's Lemma

$$\int_a^b f(t)\mathrm{e}^{-xt}\mathrm{d}t$$

The exponential decay means that the value of the integral is determined by what happens at, or near, $t = a$. Thus, for $t \to a^+$, assume that

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$$f(t) \sim f_0(t-a)^\alpha + f_1(t-a)^\beta,$$

where $-1 < \alpha < \beta$. Also, assume that $-\infty < a$. In this case,

$$\int_a^b f(t)\mathrm{e}^{-xt}\mathrm{d}t \sim \left[\frac{f_0\Gamma(1+\alpha)}{x^{1+\alpha}} + \frac{f_1\Gamma(1+\beta)}{x^{1+\beta}}\right]\mathrm{e}^{-ax} \quad \text{as } x \to \infty.$$

In particular, if $f(t) \sim f(a) + (t-a)f'(a) + \cdots$, where $f(a)$ and $f'(a)$ are nonzero, then

$$\int_a^b f(t)\mathrm{e}^{-xt}\mathrm{d}t \sim \left[\frac{f(a)}{x} + \frac{f'(a)}{x^2}\right]\mathrm{e}^{-ax} \quad \text{as } x \to \infty.$$

C.3 Laplace's Approximation

$$\int_a^b f(t)\mathrm{e}^{-xg(t)}\mathrm{d}t.$$

The exponential decay means that the value of the integral is determined by what happens at, or near, points where $g(t)$ has a minimum value.

(a) The minimum of $g(t)$ for $a \le t \le b$ occurs only at $t = t_0$, where $a < t_0 < b$. Assuming that $f(t_0) \neq 0$ and $g(t)$ is a smooth function with $g''(t_0) > 0$, then

$$\int_a^b f(t)\mathrm{e}^{-xg(t)}\mathrm{d}t \sim f(t_0)\sqrt{\frac{2\pi}{xg''(t_0)}}\,\mathrm{e}^{-xg(t_0)} \quad \text{as } x \to \infty.$$

More generally, for $t \to t_0$, assume that

$$f(t) \sim f_0(t-t_0)^\alpha \quad \text{for } \alpha \ge 0 \text{ and } f_0 \neq 0,$$
$$g(t) \sim g_0 + g_1(t-t_0)^{2m} \quad \text{for } m \text{ a positive integer and } g_1 > 0.$$

In this case,

$$\int_a^b f(t)\mathrm{e}^{-xg(t)}\mathrm{d}t \sim \frac{f_0}{m}\Gamma(\kappa)\left(\frac{1}{xg_1}\right)^\kappa \mathrm{e}^{-xg_0} \quad \text{as } x \to \infty,$$

where

$$\kappa = \frac{1+\alpha}{2m}.$$

(b) The minimum of $g(t)$ for $a \le t \le b$ occurs only at $t = a$ and $a > -\infty$. Assuming that $f(a) \neq 0$ and $g'(t) > 0$ for $a \le t \le b$, then

$$\int_a^b f(t)\mathrm{e}^{-xg(t)}\mathrm{d}t \sim \frac{f(a)}{g'(a)x}\mathrm{e}^{-xg(a)} \quad \text{as } x \to \infty.$$

More generally, for $t \to a^+$, assume that

$$f(t) \sim f_0(t-a)^\alpha \quad \text{for } \alpha > -1 \text{ and } f_0 \neq 0,$$
$$g(t) \sim g_0 + g_1(t-a)^\lambda \quad \text{for } \lambda > 1+\alpha \text{ and } g_1 > 0.$$

In this case,

$$\int_a^b f(t)\mathrm{e}^{-xg(t)}\mathrm{d}t \sim \frac{f_0}{\lambda}\Gamma(\kappa)\left(\frac{1}{xg_1}\right)^\kappa \mathrm{e}^{-xg_0} \quad \text{as } x \to \infty,$$

where $\kappa = (1+\alpha)/\lambda$.

(c) The minimum of $g(t)$ for $a \le t \le b$ occurs only at $t = b$ and $b < \infty$. Assuming that $f(b) \neq 0$ and $g'(t) < 0$ for $a \le t \le b$, then

$$\int_a^b f(t)\mathrm{e}^{-xg(t)}\mathrm{d}t \sim -\frac{f(b)}{g'(b)x}\mathrm{e}^{-xg(b)} \quad \text{as } x \to \infty.$$

C.4 Stationary Phase Approximation

$$\int_a^b f(t)\mathrm{e}^{\mathrm{i}xg(t)}\mathrm{d}t.$$

The fast oscillations mean that the value of the integral is determined by what happens at, or near, points where $g(t)$ has a stationary point. It is assumed that $g(t)$ has only one stationary point. Specifically, in the interval $a \le t \le b$, $g'(t) = 0$ only at $t = t_0$, which is a finite point (i.e., $-\infty < t_0 < \infty$).

Assuming that $f(t_0) \neq 0$, $g''(t_0) \neq 0$, and $a < t_0 < b$, then

$$\int_a^b f(t)\mathrm{e}^{\mathrm{i}xg(t)}\mathrm{d}t \sim f(t_0)\sqrt{\frac{2\pi}{x|g''(t_0)|}}\,\mathrm{e}^{\mathrm{i}\left(xg(t_0)\pm\frac{\pi}{4}\right)} \quad \text{as } x \to \infty,$$

where $+$ is used when $g''(t_0) > 0$ and $-$ when $g''(t_0) < 0$. If $t_0 = a$ or $t_0 = b$, then the approximation is

$$\int_a^b f(t)\mathrm{e}^{\mathrm{i}xg(t)}\mathrm{d}t \sim f(t_0)\sqrt{\frac{\pi}{2x|g''(t_0)|}}\,\mathrm{e}^{\mathrm{i}\left(xg(t_0)\pm\frac{\pi}{4}\right)} \quad \text{as } x \to \infty,$$

where $+$ is used when $g''(t_0) > 0$ and $-$ when $g''(t_0) < 0$.

More generally, in the case where $t_0 = a$, for $t \to a^+$ assume that

$$f(t) \sim f_0(t-a)^\alpha \qquad \text{for} \;\; \alpha \geq 0 \text{ and } f_0 \neq 0,$$
$$g(t) \sim g_0 + g_1(t-a)^\lambda \quad \text{for} \;\; \lambda > 1+\alpha.$$

In this case,

$$\int_a^b f(t)\mathrm{e}^{\mathrm{i}xg(t)}\mathrm{d}t \sim \frac{f_0}{\lambda}\Gamma(\kappa)\left(\frac{1}{x|g_1|}\right)^\kappa \mathrm{e}^{\mathrm{i}(xg_0 \pm \frac{1}{2}\pi\kappa)} \quad \text{as} \;\; x \to \infty,$$

where $\kappa = (1+\alpha)/\lambda$. In the preceding expression, $+$ is used when $g_1 > 0$ and $-$ when $g_1 < 0$.

Appendix D
Second-Order Difference Equations

We will be interested mostly in second-order linear difference equations; an example is

$$y_{n+1} + ay_n + by_{n-1} = f_n \quad \text{for } n = 1, 2, 3, \ldots, \tag{D.1}$$

where $b \neq 0$. The theory and methods developed for such equations are very similar to what is found for second-order differential equations. For example, the general solution has the form $y_n = Y_n + Z_n$, where Y_n is the general solution of the associated homogeneous equation (where $f_n = 0$) and Z_n is a particular solution of the inhomogeneous equation. The specifics of this can be found in Elaydi (2005).

To determine Y_n, one assumes that $Y_n = r^n$. Substituting this into (D.1) and setting $f_n = 0$, the equation reduces to solving $r^2 + ar + b = 0$. The roots of this equation are

$$r_\pm = \frac{1}{2}\left[-a \pm \sqrt{a^2 - 4b}\right] .$$

With this, the general solution of the associated homogeneous equation is

$$Y_n = \begin{cases} \alpha r_+^n + \beta r_-^n & \text{if} \quad a^2 \neq 4b, \\ \alpha r^n + \beta n r^n & \text{if} \quad a^2 = 4b, \end{cases} \tag{D.2}$$

where α and β are arbitrary constants and $r = -a/2$. In the special case where $a^2 < 4b$, Y_n can be written as

$$Y_n = A\rho^n \cos(n\theta + \phi) ,$$

where A and ϕ are arbitrary constants, $\rho = \sqrt{b}$, and $\cos\theta = -a/(2\rho)$. It is assumed that $0 < \theta < \pi$.

A particular solution can be found using a variety of methods, including reduction of order or the z-transform. However, writing $y_{n+1} = f_n - ay_n - by_{n-1}$ and then using this to determine the first few terms it is evident that

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there is a particular solution of the form

$$Z_n = \sum_{i=1}^{n-1} f_{n-i} q_i \quad \text{for } n = 2, 3, 4, \dots , \tag{D.3}$$

where $q_1 = 1$, $q_2 + aq_1 = 0$, and $q_i + aq_{i-1} + bq_{i-2} = 0$. The latter is the associated homogeneous equation, and so (D.2) applies. From this and the conditions on q_1 and q_2, one finds that

$$q_i = \begin{cases} \dfrac{r_+^i - r_-^i}{r_+ - r_-} & \text{if} \quad a^2 \neq 4b, \\ \\ i r^{i-1} & \text{if} \quad a^2 = 4b. \end{cases}$$

Note that, by construction, the q_i satisfy the homogeneous equation, with $q_{-1} = -1/b$, $q_0 = 0$, and $q_1 = 1$. Also, in the case where $a^2 < 4b$, q_i can be written as

$$q_i = \frac{\rho^{i-1} \sin(i\theta)}{\sin(\theta)} ,$$

where $\rho = \sqrt{b}$ and $\cos\theta = -a/(2\rho)$. This is well defined because $0 < \theta < \pi$.

The complete solution depends on the particular problem being solved, and the most commonly studied are initial-value problems and boundary-value problems.

D.1 Initial-Value Problems

For an initial-value problem, (D.1) is to be satisfied, and y_0 and y_1 are prescribed. We have that the general solution has the form $y_n = Y_n + Z_n$, where $Z_0 = Z_1 = 0$. From the requirement that $Y_0 = y_0$ and $Y_1 = y_1$ it follows that the solution is

$$y_n = y_1 q_n - b y_0 q_{n-1} + \sum_{i=1}^{n-1} f_{n-i} q_i \quad \text{for } n = 0, 1, 2, 3, \dots .$$

The sum is understood to be zero in the case where $n = 0$ or $n = 1$.

In the special case where $a^2 < 4b$, note that

$$\sum_{i=1}^{n-1} f_{n-i} q_i = \frac{1}{\sin\theta} \sum_{i=1}^{n-1} f_{n-i} \rho^{i-1} \sin(i\theta).$$

In some cases it is possible to sum this series, although this usually requires knowing the right identities. Two of particular value are

$$\sum_{i=0}^{n-1} \sin(\alpha + \mathrm{i}\beta) = \frac{\sin(n\beta/2)}{\sin(\beta/2)} \sin\left(\alpha + \frac{1}{2}(n-1)\beta\right)$$

and

$$\sum_{i=0}^{n-1} \cos(\alpha + \mathrm{i}\beta) = \frac{\sin(n\beta/2)}{\sin(\beta/2)} \cos\left(\alpha + \frac{1}{2}(n-1)\beta\right),$$

where β is not an integer multiple of 2π. As an example, if $\rho = 1$ and $f_n = \cos(n\theta)$, then

$$\sum_{i=1}^{n-1} f_{n-i}q_i = \frac{1}{2}(n-1)\frac{\sin(n\theta)}{\sin\theta}.$$

Similarly, if $\rho = 1$ and $f_n = \sin(n\theta)$, then

$$\sum_{i=1}^{n-1} f_{n-i}q_i = \frac{1}{2\sin\theta}\left[-(n-1)\cos(n\theta) + \csc(\theta)\sin((n-1)\theta)\right].$$

D.2 Boundary-Value Problems

For a boundary-value problem, (D.1) is to be satisfied for $i = 1, 2, \ldots, N$, and y_0 and y_{N+1} are prescribed. In this case the solution is

$$y_n = y_0 Q_n + \sum_{i=1}^{n-1} f_{n-i}q_i + \frac{q_n}{q_{N+1}}\left[y_{N+1} - \sum_{i=1}^{N} f_{N+1-i}q_i\right] \quad \text{for } n = 0, 1, 2, \ldots, N+1, \qquad \text{(D.4)}$$

where the first sum is understood to be zero in the case where $n = 0$ or $n = 1$. Also,

$$Q_n = \begin{cases} \dfrac{r_+^{N+1}r_-^n - r_+^n r_-^{N+1}}{r_+^{N+1} - r_-^{N+1}} & \text{if} \quad a^2 \neq 4b, \\[2ex] \left(1 - \dfrac{n}{N+1}\right) r^n & \text{if} \quad a^2 = 4b. \end{cases}$$

In the special case where $a^2 < 4b$, the preceding expressions can be written as

$$Q_n = \rho^n \frac{\sin(N+1-n)\theta}{\sin[(N+1)\theta]}$$

and

$$\frac{q_n}{q_{N+1}} = \rho^{n-N-1} \, \frac{\sin(n\theta)}{\sin[(N+1)\theta]} \, ,$$

where $\rho = \sqrt{b}$ and $\cos\theta = -a/(2\rho)$. This shows that the solution is well defined only so long as $(N+1)\theta \neq j\pi$ for $j = 1, 2, \cdots, N$.

Appendix E
Delay Equations

E.1 Differential Delay Equations

An example of a linear first-order delay equation is

$$y'(t) = ay(t) + by(t-\tau) \quad \text{for } t > 0, \tag{E.1}$$

where τ is a positive constant. It is assumed that $y(t) = \chi(t)$, for $-\tau \le t \le 0$, is known. This equation can be solved using the Laplace transform. To see this, note that

$$\int_0^\infty y(t-\tau)\mathrm{e}^{-st}\mathrm{d}t = \mathrm{e}^{-\tau s}Y(s) + Z(s),$$

where $Y(s)$ is the Laplace transform of $y(t)$ and

$$Z(s) = \mathrm{e}^{-\tau s}\int_{-\tau}^0 \chi(r)\mathrm{e}^{-rs}dr.$$

With this, then from (E.1) we have that

$$Y(s) = \frac{y(0) + bZ(s)}{s - a - b\mathrm{e}^{-\tau s}} .$$

Using the definition of the inverse transform,

$$y(t) = \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{y(0) + bZ(s)}{s - a - b\mathrm{e}^{-\tau s}}\mathrm{e}^{st}\mathrm{d}s.$$

Assuming that $Z(s)$ is well behaved, from Cauchy's residue theorem we have that

$$y(t) = \sum_n \frac{y(0) + bZ(s_n)}{1 + \tau b\mathrm{e}^{-\tau s_n}}\mathrm{e}^{s_n t}, \tag{E.2}$$

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where the sum is over all roots of the equation $s = a+b\mathrm{e}^{-\tau s}$. This assumes the roots are simple. If not, then the terms in the preceding series must be modified using the formula for a higher-order pole. A more complete presentation of this method can be found in Pinney (1958).

The critical observation in the foregoing discussion is the form of the general solution in (E.2). This shows that to find the solution of a delay equation like that in (E.1), we simply assume the solution has the form $y = \mathrm{e}^{\lambda t}$ and then use the delay equation to find what equation λ satisfies. For (E.1) this is $\lambda = a + b\mathrm{e}^{-\tau\lambda}$. We then sum over the roots of this equation to obtain a general solution of the form

$$y = \sum_n a_n \mathrm{e}^{\lambda_n t}.$$

This assumes the roots are simple. If a root has order $m+1$, then its contribution to the preceding series will be of the form $(a_{n1}+a_{n2}t+\cdots+a_{nm}t^m)\mathrm{e}^{\lambda_n t}$.

The preceding discussion can be extended to equations of the form

$$y''(x) = ay(x-h) + by(x) + cy(x+h),$$

where h is a positive constant. This is an example of what is called an advance-delay equation. Depending on what boundary or initial conditions are imposed, one can use the Laplace of Fourier transform to find the solution. However, the conclusion is the same as before. That is, the general solution can be obtained by simply assuming the solution has the form $y = \mathrm{e}^{\lambda x}$ and then using the differential-difference equation to determine what equation λ satisfies. One then sums over the roots of this equation to obtain a general solution.

E.2 Integrodifferential Delay Equations

The equation of interest here has the general form

$$y'(t) = F(t, y(t), z(t)) \quad \text{for } t > 0, \tag{E.3}$$

where

$$z(t) = \int_{-\infty}^{t} K(t-\tau)y(\tau)\mathrm{d}\tau.$$

It is assumed that $y(t) = \chi(t)$, for $t \le 0$, is known. Because of the improper integral, certain assumptions must be made about the kernel $K(t)$ and initial data $\chi(t)$. In particular, $K(t)$ is assumed to be smooth and $\int_0^\infty |K(s)|\mathrm{d}s$ is assumed to be finite. In addition, it is assumed that

$$z_0(t) = \int_{-\infty}^{0} K(t-\tau)\chi(\tau)\mathrm{d}\tau$$

is well defined.

An example is the Volterra delay equation that comes up in population modeling (Cushing, 1977):

$$y'(t) = ry(t)\left[Y - \int_{-\infty}^{t} K(t-\tau)y(\tau)\mathrm{d}\tau\right], \tag{E.4}$$

where Y is a positive constant and $K(t) = t\mathrm{e}^{-t}$. In this case, $F(t, y, z) = ry(Y - z)$.

These equations arise frequently in applications but are not usually studied in graduate mathematics programs. The objective here is to show that in some situations they can be written in more familiar terms, namely, as a system of differential equations. This does not necessarily make them easier to solve, but it does open up a number of possibilities on how to study such problems. For a more expansive discussion, including some of the theory, the book by Arino et al. (2006) may be consulted.

E.2.1 Basis Function Approach

The assumption is that it is possible to write

$$K(t) = k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t),$$

where $f_1(t), f_2(t), \ldots, f_n(t)$ are closed under differentiation. This means that given any $f_i(t)$, there are constants $a_{i1}, a_{i2}, \ldots, a_{in}$ such that

$$f_i'(t) = a_{i1} f_1(t) + a_{i2} f_2(t) + \cdots + a_{in} f_n(t).$$

As an example, if $K(t) = t\mathrm{e}^{-t}$, then we can take $f_1(t) = t\mathrm{e}^{-t}$ and $f_2(t) = \mathrm{e}^{-t}$. Note that the closed-under-differentiation assumption is the same one as was made when using the method of undetermined coefficients to find a particular solution of a linear differential equation.

To write (E.3) as a first-order system, let

$$y_i(t) = \int_0^t f_i(t-\tau)y(\tau)\mathrm{d}\tau.$$

In this case,

$$y_i'(t) = f_i(0)y(t) + a_{i1}y_1(t) + a_{i2}y_2(t) + \cdots + a_{in}y_n(t).$$

Therefore, (E.3) can be written as

$$\begin{aligned}
y' &= F(t, y, k_1y_1 + k_2y_2 + \cdots + k_ny_n + z_0),\\
y_1' &= f_1(0)y(t) + a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n,\\
y_2' &= f_2(0)y(t) + a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n,\\
&\vdots \qquad\quad \vdots\\
y_n' &= f_n(0)y(t) + a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n.
\end{aligned}$$

The associated initial conditions are $y(0) = \chi(0)$ and $y_i(0) = 0$ for $i = 1, 2, \ldots, n$.

Applying this to (E.3), in the case where $K(t) = t\mathrm{e}^{-t}$, yields

$$\begin{aligned}
y' &= F(t, y, y_1 + z_0),\\
y_1' &= -y_1 + y_2,\\
y_2' &= y - y_2.
\end{aligned}$$

In the particular case of the linear integrodifferential equation where $F(t, y, z) = ay + bz$, for a and b constants, the preceding system can be written as

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g},$$

where

$$\mathbf{A} = \begin{pmatrix} a & b & 0\\ 0 & -1 & 1\\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{g} = \begin{pmatrix} bz_0\\ 0\\ 0 \end{pmatrix}.$$

This also shows that the general solution of a linear first-order integrodifferential equation can contain more than one arbitrary constant, and the exact number depends on the kernel.

E.2.2 Differential Equation Approach

It is possible to express the basis function approach in another form. The assumption is that $K(t)$ satisfies a constant-coefficient differential equation. The easiest way to explain this is to look at a couple of examples.

1. $K' + aK = 0$

   In this case,

   $$z' = K_0 y - \frac{1}{a}z,$$

   where $K_0 = K(0)$. With this, (E.3) can be written as

$$
\begin{aligned}
y' &= F(t, y, z + z_0), \\
z' &= K_0 y - \frac{1}{a} z.
\end{aligned}
$$

As an example, if $K(t) = \mathrm{e}^{-t}$, then $a = 1$, and the system is

$$
\begin{aligned}
y' &= F(t, y, z + z_0), \\
z' &= y - z.
\end{aligned}
$$

2. $K'' + aK' + bK = 0$

In this case, letting

$$
z_1 = \int_0^t K(t - \tau) y(\tau) \mathrm{d}\tau,
$$

and $z_2 = z_1'$, then (E.3) can be written as

$$
\begin{aligned}
y' &= F(t, y, z_1 + z_0), \\
z_1' &= z_2, \\
z_2' &= (K_0' + aK_0) y - b z_1 - a z_2 + K_0 F(t, y, z_1 + z_0),
\end{aligned}
$$

where $K_0' = K'(0)$.

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