

Appendix A

Special Functions

A.1 Bessel Functions and J -Functions

The properties of Bessel function are summarized in standard references [1, 3, 8].

A.1.1 Ordinary Bessel Functions

Ordinary Bessel functions, $J_\nu(z)$, have the following properties.
Differential equation:

$$z^2 J_\nu''(z) + z J_\nu'(z) + (z^2 - \nu^2) J_\nu(z) = 0, \tag{A.1.1}$$

where a prime denotes differentiation with respect to z .

Power series:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} (z/2)^{2k + \nu}. \tag{A.1.2}$$

Recursion relations:

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2 \frac{\nu}{z} J_\nu(z), \tag{A.1.3}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2 J_\nu'(z). \tag{A.1.4}$$

Generating function:

$$e^{iz \sin \phi} = \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(z). \tag{A.1.5}$$

Sum rules:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} J_n^2(z) &= 1, & \sum_{n=-\infty}^{\infty} nJ_n^2(z) &= 0, & \sum_{n=-\infty}^{\infty} J_n(z)J_n'(z) &= 0, \\ \sum_{n=-\infty}^{\infty} n^2J_n^2(z) &= \frac{1}{2}z^2, & \sum_{n=-\infty}^{\infty} J_n'^2(z) &= \frac{1}{2}. \end{aligned} \quad (\text{A.1.6})$$

A.1.2 Modified Bessel Functions $I_\nu(z)$

Differential equation:

$$I_\nu''(z) + \frac{1}{z}I_\nu'(z) - \left(1 + \frac{\nu^2}{z^2}\right)I_\nu(z) = 0, \quad (\text{A.1.7})$$

Power series:

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k + \nu + 1)} (z/2)^{2k + \nu}. \quad (\text{A.1.8})$$

Recursion relations:

$$\begin{aligned} I_{\nu-1}(z) - I_{\nu+1}(z) &= 2(\nu/z)I_\nu(z), \\ I_{\nu-1}(z) + I_{\nu+1}(z) &= 2I_\nu'(z). \end{aligned} \quad (\text{A.1.9})$$

Generating function:

$$e^{z \cos \phi} = \sum_{s=-\infty}^{\infty} I_s(z)e^{\pm is\phi}, \quad (\text{A.1.10})$$

A.1.3 Macdonald Functions $K_\nu(z)$

Differential equation:

$$\frac{d^2}{dz^2}K_\nu(z) + \frac{1}{z}\frac{d}{dz}K_\nu(z) - \left(1 + \frac{\nu^2}{z^2}\right)K_\nu(z) = 0, \quad (\text{A.1.11})$$

Recursion relations:

$$\begin{aligned} K_{\nu-1}(z) - K_{\nu+1}(z) &= -2(\nu/z)K_{\nu}(z), \\ K_{\nu-1}(z) + K_{\nu+1}(z) &= -2K'_{\nu}(z). \end{aligned} \tag{A.1.12}$$

The recursion relations imply $K_{-\nu}(z) = K_{\nu}(z)$ and

$$\frac{1}{z} \frac{d}{dz} [z^{\pm\nu} K_{\nu}(z)] = -z^{\pm\nu-1} K_{\nu\mp 1}(z). \tag{A.1.13}$$

Expansion of $K_{\nu}(z)$ for small z is

$$K_{\nu}(z) \approx 2^{\nu-1} \Gamma(\nu) z^{-\nu}. \tag{A.1.14}$$

The asymptotic expansion for large z is

$$K_{\nu}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128z^2} + \dots\right). \tag{A.1.15}$$

Integral representation:

$$K_{\nu}(x) = \frac{(x/2)^{\nu} \Gamma(\frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} \int_0^{\infty} d\chi \sinh^{2\nu} \chi e^{-x \cosh \chi} \tag{A.1.16}$$

The Gamma function satisfies

$$\Gamma(x + 1) = x\Gamma(x), \quad \Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \pi^{1/2}. \tag{A.1.17}$$

The integral (A.1.16) also applies when ν is negative, and then $K_{-\nu}(x) = K_{\nu}(x)$ implies

$$K_{\nu}(x) = \frac{(x/2)^{-\nu} \Gamma(\nu + \frac{1}{2}) \cos \pi\nu}{\Gamma(\frac{1}{2})} \int_0^{\infty} d\chi \frac{e^{-x \cosh \chi}}{\sinh^{2\nu} \chi}, \tag{A.1.18}$$

$$\Gamma\left(\frac{1}{2} + \nu\right) \Gamma\left(\frac{1}{2} - \nu\right) = \frac{\pi}{\cos \pi\nu}. \tag{A.1.19}$$

An integral identity due to Schwinger is

$$\int_0^{\infty} d\xi \xi^2 K_{\mu}^2(\xi) = \frac{\pi^2(1 - 4\mu^2)}{32 \cos \pi\mu}. \tag{A.1.20}$$

A.1.4 Airy Functions

The two Airy functions that appear are defined by

$$\text{Ai}(z) = \frac{1}{\pi} \int_0^{\infty} dt \cos\left(zt + \frac{1}{3}t^3\right), \quad \text{Gi}(z) = \frac{1}{\pi} \int_0^{\infty} dt \sin\left(zt + \frac{1}{3}t^3\right). \quad (\text{A.1.21})$$

For $z > 0$ one has

$$\text{Ai}(z) = \frac{1}{\pi} \left(\frac{z}{3}\right)^{1/2} K_{1/3}(\zeta), \quad \text{Ai}'(z) = -\frac{z}{\pi\sqrt{3}} K_{2/3}(\zeta), \quad (\text{A.1.22})$$

with $\zeta = 2z^{3/2}/3$.

The approximations available for $\text{Gi}(z)$ are for large and small z . The leading terms in the asymptotic expansion for $z \gg 1$ are [5]

$$\begin{aligned} \text{Gi}(z) &\sim \frac{1}{\pi} \left(\frac{1}{z} + \frac{2}{z^4} + \dots\right), & \text{Gi}'(z) &\sim \frac{1}{\pi} \left(-\frac{1}{z^2} + \dots\right), \\ \int^z dz' \text{Gi}(z') &\sim \frac{1}{\pi} \left(\ln z + \frac{2C + \ln 3}{3} - \frac{2}{3z^3} + \dots\right), \end{aligned} \quad (\text{A.1.23})$$

where $C = 0.577\dots$ is Euler's constant. The expansion for $z \ll 1$ gives

$$\begin{aligned} \text{Gi}(z) &= \frac{1}{\pi} \left[\frac{3^{1/3}}{2} \Gamma(4/3) + \frac{3^{2/3}}{4} \Gamma(5/3) z - \frac{z^2}{2} + \dots \right], \\ \text{Gi}(0) &= 0.205, \quad \text{Gi}'(0) = 0.149. \end{aligned} \quad (\text{A.1.24})$$

Rothman [5] found that the asymptotic expansion is accurate for $z \gtrsim 8$ and tabulated the functions for lower z .

A.1.5 J-Functions

Definition

The J -functions used here are defined by, for $\nu \geq 0$,

$$J_{\nu}^n(x) = \left(\frac{n!}{(n+\nu)!}\right)^{1/2} e^{-x/2} x^{\nu/2} L_n^{\nu}(x). \quad (\text{A.1.25})$$

By requiring $J_\nu^n(x) = (-)^\nu J_{-\nu}^{n+\nu}(x)$, for $\nu < 0$ one has

$$J_\nu^n(x) = (-)^\nu \left(\frac{(n - |\nu|)!}{n!} \right)^{1/2} e^{-x/2} x^{|\nu|/2} L_n^{|\nu|}(x), \quad (\text{A.1.26})$$

with $L_n^\nu(x)$ the generalized Laguerre polynomial, defined by

$$L_n^\nu(x) = \frac{e^x x^{-\nu}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\nu}) = \sum_{k=0}^n \frac{(n + \nu)! (-x)^k}{(n - k)! (k + \nu)! k!}. \quad (\text{A.1.27})$$

Sokolov and Ternov Function

The function defined by Sokolov and Ternov [6, 7] is related to (A.1.25) by

$$I_{n,n'}(x) = J_{n-n'}^{n'}(x). \quad (\text{A.1.28})$$

Recursion Relations

The J -functions satisfy recursion relations

$$x^{1/2} J_{\nu+1}^{n-1}(x) = (n + \nu)^{1/2} J_\nu^{n-1}(x) - n^{1/2} J_\nu^n(x), \quad (\text{A.1.29})$$

$$x^{1/2} J_{\nu-1}^n(x) = -n^{1/2} J_\nu^{n-1}(x) + (n + \nu)^{1/2} J_\nu^n(x), \quad (\text{A.1.30})$$

and also

$$\nu J_\nu^{n-1}(x) = x^{1/2} [(n + \nu)^{1/2} J_{\nu+1}^{n-1}(x) + n^{1/2} J_{\nu-1}^n(x)], \quad (\text{A.1.31})$$

$$\nu J_\nu^n(x) = x^{1/2} [n^{1/2} J_{\nu+1}^{n-1}(x) + (n + \nu)^{1/2} J_{\nu-1}^n(x)]. \quad (\text{A.1.32})$$

A further pair of relations that is similar to the recursion relations for Bessel functions is

$$(x + \nu) J_\nu^n(x) = [x(n + \nu)]^{1/2} J_{\nu-1}^n(x) + [x(n + \nu + 1)]^{1/2} J_{\nu+1}^n(x), \quad (\text{A.1.33})$$

$$2x \frac{d}{dx} J_\nu^n(x) = [x(n + \nu)]^{1/2} J_{\nu-1}^n(x) - [x(n + \nu + 1)]^{1/2} J_{\nu+1}^n(x). \quad (\text{A.1.34})$$

Relations Involving J -Functions

With $\nu = n - n'$ $p_n = (2neB)^{1/2}$, $x = k_\perp^2/2eB$, relations (A.1.33) and (A.1.34) become

$$\begin{aligned}
p_{n'} J_{n'-n}^n(x) &= p_n J_{n'-n}^{n-1}(x) + k_{\perp} J_{n'-n-1}^n(x), \\
p_{n'} J_{n'-n}^{n-1}(x) &= p_n J_{n'-n}^n(x) + k_{\perp} J_{n'-n+1}^{n-1}(x).
\end{aligned} \tag{A.1.35}$$

The following identities result from squares of the relations (A.1.35):

$$\begin{aligned}
(p_{n'}^2 + p_n^2)[(J_{n'-n}^{n-1})^2 + (J_{n'-n}^n)^2] - 4p_{n'} p_n J_{n'-n}^{n-1} J_{n'-n}^n \\
= k_{\perp}^2 [(J_{n'-n+1}^{n-1})^2 + (J_{n'-n-1}^n)^2],
\end{aligned} \tag{A.1.36}$$

$$(p_{n'}^2 - p_n^2)[(J_{n'-n}^{n-1})^2 - (J_{n'-n}^n)^2] = k_{\perp}^2 [(J_{n'-n+1}^{n-1})^2 - (J_{n'-n-1}^n)^2], \tag{A.1.37}$$

$$\begin{aligned}
(p_{n'}^2 + p_n^2)[(J_{n'-n}^{n-1})^2 - (J_{n'-n}^n)^2] &= 2p_n k_{\perp} [J_{n'-n}^{n-1} J_{n'-n-1}^n - J_{n'-n}^n J_{n'-n+1}^{n-1}] \\
&+ k_{\perp}^2 [(J_{n'-n+1}^{n-1})^2 - (J_{n'-n-1}^n)^2],
\end{aligned} \tag{A.1.38}$$

$$\begin{aligned}
(p_{n'}^2 - p_n^2)[(J_{n'-n}^{n-1})^2 + (J_{n'-n}^n)^2] &= 2p_n k_{\perp} [J_{n'-n}^{n-1} J_{n'-n-1}^n + J_{n'-n}^n J_{n'-n+1}^{n-1}] \\
&+ k_{\perp}^2 [(J_{n'-n+1}^{n-1})^2 + (J_{n'-n-1}^n)^2].
\end{aligned} \tag{A.1.39}$$

In evaluating the response tensor in the summed form (9.1.20) some tensorial components are multiplied by $(pk)_{nn'} = \frac{1}{2}[(k^2)_{\parallel} + p_n^2 - p_{n'}^2]$, and (A.1.37), (A.1.39) allow one to rewrite some of the terms that are multiplied by $p_{n'}^2 - p_n^2$. Other terms that are multiplied by $p_{n'}^2 - p_n^2$ can be rewritten using

$$(p_{n'}^2 - p_n^2) J_{n'-n}^{n-1} = k_{\perp} [p_n J_{n'-n-1}^n + p_{n'} J_{n'-n+1}^{n-1}], \tag{A.1.40}$$

$$(p_{n'}^2 - p_n^2) J_{n'-n}^n = k_{\perp} [p_n J_{n'-n+1}^{n-1} + p_{n'} J_{n'-n-1}^n]. \tag{A.1.41}$$

The remaining terms that are multiplied by $p_{n'}^2 - p_n^2$ involve the square and products of $J_{n'-n+1}^{n-1}$, $J_{n'-n-1}^n$, and these can be rewritten by first expressing these in terms of $J_{n'-n}^{n-1}$, $J_{n'-n}^n$ using (A.1.36)–(A.1.39), but no major simplifications occur.

Sum Rules

The sum rules

$$\sum_{n'=0}^{\infty} J_{n-n'}^{n'}(x) J_{n''-n'}^{n'}(x) = \delta^{nn''}, \tag{A.1.42}$$

$$\sum_{n'=0}^{\infty} (n' - n) [J_{n-n'}^{n'}(x)]^2 = x, \tag{A.1.43}$$

were derived by Quinn and Rodriguez [4] and Sokolov and Ternov [6].

Orthogonality Relation

$$\int_0^\infty dx J_\nu^n(x) J_\nu^{n'}(x) = \delta^{nn'}. \quad (\text{A.1.44})$$

Integral Identities

$$\int_0^\infty dx x^{1/2} [J_\nu^n(x)]^2 = (n + \nu + 1)^{1/2} \left(1 + \frac{n + \frac{1}{2}}{4(n + \nu + 1)} \right), \quad (\text{A.1.45})$$

$$\int_0^\infty dx x [J_\nu^n(x)]^2 = 2n + \nu + \frac{3}{2}, \quad (\text{A.1.46})$$

Particular Values

For $\nu \geq 0$, one has

$$J_\nu^0(x) = (-)^{\nu} J_{-\nu}^{\nu+1}(x) = \frac{x^{\nu/2} e^{-x/2}}{(\nu!)^{1/2}}, \quad (\text{A.1.47})$$

$$J_\nu^1(x) = (-)^{\nu} J_{-\nu}^{\nu+1}(x) = \frac{x^{\nu/2} e^{-x/2}}{((\nu + 1)!)^{1/2}} (\nu + 1 - x), \quad (\text{A.1.48})$$

$$J_\nu^2(x) = (-)^{\nu} J_{-\nu}^{\nu+2}(x) = \frac{x^{\nu/2} e^{-x/2}}{(2!(\nu + 2)!)^{1/2}} \\ \times [(\nu + 1)(\nu + 2) - 2(\nu + 2)x + x^2], \quad (\text{A.1.49})$$

$$J_\nu^3(x) = (-)^{\nu} J_{-\nu}^{\nu+3}(x) = \frac{x^{\nu/2} e^{-x/2}}{(3!(\nu + 3)!)^{1/2}} [(\nu + 1)(\nu + 2)(\nu + 3) \\ - 3(\nu + 2)(\nu + 3)x + 3(\nu + 3)x^2 - x^3]. \quad (\text{A.1.50})$$

Expansion in x

For $x \ll 1$, the J -functions may be approximated by the leading term in their expansion in powers of x :

$$J_{n'-n}^n(x) = \left(\frac{n'!}{n!} \right)^{1/2} \frac{x^{(n'-n)/2}}{(n'-n)!} \left[1 - \frac{n' + n + 1}{2(n' - n + 1)} x + \dots \right], \quad (\text{A.1.51})$$

which applies for $n' \geq n$. The limit $x \rightarrow 0$ gives

$$J_0^n(0) = 1, \quad J_\nu^n(0) = 0 \quad \text{for } \nu \neq 0. \quad (\text{A.1.52})$$

Approximation by Bessel Functions

The expansion of the J -functions in terms of Bessel functions,

$$J_\nu^n \left(\frac{z^2}{4n} \right) = \left[\frac{(n+\nu)!}{n!n^\nu} \right]^{1/2} \sum_{a=0}^{\infty} b_a \left(\frac{z}{2n} \right)^a J_{\nu+a}(z),$$

$$b_0 = 1, \quad b_1 = -\frac{1}{2}(\nu+1), \quad b_2 = \frac{1}{8}(\nu+1)(\nu+2),$$

$$(a+1)b_{a+1} = -\frac{1}{2}(\nu+1)b_a + \frac{1}{4}(\nu+a)b_{a-1} - \frac{1}{4}nb_{a-2}, \quad (\text{A.1.53})$$

converges rapidly for sufficiently large n .

In taking the nonquantum limit, one takes the limit $\hbar \rightarrow 0$, with $n \rightarrow \infty$ so that $p_n = (2neB\hbar)^{1/2} \rightarrow p_\perp$ remains finite; the ratio $a/n = (n-n')/n$ is regarded as of order \hbar . To first order in \hbar one has

$$J_{n-n'}^n(x) = J_a(z) - \frac{1}{2}(a+1) \frac{\hbar k_\perp}{p_\perp} J_{a+1}(z). \quad (\text{A.1.54})$$

The J -functions with upper index $n-1$ and n differ at first order in \hbar :

$$J_{n'-n}^{n-1}(x) - J_{n-n'}^n(x) = -\frac{\hbar k_\perp}{p_\perp} J'_a(z). \quad (\text{A.1.55})$$

Related identities (with arguments x and z omitted) are

$$(J_{n'-n}^{n-1})^2 + (J_{n'-n}^n)^2 = J_a^2 - \frac{2a\hbar k_\perp}{p_\perp} J'_a J_a,$$

$$J_{n'-n}^{n-1} J_{n'-n}^n = J_a^2 - \frac{a\hbar k_\perp}{p_\perp} J'_a J_a,$$

$$(J_{n'-n+1}^{n-1})^2 + (J_{n'-n+1}^n)^2 = \sum_{\eta=\pm 1} J_{a-\eta}^2 \left(1 + \eta \frac{a(a-\eta)eB}{p_\perp^2} \right) + \frac{2a\hbar k_\perp}{p_\perp} J'_a J_a,$$

$$J_{n'-n+1}^{n-1} J_{n'-n+1}^n = J_{a+1} J_{a-1} \left(1 + \frac{aeB}{p_\perp^2} \right) - \frac{a\hbar k_\perp}{p_\perp} J'_a J_a,$$

$$(J_{n'-n}^{n-1})^2 - (J_{n'-n}^n)^2 = -\frac{2a\hbar k_\perp}{p_\perp} J'_a J_a,$$

$$(J_{n'-n+1}^{n-1})^2 - (J_{n'-n-1}^n)^2 = \sum_{\eta=\pm 1} \eta J_{a-\eta}^2 \left(1 + \eta \frac{a(a-\eta)eB}{\rho_{\perp}^2} \right) + \frac{a}{n} J_a^2. \quad (\text{A.1.56})$$

A.2 Relativistic Plasma Dispersion Functions

A.2.1 Relativistic Thermal Function $T(z, \rho)$

The function $T(z, \rho)$, defined by (2.4.29), has alternative integral representations:

$$\begin{aligned} T(z, \rho) &= -\rho \int_0^{\infty} d\chi \sinh \chi e^{-\rho \cosh \chi} \ln \left(\frac{z + \tanh \chi}{z - \tanh \chi} \right) \\ &= 2z \int_0^{\infty} d\chi \frac{e^{-\rho \cosh \chi}}{(1-z^2) \cosh^2 \chi - 1} \\ &= -\frac{2\rho}{1-z^2} \int^z d\zeta \frac{K_1(\rho R)}{R}, \end{aligned} \quad (\text{A.2.1})$$

with $R = [(1-\zeta^2)(1-z^2)]^{1/2}$.

The function $T(z, \rho)$ satisfies the partial differential equations [2]:

$$(1-z^2) \frac{\partial^2}{\partial \rho^2} T(z, \rho) = 2zK_0(\rho) + T(z, \rho), \quad (\text{A.2.2})$$

$$\begin{aligned} z(1-z^2)^3 T''(z, \rho) - (1-z^2)^2(1+2z^2) T'(z, \rho) - \rho^2 z^3 T(z, \rho) \\ = 2z^2 \rho^2 K_0(\rho) + 2(1-z^2) \rho K_1(\rho), \end{aligned} \quad (\text{A.2.3})$$

$$z \frac{\partial}{\partial \rho} T(z, \rho) = 2K_1(\rho) + \frac{(1-z^2)}{\rho} T'(z, \rho), \quad (\text{A.2.4})$$

with $T'(z, \rho) = \partial T(z, \rho) / \partial z$, $T''(z, \rho) = \partial^2 T(z, \rho) / \partial z^2$.

A.2.2 Trubnikov Functions

Trubnikov functions are defined by

$$t_v^n(z, \rho) = (k\tilde{u})^{n+1} \int_0^{\infty} d\xi \xi^n \frac{K_\nu(r(\xi))}{r^\nu(\xi)}, \quad (\text{A.2.5})$$

with $r(\xi)$ given by (2.4.10), and where the power of $k\tilde{u}$ is included so that the integral is dimensionless. They satisfy the recursion relations

$$t_{\nu+1}^{n+1}(z, \rho) = \frac{i\rho z^2}{1-z^2} t_{\nu+1}^n(z, \rho) + \frac{z^2}{1-z^2} \begin{cases} \frac{K_\nu(\rho)}{\rho^\nu} & \text{for } n = 0, \\ n t_\nu^{n-1}(z, \rho) & \text{for } n > 0, \end{cases} \quad (\text{A.2.6})$$

$$\frac{\partial t_\nu^n(z, \rho)}{\partial \rho} = -i\rho t_{\nu+1}^n(z, \rho) - i t_{\nu+1}^{n+1}(z, \rho). \quad (\text{A.2.7})$$

Two further identities are

$$t_{\nu+1}^n(z, \rho) = -\frac{1-z^2}{\rho} \frac{\partial t_\nu^n(z, \rho)}{\partial \rho} + \frac{i z^2}{\rho} \begin{cases} \frac{K_\nu(\rho)}{\rho^\nu} & \text{for } n = 0, \\ n t_\nu^{n-1}(z, \rho) & \text{for } n > 0, \end{cases} \quad (\text{A.2.8})$$

$$t_{\nu+1}^{n+2}(z, \rho) = z^3 \frac{\partial t_\nu^n(z, \rho)}{\partial z}. \quad (\text{A.2.9})$$

The relation to $T(z, \rho)$ follows from

$$t_0^0(z, \rho) = \frac{i z}{2} \frac{\partial T(z, \rho)}{\partial \rho} = \frac{i}{2} \left[2K_1(\rho) + \frac{(1-z^2)}{\rho} T'(z, \rho) \right], \quad (\text{A.2.10})$$

$$t_1^0(z, \rho) = -\frac{i z}{2\rho} T(z, \rho). \quad (\text{A.2.11})$$

The functions for higher n are generated from these using (A.2.6).

A.2.3 Shkarofsky and Dnestrovskii Functions

The generalized Shkarofsky functions are defined by (2.5.28) for real q , integer $r \geq 0$ and complex z, a with $\text{Im}(z-a) > 0$ by

$$\begin{aligned} \mathcal{F}_{q,r}(z, a) &= -i \int_0^\infty dt \frac{(it)^r}{(1-it)^q} \exp \left[izt - \frac{at^2}{1-it} \right] \\ &= -i e^{-a} \int_0^\infty dt \frac{(it)^r}{(1-it)^q} \exp \left[i(z-a)t + \frac{a}{1-it} \right]. \end{aligned} \quad (\text{A.2.12})$$

The definition is extended to $\text{Im}(z-a) < 0$ by analytic continuation. Generalized Dnestrovskii functions are defined by (2.5.34), viz. $F_{q,r}(z) = \mathcal{F}_{q,r}(z, 0)$. The usual Shkarofsky functions, $\mathcal{F}_q(z, a) = \mathcal{F}_{q,0}(z, a)$, and Dnestrovskii functions, $F_q(z) = F_{q,0}(z)$, are the special cases $r = 0$.

The Shkarofsky functions and the Dnestrovskii functions are related by an expansion in modified Bessel functions:

$$\mathcal{F}_q(z, a) = \sum_{s=-\infty}^{\infty} e^{-2a} I_s(2a) F_{q-s}(z). \tag{A.2.13}$$

Recursion Relations and Differential Equations

Recursion relations satisfied by the Shkarofsky functions are

$$a\mathcal{F}_{q-2}(z, a) = 1 + (a - z)\mathcal{F}_q(z, a) - q\mathcal{F}_{q+1}(z, a), \tag{A.2.14}$$

$$\mathcal{F}'_q(z, a) = \mathcal{F}_q(z, a) - \mathcal{F}_{q-1}(z, a), \tag{A.2.15}$$

$$\mathcal{F}''_q(z, a) = \mathcal{F}_q(z, a) - 2\mathcal{F}_{q-1}(z, a) + \mathcal{F}_{q-2}(z, a), \tag{A.2.16}$$

where a prime denotes a derivative with respect to z . Eliminating $\mathcal{F}_{q-1}(z, a)$ and $\mathcal{F}_{q-2}(z, a)$ between these gives a second order differential equation satisfied by the Shkarofsky functions:

$$(a - z)\mathcal{F}''_q(z, a) - [2(a - z) - q - 2]\mathcal{F}'_q(z, a) - (z + q - 2)\mathcal{F}_q(z, a) + 1 = 0. \tag{A.2.17}$$

Recursion relations for the Dnestrovskii functions follow from (A.2.14) and (A.2.15) for $a = 0$:

$$(q - 1)F_q(z) = 1 - zF_{q-1}(z), \tag{A.2.18}$$

$$F'_q(z) = F_q(z) - F_{q-1}(z). \tag{A.2.19}$$

Eliminating $F_{q-1}(z)$ between these gives a first order differential equation satisfied by the Dnestrovskii functions:

$$zF'_q(z) = (z + q - 1)F_q(z) - 1. \tag{A.2.20}$$

The function $F_q(z)$ also satisfies (A.2.17) with $a = 0$. Equation (A.2.19) integrates to give

$$F_q(z) = z^{q-1}e^z\Gamma(1 - q, z), \quad \Gamma(q, z) = \int_z^{\infty} d\xi \zeta^{q-1}e^{-\zeta}, \tag{A.2.21}$$

where $\Gamma(q, z)$ is the incomplete gamma function.

Limiting Cases

The expansion of the Dnestrovskii functions for small arguments z follows from (A.2.21) and the relevant expansion of the incomplete gamma function:

$$\begin{aligned}
 F_q(z) &= z^{q-1} e^z \Gamma(1-q) - \sum_j^{\infty} \frac{z^j \Gamma(1-q)}{\Gamma(j+q-1)j!} \\
 &= z^{q-1} e^z \Gamma(1-q) - e^z \sum_j^{\infty} \frac{(-z)^j \Gamma(1-q)}{\Gamma(j+2-q)}. \quad (\text{A.2.22})
 \end{aligned}$$

For real, positive z there is an expansion in generalized Laguerre polynomials:

$$F_q(z) = \sum_{j=0}^{\infty} \frac{L_j^{(1-q)}(z)}{j+1}. \quad (\text{A.2.23})$$

For large argument, $|z| \gg 1$, the limit

$$F_q(z) \sim \sum_{j=0}^{\infty} (-1)^j z^{-1-j} \Gamma(q+j) \quad (\text{A.2.24})$$

applies for $\arg(z) < 3\pi/2$.

Half-Integer q

In evaluating (2.5.27) in terms of Shkarofsky functions, the function and its derivative with $q = 5/2$ appear. The expansion (2.5.38) then leads to Dnestrovskii functions with half-integer q . For q a positive half-integer, the Dnestrovskii functions are expressible in terms of the plasma dispersion function

$$Z(y) = \pi^{-1/2} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t-y} = -\frac{\phi(y)}{y} + i\pi^{1/2} e^{-y^2}, \quad (\text{A.2.25})$$

The relevant form is

$$\Gamma(q)F_q(z) = \sum_{j=0}^{q-3/2} (-z)^j \Gamma(q-1-j) + \pi^{1/2} (-z)^{q-3/2} [iz^{1/2} e^z Z(iz^{1/2})]. \quad (\text{A.2.26})$$

Expansions for small and large arguments are

$$\Gamma(q)F_q(z) = \begin{cases} \sum_{j=0}^{\infty} (-z)^j \Gamma(q-1-j) - i\pi(-z)^{q-1} e^z & \text{for } |z|^2 \ll 1, \\ -\sum_{j=0}^{\infty} \Gamma(q+j)(-z)^{-1-j} - i\sigma\pi(-z)^{q-1} e^z & \text{for } |z| \gg 1, \end{cases} \quad (\text{A.2.27})$$

with $\sigma = 0$ for $\arg z < \pi$, $\sigma = 1$ for $\arg z = \pi$ and $\sigma = 2$ for $\pi < \arg z < 2\pi$.

A.3 Dirac Algebra

In this section some results associated with the properties of Dirac matrices are summarized.

A.3.1 Definitions and the Standard Representation

The Dirac matrices are defined to satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad (\text{A.3.1})$$

where the unit Dirac matrix is implicit on the right hand side. The Dirac Hamiltonian is

$$\hat{H} = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m, \quad \boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}, \quad \beta = \gamma^0. \quad (\text{A.3.2})$$

The requirement that the Dirac Hamiltonian be self-adjoint implies

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \quad (\text{A.3.3})$$

Standard Representation

The specific choice for the Dirac matrices used here is referred to as the standard representation. It corresponds to

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.3.4})$$

A convenient way of writing these and other 4×4 matrices is in terms of block matrices. Let $\mathbf{0}$ and $\mathbf{1}$ be the null and unit 2×2 matrices. One writes

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}, \quad \rho_x = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

$$\rho_y = \begin{pmatrix} \mathbf{0} & -i\mathbf{1} \\ i\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \rho_z = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad (\text{A.3.5})$$

where the 2×2 matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.3.6})$$

are the usual Pauli matrices. In this representation one has

$$\gamma^\mu = [\rho_z, i\rho_y \boldsymbol{\Sigma}], \quad \boldsymbol{\alpha} = \rho_x \boldsymbol{\sigma}, \quad \beta = \rho_z. \quad (\text{A.3.7})$$

Dirac Matrices $\sigma^{\mu\nu}$ and γ^5

Two additional Dirac matrices that play an important role in the theory are

$$\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu], \quad (\text{A.3.8})$$

which plays the role of a spin angular momentum, and

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \quad (\text{A.3.9})$$

which satisfies the relations

$$\gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0, \quad (\gamma^5)^2 = 1, \quad (\gamma^5)^\dagger = \gamma^5. \quad (\text{A.3.10})$$

One also has

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 = -i\epsilon^{\mu\nu\rho\sigma}. \quad (\text{A.3.11})$$

In the standard representation one has $\gamma^5 = -\rho_x$. The spin 4-tensor $\sigma^{\mu\nu}$, defined by (A.3.8), has components

$$\sigma^{\mu\nu} = \begin{pmatrix} 0 & \alpha_x & \alpha_y & \alpha_z \\ -\alpha_x & 0 & -i\sigma_z & i\sigma_y \\ -\alpha_y & i\sigma_z & 0 & -i\sigma_x \\ -\alpha_z & -i\sigma_y & i\sigma_x & 0 \end{pmatrix}. \quad (\text{A.3.12})$$

A.3.2 Basic Set of Dirac Matrices

There are 16 independent 4×4 matrices and for the Dirac matrices it is sometimes convenient to choose a set of 16 basis vectors. A specific choice of 16 independent

matrices is the set

$$\gamma^A = \left[1, \gamma^\mu, i\sigma^{\mu\nu}, i\gamma^\mu\gamma^5, \gamma^5 \right]. \tag{A.3.13}$$

This choice involves a scalar and a pseudo scalar ($1, \gamma^5$), a 4-vector and a pseudo 4-vector ($\gamma^\mu, i\gamma^\mu\gamma^5$) and an antisymmetric second rank 4-tensor ($\sigma^{\mu\nu}$). These have 1, 1, 4, 4, and 6 components, respectively. This set is chosen such that the analogous set, γ_A with indices down, $\gamma_A = [1, \gamma_\mu, i\sigma_{\mu\nu}, i\gamma_\mu\gamma^5, \gamma^5]$ satisfy

$$\gamma^A\gamma_A = 1 \quad (\text{no sum}), \quad \gamma^A\gamma_B = \delta_B^A. \tag{A.3.14}$$

The expansion of an arbitrary Dirac matrix, O say, in this basis gives

$$O = \sum_A c_A \gamma^A, \quad c_A = \frac{1}{4} \text{Tr}[\gamma_A O]. \tag{A.3.15}$$

Traces of Products of γ -Matrices

The traces of products of γ -matrices are important in detailed calculations in QED. Consider

$$T^{\alpha_1\alpha_2\dots\alpha_n} = \text{Tr}(\gamma^{\alpha_1}\gamma^{\alpha_2}\dots\gamma^{\alpha_n}). \tag{A.3.16}$$

The trace of γ^μ is zero, as are the traces of $\sigma^{\mu\nu}, \gamma^\mu\gamma^5$ and γ^5 . The trace of a product of an odd number of γ -matrices is also zero: $T^{\alpha_1\alpha_2\dots\alpha_n} = 0$ for n odd. The trace of a product of two γ -matrices is nonzero. This trace is evaluated as follows. First the invariance of the trace of a product of matrices under cyclic permutations of the matrices implies $T^{\mu\nu} = T^{\nu\mu}$. The trace of (5.1.1) implies $T^{\mu\nu} = 4g^{\mu\nu}$, where the factor of 4 arising from the trace of the unit 4×4 matrix. Using the invariance of the trace under cyclic permutations and (5.1.1) allows one to evaluate the traces (A.3.16) for all even n . One finds

$$T^{\mu\nu} = 4g^{\mu\nu}, \quad T^{\mu\nu\rho\sigma} = 4\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} \right], \tag{A.3.17}$$

$$T^{\mu\nu\rho\sigma\alpha\beta} = 4\left[g^{\mu\nu}T^{\rho\sigma\alpha\beta} - g^{\mu\rho}T^{\nu\sigma\alpha\beta} + g^{\mu\sigma}T^{\nu\rho\alpha\beta} - g^{\mu\alpha}T^{\nu\rho\sigma\alpha} + g^{\mu\beta}T^{\nu\rho\sigma\alpha} \right], \tag{A.3.18}$$

and so on.

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