Appendix A
Special Functions

A.1 Bessel Functions and J-Functions

The properties of Bessel function are summarized in standard references [1, 3, 8].

A.1.1 Ordinary Bessel Functions

Ordinary Bessel functions, $J_\nu(z)$, have the following properties.
Differential equation:

$$z^2 J_\nu''(z) + zJ_\nu'(z) + (z^2 - \nu^2)J_\nu(z) = 0, \tag{A.1.1}$$

where a prime denotes differentiation with respect to z.
Power series:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} (z/2)^{2k+\nu}. \tag{A.1.2}$$

Recursion relations:

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\frac{\nu}{z} J_\nu(z), \tag{A.1.3}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z). \tag{A.1.4}$$

Generating function:

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(z). \tag{A.1.5}$$

D. Melrose, *Quantum Plasmadynamics: Magnetized Plasmas*, Lecture Notes in Physics 854, DOI 10.1007/978-1-4614-4045-1,

Sum rules:

$$\sum_{n=-\infty}^{\infty} J_n^2(z) = 1, \qquad \sum_{n=-\infty}^{\infty} n J_n^2(z) = 0, \qquad \sum_{n=-\infty}^{\infty} J_n(z) J_n'(z) = 0,$$

$$\sum_{n=-\infty}^{\infty} n^2 J_n^2(z) = \frac{1}{2} z^2, \qquad \sum_{n=-\infty}^{\infty} J_n'^2(z) = \frac{1}{2}. \tag{A.1.6}$$

A.1.2 Modified Bessel Functions $I_\nu(z)$

Differential equation:

$$I_\nu''(z) + \frac{1}{z} I_\nu'(z) - \left(1 + \frac{\nu^2}{z^2}\right) I_\nu(z) = 0, \tag{A.1.7}$$

Power series:

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \nu + 1)} (z/2)^{2k+\nu}. \tag{A.1.8}$$

Recursion relations:

$$I_{\nu-1}(z) - I_{\nu+1}(z) = 2(\nu/z) I_s(z),$$

$$I_{\nu-1}(z) + I_{\nu+1}(z) = 2 I_\nu'(z). \tag{A.1.9}$$

Generating function:

$$e^{z \cos\phi} = \sum_{s=-\infty}^{\infty} I_s(z) e^{\pm i s \phi}, \tag{A.1.10}$$

A.1.3 Macdonald Functions $K_\nu(z)$

Differential equation:

$$\frac{d^2}{dz^2} K_\nu(z) + \frac{1}{z} \frac{d}{dz} K_\nu(z) - \left(1 + \frac{\nu^2}{z^2}\right) K_\nu(z) = 0, \tag{A.1.11}$$

Recursion relations:

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -2(\nu/z)K_\nu(z),$$
$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_\nu(z). \tag{A.1.12}$$

The recursion relations imply $K_{-\nu}(z) = K_\nu(z)$ and

$$\frac{1}{z}\frac{d}{dz}\big[z^{\pm\nu}K_\nu(z)\big] = -z^{\pm\nu-1}K_{\nu\mp1}(z). \tag{A.1.13}$$

Expansion of $K_\nu(z)$ for small z is

$$K_\nu(z) \approx 2^{\nu-1}\Gamma(\nu)\, z^{-\nu}. \tag{A.1.14}$$

The asymptotic expansion for large z is

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1 + \frac{4\nu^2-1}{8z} + \frac{(4\nu^2-1)(4\nu^2-9)}{128z^2} + \cdots\right). \tag{A.1.15}$$

Integral representation:

$$K_\nu(x) = \frac{(x/2)^\nu\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty d\chi\, \sinh^{2\nu}\chi\, e^{-x\cosh\chi} \tag{A.1.16}$$

The Gamma function satisfies

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(1) = 1, \qquad \Gamma\left(\tfrac{1}{2}\right) = \pi^{1/2}. \tag{A.1.17}$$

The integral (A.1.16) also applies when ν is negative, and then $K_{-\nu}(x) = K_\nu(x)$ implies

$$K_\nu(x) = \frac{(x/2)^{-\nu}\Gamma(\nu+\frac{1}{2})\cos\pi\nu}{\Gamma(\frac{1}{2})}\int_0^\infty d\chi\,\frac{e^{-x\cosh\chi}}{\sinh^{2\nu}\chi}, \tag{A.1.18}$$

$$\Gamma\left(\tfrac{1}{2}+\nu\right)\Gamma\left(\tfrac{1}{2}-\nu\right) = \frac{\pi}{\cos\pi\nu}. \tag{A.1.19}$$

An integral identity due to Schwinger is

$$\int_0^\infty d\xi\,\xi^2 K_\mu^2(\xi) = \frac{\pi^2(1-4\mu^2)}{32\cos\pi\mu}. \tag{A.1.20}$$

A.1.4 Airy Functions

The two Airy functions that appear are defined by

$$\mathrm{Ai}\,(z) = \frac{1}{\pi}\int_0^{\infty} dt\ \cos\left(zt + \tfrac{1}{3}t^3\right), \qquad \mathrm{Gi}\,(z) = \frac{1}{\pi}\int_0^{\infty} dt\ \sin\left(zt + \tfrac{1}{3}t^3\right). \tag{A.1.21}$$

For $z > 0$ one has

$$\mathrm{Ai}\,(z) = \frac{1}{\pi}\left(\frac{z}{3}\right)^{1/2} K_{1/3}(\zeta), \qquad \mathrm{Ai}'\,(z) = -\frac{z}{\pi\sqrt{3}}K_{2/3}(\zeta), \tag{A.1.22}$$

with $\zeta = 2z^{3/2}/3$.

The approximations available for $\mathrm{Gi}\,(z)$ are for large and small z. The leading terms in the asymptotic expansion for $z \gg 1$ are [5]

$$\mathrm{Gi}\,(z) \sim \frac{1}{\pi}\left(\frac{1}{z} + \frac{2}{z^4} + \cdots\right), \qquad \mathrm{Gi}'\,(z) \sim \frac{1}{\pi}\left(-\frac{1}{z^2} + \cdots\right),$$
$$\int^z dz'\,\mathrm{Gi}\,(z') \sim \frac{1}{\pi}\left(\ln z + \frac{2C + \ln 3}{3} - \frac{2}{3z^3} + \cdots\right), \tag{A.1.23}$$

where $C = 0.577\cdots$ is Euler's constant. The expansion for $z \ll 1$ gives

$$\mathrm{Gi}\,(z) = \frac{1}{\pi}\left[\frac{3^{1/3}}{2}\,\Gamma(4/3) + \frac{3^{2/3}}{4}\,\Gamma(5/3)\,z - \frac{z^2}{2} + \cdots\right],$$
$$\mathrm{Gi}\,(0) = 0.205, \quad \mathrm{Gi}'\,(0) = 0.149. \tag{A.1.24}$$

Rothman [5] found that the asymptotic expansion is accurate for $z \gtrsim 8$ and tabulated the functions for lower z.

A.1.5 J-Functions

Definition

The J-functions used here are defined by, for $\nu \geq 0$,

$$J_\nu^n(x) = \left(\frac{n!}{(n+\nu)!}\right)^{1/2} e^{-x/2} x^{\nu/2}\, L_n^\nu(x). \tag{A.1.25}$$

By requiring $J_\nu^n(x) = (-)^\nu J_{-\nu}^{n+\nu}(x)$, for $\nu < 0$ one has

$$J_\nu^n(x) = (-)^\nu \left(\frac{(n-|\nu|)!}{n!}\right)^{1/2} e^{-x/2} x^{|\nu|/2} L_n^{|\nu|}(x), \tag{A.1.26}$$

with $L_n^\nu(x)$ the generalized Laguerre polynomial, defined by

$$L_n^\nu(x) = \frac{e^x x^{-\nu}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\nu}) = \sum_{k=0}^{n} \frac{(n+\nu)!\,(-x)^k}{(n-k)!(k+\nu)!k!}. \tag{A.1.27}$$

Sokolov and Ternov Function

The function defined by Sokolov and Ternov [6, 7] is related to (A.1.25) by

$$I_{n,n'}(x) = J_{n-n'}^{n'}(x). \tag{A.1.28}$$

Recursion Relations

The J-functions satisfy recursion relations

$$x^{1/2} J_{\nu+1}^{n-1}(x) = (n+\nu)^{1/2} J_\nu^{n-1}(x) - n^{1/2} J_\nu^n(x), \tag{A.1.29}$$

$$x^{1/2} J_{\nu-1}^{n}(x) = -n^{1/2} J_\nu^{n-1}(x) + (n+\nu)^{1/2} J_\nu^n(x), \tag{A.1.30}$$

and also

$$\nu J_\nu^{n-1}(x) = x^{1/2}\big[(n+\nu)^{1/2} J_{\nu+1}^{n-1}(x) + n^{1/2} J_{\nu-1}^n(x)\big], \tag{A.1.31}$$

$$\nu J_\nu^{n}(x) = x^{1/2}\big[n^{1/2} J_{\nu+1}^{n-1}(x) + (n+\nu)^{1/2} J_{\nu-1}^n(x)\big]. \tag{A.1.32}$$

A further pair of relations that is similar to the recursion relations for Bessel functions is

$$(x+\nu) J_\nu^n(x) = [x(n+\nu)]^{1/2} J_{\nu-1}^n(x) + [x(n+\nu+1)]^{1/2} J_{\nu+1}^n(x), \tag{A.1.33}$$

$$2x\frac{d}{dx} J_\nu^n(x) = [x(n+\nu)]^{1/2} J_{\nu-1}^n(x) - [x(n+\nu+1)]^{1/2} J_{\nu+1}^n(x). \tag{A.1.34}$$

Relations Involving *J*-Functions

With $\nu = n - n'$ $p_n = (2neB)^{1/2}$, $x = k_\perp^2/2eB$, relations (A.1.33) and (A.1.34) become

$$p_{n'} J^{n}_{n'-n}(x) = p_n J^{n-1}_{n'-n}(x) + k_\perp J^{n}_{n'-n-1}(x),$$
$$p_{n'} J^{n-1}_{n'-n}(x) = p_n J^{n}_{n'-n}(x) + k_\perp J^{n-1}_{n'-n+1}(x). \qquad \text{(A.1.35)}$$

The following identities result from squares of the relations (A.1.35):

$$(p_{n'}^2 + p_n^2)[(J^{n-1}_{n'-n})^2 + (J^{n}_{n'-n})^2] - 4p_{n'} p_n J^{n-1}_{n'-n} J^{n}_{n'-n}$$
$$= k_\perp^2 [(J^{n-1}_{n'-n+1})^2 + (J^{n}_{n'-n-1})^2], \qquad \text{(A.1.36)}$$

$$(p_{n'}^2 - p_n^2)[(J^{n-1}_{n'-n})^2 - (J^{n}_{n'-n})^2] = k_\perp^2 [(J^{n-1}_{n'-n+1})^2 - (J^{n}_{n'-n-1})^2], \qquad \text{(A.1.37)}$$

$$(p_{n'}^2 + p_n^2)[(J^{n-1}_{n'-n})^2 - (J^{n}_{n'-n})^2] = 2p_n k_\perp [J^{n-1}_{n'-n} J^{n}_{n'-n-1} - J^{n}_{n'-n} J^{n-1}_{n'-n+1}]$$
$$+ k_\perp^2 [(J^{n-1}_{n'-n+1})^2 - (J^{n}_{n'-n-1})^2], \qquad \text{(A.1.38)}$$

$$(p_{n'}^2 - p_n^2)[(J^{n-1}_{n'-n})^2 + (J^{n}_{n'-n})^2] = 2p_n k_\perp [J^{n-1}_{n'-n} J^{n}_{n'-n-1} + J^{n}_{n'-n} J^{n-1}_{n'-n+1}]$$
$$+ k_\perp^2 [(J^{n-1}_{n'-n+1})^2 + (J^{n}_{n'-n-1})^2]. \qquad \text{(A.1.39)}$$

In evaluating the response tensor in the summed form (9.1.20) some tensorial components are multiplied by $(pk)_{nn'} = \frac{1}{2}[(k^2)_\parallel + p_n^2 - p_{n'}^2]$, and (A.1.37), (A.1.39) allow one to rewrite some of the terms that are multiplied by $p_{n'}^2 - p_n^2$. Other terms that are multiplied by $p_{n'}^2 - p_n^2$ can be rewritten using

$$(p_{n'}^2 - p_n^2) J^{n-1}_{n'-n} = k_\perp [p_n J^{n}_{n'-n-1} + p_{n'} J^{n-1}_{n'-n+1}], \qquad \text{(A.1.40)}$$

$$(p_{n'}^2 - p_n^2) J^{n}_{n'-n} = k_\perp [p_n J^{n-1}_{n'-n+1} + p_{n'} J^{n}_{n'-n-1}]. \qquad \text{(A.1.41)}$$

The remaining terms that are multiplied by $p_{n'}^2 - p_n^2$ involve the square and products of $J^{n-1}_{n'-n+1}$, $J^{n}_{n'-n-1}$, and these can be rewritten by first expressing these in terms of $J^{n-1}_{n'-n}$, $J^{n}_{n'-n}$ using (A.1.36)–(A.1.39), but no major simplifications occur.

Sum Rules

The sum rules

$$\sum_{n'=0}^{\infty} J^{n'}_{n-n'}(x) J^{n'}_{n''-n'}(x) = \delta^{nn''}, \qquad \text{(A.1.42)}$$

$$\sum_{n'=0}^{\infty} (n'-n)[J^{n'}_{n-n'}(x)]^2 = x, \qquad \text{(A.1.43)}$$

were derived by Quinn and Rodriguez [4] and Sokolov and Ternov [6].

Orthogonality Relation

$$\int_0^\infty dx\, J_\nu^n(x) J_\nu^{n'}(x) = \delta^{nn'}. \tag{A.1.44}$$

Integral Identities

$$\int_0^\infty dx x^{1/2}\, [J_\nu^n(x)]^2 = (n+\nu+1)^{1/2} \left(1 + \frac{n+\frac{1}{2}}{4(n+\nu+1)}\right), \tag{A.1.45}$$

$$\int_0^\infty dx\, x\, [J_\nu^n(x)]^2 = 2n+\nu+\tfrac{3}{2}, \tag{A.1.46}$$

Particular Values

For $\nu \geq 0$, one has

$$J_\nu^0(x) = (-)^\nu J_{-\nu}^{\nu+1}(x) = \frac{x^{\nu/2} e^{-x/2}}{(\nu!)^{1/2}}, \tag{A.1.47}$$

$$J_\nu^1(x) = (-)^\nu J_{-\nu}^{\nu+1}(x) = \frac{x^{\nu/2} e^{-x/2}}{((\nu+1)!)^{1/2}}(\nu+1-x), \tag{A.1.48}$$

$$J_\nu^2(x) = (-)^\nu J_{-\nu}^{\nu+2}(x) = \frac{x^{\nu/2} e^{-x/2}}{(2!(\nu+2)!)^{1/2}} \times[(\nu+1)(\nu+2) - 2(\nu+2)x + x^2], \tag{A.1.49}$$

$$J_\nu^3(x) = (-)^\nu J_{-\nu}^{\nu+3}(x) = \frac{x^{\nu/2} e^{-x/2}}{(3!(\nu+3)!)^{1/2}}[(\nu+1)(\nu+2)(\nu+3) - 3(\nu+2)(\nu+3)x + 3(\nu+3)x^2 - x^3]. \tag{A.1.50}$$

Expansion in *x*

For $x \ll 1$, the J-functions may be approximated by the leading term in their expansion in powers of x:

$$J_{n'-n}^n(x) = \left(\frac{n'!}{n!}\right)^{1/2} \frac{x^{(n'-n)/2}}{(n'-n)!}\left[1 - \frac{n'+n+1}{2(n'-n+1)}x + \cdots\right], \tag{A.1.51}$$

which applies for $n' \geq n$. The limit $x \to 0$ gives

$$J_0^n(0) = 1, \qquad J_\nu^n(0) = 0 \quad \text{for } \nu \neq 0. \tag{A.1.52}$$

Approximation by Bessel Functions

The expansion of the J-functions in terms of Bessel functions,

$$J_\nu^n\left(\frac{z^2}{4n}\right) = \left[\frac{(n+\nu)!}{n!n^\nu}\right]^{1/2} \sum_{a=0}^{\infty} b_a \left(\frac{z}{2n}\right)^a J_{\nu+a}(z),$$

$$b_0 = 1, \quad b_1 = -\tfrac{1}{2}(\nu+1), \quad b_2 = \tfrac{1}{8}(\nu+1)(\nu+2),$$

$$(a+1)b_{a+1} = -\tfrac{1}{2}(\nu+1)b_a + \tfrac{1}{4}(\nu+a)b_{a-1} - \tfrac{1}{4}nb_{a-2}, \tag{A.1.53}$$

converges rapidly for sufficiently large n.

In taking the nonquantum limit, one takes the limit $\hbar \to 0$, with $n \to \infty$ so that $p_n = (2neB\hbar)^{1/2} \to p_\perp$ remains finite; the ratio $a/n = (n-n')/n$ is regarded as of order $\hbar$. To first order in $\hbar$ one has

$$J_{n-n'}^n(x) = J_a(z) - \tfrac{1}{2}(a+1)\frac{\hbar k_\perp}{p_\perp} J_{a+1}(z). \tag{A.1.54}$$

The J-functions with upper index $n-1$ and n differ at first order in $\hbar$:

$$J_{n'-n}^{n-1}(x) - J_{n-n'}^n(x) = -\frac{\hbar k_\perp}{p_\perp} J_a'(z). \tag{A.1.55}$$

Related identities (with arguments x and z omitted) are

$$(J_{n'-n}^{n-1})^2 + (J_{n'-n}^n)^2 = J_a^2 - \frac{2a\hbar k_\perp}{p_\perp} J_a' J_a,$$

$$J_{n'-n}^{n-1} J_{n'-n}^n = J_a^2 - \frac{a\hbar k_\perp}{p_\perp} J_a' J_a,$$

$$(J_{n'-n+1}^{n-1})^2 + (J_{n'-n-1}^n)^2 = \sum_{\eta=\pm1} J_{a-\eta}^2 \left(1 + \eta\frac{a(a-\eta)eB}{p_\perp^2}\right) + \frac{2a\hbar k_\perp}{p_\perp} J_a' J_a,$$

$$J_{n'-n+1}^{n-1} J_{n'-n-1}^n = J_{a+1} J_{a-1}\left(1 + \frac{aeB}{p_\perp^2}\right) - \frac{a\hbar k_\perp}{p_\perp} J_a' J_a,$$

$$(J_{n'-n}^{n-1})^2 - (J_{n'-n}^n)^2 = -\frac{2a\hbar k_\perp}{p_\perp} J_a' J_a,$$

$$(J^{n-1}_{n'-n+1})^2 - (J^{n}_{n'-n-1})^2 = \sum_{\eta=\pm1} \eta J^2_{a-\eta}\left(1 + \eta\frac{a(a-\eta)eB}{p_\perp^2}\right) + \frac{a}{n}J_a^2. \tag{A.1.56}$$

A.2 Relativistic Plasma Dispersion Functions

A.2.1 Relativistic Thermal Function $T(z, \rho)$

The function $T(z, \rho)$, defined by (2.4.29), has alternative integral representations:

$$\begin{aligned} T(z,\rho) &= -\rho\int_0^\infty d\chi\, \sinh\chi\, e^{-\rho\cosh\chi}\, \ln\left(\frac{z+\tanh\chi}{z-\tanh\chi}\right) \\ &= 2z\int_0^\infty d\chi\, \frac{e^{-\rho\cosh\chi}}{(1-z^2)\cosh^2\chi - 1} \\ &= -\frac{2\rho}{1-z^2}\int^z d\zeta\, \frac{K_1(\rho R)}{R}, \end{aligned} \tag{A.2.1}$$

with $R = [(1-\zeta^2)(1-z^2)]^{1/2}$.

The function $T(z, \rho)$ satisfies the partial differential equations [2]:

$$(1-z^2)\,\frac{\partial^2}{\partial\rho^2}\,T(z,\rho) = 2zK_0(\rho) + T(z,\rho), \tag{A.2.2}$$

$$\begin{aligned} z(1-z^2)^3\, T''(z,\rho) - (1-z^2)^2(1+2z^2)\, T'(z,\rho) - \rho^2 z^3 T(z,\rho) \\ = 2z^2\rho^2 K_0(\rho) + 2(1-z^2)\rho K_1(\rho), \end{aligned} \tag{A.2.3}$$

$$z\frac{\partial}{\partial\rho}\,T(z,\rho) = 2K_1(\rho) + \frac{(1-z^2)}{\rho}\,T'(z,\rho), \tag{A.2.4}$$

with $T'(z,\rho) = \partial T(z,\rho)/\partial z$, $T''(z,\rho) = \partial^2 T(z,\rho)/\partial z^2$.

A.2.2 Trubnikov Functions

Trubnikov functions are defined by

$$t^n_\nu(z,\rho) = (k\tilde{u})^{n+1}\int_0^\infty d\xi\, \xi^n\, \frac{K_\nu\big(r(\xi)\big)}{r^\nu(\xi)}, \tag{A.2.5}$$

with $r(\xi)$ given by (2.4.10), and where the power of $k\tilde{u}$ is included so that the integral is dimensionless. They satisfy the recursion relations

$$t_{\nu+1}^{n+1}(z,\rho) = \frac{i\rho z^2}{1-z^2}\, t_{\nu+1}^{n}(z,\rho) + \frac{z^2}{1-z^2} \begin{cases} \dfrac{K_\nu(\rho)}{\rho^\nu} & \text{for } n=0, \\ n t_\nu^{n-1}(z,\rho) & \text{for } n>0, \end{cases} \tag{A.2.6}$$

$$\frac{\partial t_\nu^n(z,\rho)}{\partial\rho} = -i\rho\, t_{\nu+1}^n(z,\rho) - i\, t_{\nu+1}^{n+1}(z,\rho). \tag{A.2.7}$$

Two further identities are

$$t_{\nu+1}^{n}(z,\rho) = -\frac{1-z^2}{\rho}\,\frac{\partial t_\nu^n(z,\rho)}{\partial\rho} + \frac{iz^2}{\rho} \begin{cases} \dfrac{K_\nu(\rho)}{\rho^\nu} & \text{for } n=0, \\ n\, t_\nu^{n-1}(z,\rho) & \text{for } n>0, \end{cases} \tag{A.2.8}$$

$$t_{\nu+1}^{n+2}(z,\rho) = z^3\,\frac{\partial t_\nu^n(z,\rho)}{\partial z}. \tag{A.2.9}$$

The relation to $T(z,\rho)$ follows from

$$t_0^0(z,\rho) = \frac{iz}{2}\frac{\partial T(z,\rho)}{\partial\rho} = \frac{i}{2}\left[2K_1(\rho) + \frac{(1-z^2)}{\rho}\, T'(z,\rho)\right], \tag{A.2.10}$$

$$t_1^0(z,\rho) = -\frac{iz}{2\rho}\, T(z,\rho). \tag{A.2.11}$$

The functions for higher n are generated from these using (A.2.6).

A.2.3 Shkarofsky and Dnestrovskii Functions

The generalized Shkarofsky functions are defined by (2.5.28) for real q, integer $r \geq 0$ and complex z, a with $\mathrm{Im}\,(z-a) > 0$ by

$$\begin{aligned} \mathcal{F}_{q,r}(z,a) &= -i\int_0^\infty dt\,\frac{(it)^r}{(1-it)^q}\exp\left[izt - \frac{at^2}{1-it}\right] \\ &= -ie^{-a}\int_0^\infty dt\,\frac{(it)^r}{(1-it)^q}\exp\left[i(z-a)t + \frac{a}{1-it}\right]. \end{aligned} \tag{A.2.12}$$

The definition is extended to $\mathrm{Im}\,(z-a) < 0$ by analytic continuation. Generalized Dnestrovskii functions are defined by (2.5.34), viz. $F_{q,r}(z) = \mathcal{F}_{q,r}(z,0)$. The usual Shkarofsky functions, $\mathcal{F}_q(z,a) = \mathcal{F}_{q,0}(z,a)$, and Dnestrovskii functions, $F_q(z) = F_{q,0}(z)$, are the special cases $r = 0$.

The Shkarofsky functions and the Dnestrovskii functions are related by an expansion in modified Bessel functions:

$$\mathcal{F}_q(z,a) = \sum_{s=-\infty}^{\infty} e^{-2a}\, I_s(2a)\, F_{q-s}(z). \tag{A.2.13}$$

Recursion Relations and Differential Equations

Recursion relations satisfied by the Shkarofsky functions are

$$a\mathcal{F}_{q-2}(z,a) = 1 + (a-z)\mathcal{F}_q(z,a) - q\mathcal{F}_{q+1}(z,a), \tag{A.2.14}$$

$$\mathcal{F}_q'(z,a) = \mathcal{F}_q(z,a) - \mathcal{F}_{q-1}(z,a), \tag{A.2.15}$$

$$\mathcal{F}_q''(z,a) = \mathcal{F}_q(z,a) - 2\mathcal{F}_{q-1}(z,a) + \mathcal{F}_{q-2}(z,a), \tag{A.2.16}$$

where a prime denotes a derivative with respect to z. Eliminating $\mathcal{F}_{q-1}(z,a)$ and $\mathcal{F}_{q-2}(z,a)$ between these gives a second order differential equation satisfied by the Shkarofsky functions:

$$(a-z)\mathcal{F}_q''(z,a) - [2(a-z)-q-2]\mathcal{F}_q'(z,a) - (z+q-2)\mathcal{F}_q(z,a) + 1 = 0. \tag{A.2.17}$$

Recursion relations for the Dnestrovskii functions follow from (A.2.14) and (A.2.15) for $a = 0$:

$$(q-1)F_q(z) = 1 - zF_{q-1}(z), \tag{A.2.18}$$

$$F_q'(z) = F_q(z) - F_{q-1}(z). \tag{A.2.19}$$

Eliminating $F_{q-1}(z)$ between these gives a first order differential equation satisfied by the Dnestrovskii functions:

$$zF_q'(z) = (z+q-1)F_q(z) - 1. \tag{A.2.20}$$

The function $F_q(z)$ also satisfies (A.2.17) with $a = 0$. Equation (A.2.19) integrates to give

$$F_q(z) = z^{q-1}e^z\Gamma(1-q,z), \qquad \Gamma(q,z) = \int_z^{\infty} d\zeta\, \zeta^{q-1}e^{-\zeta}, \tag{A.2.21}$$

where $\Gamma(q,z)$ is the incomplete gamma function.

Limiting Cases

The expansion of the Dnestrovskii functions for small arguments z follows from (A.2.21) and the relevant expansion of the incomplete gamma function:

$$F_q(z) = z^{q-1} e^z \Gamma(1-q) - \sum_j^{\infty} \frac{z^j \Gamma(1-q)}{\Gamma(j+q-1) j!}$$
$$= z^{q-1} e^z \Gamma(1-q) - e^z \sum_j^{\infty} \frac{(-z)^j \Gamma(1-q)}{\Gamma(j+2-q)}. \tag{A.2.22}$$

For real, positive z there is an expansion in generalized Laguerre polynomials:

$$F_q(z) = \sum_{j=0}^{\infty} \frac{L_j^{(1-q)}(z)}{j+1}. \tag{A.2.23}$$

For large argument, $|z| \gg 1$, the limit

$$F_q(z) \sim \sum_{j=0}^{\infty} (-1)^j z^{-1-j} \Gamma(q+j) \tag{A.2.24}$$

applies for $\arg(z) < 3\pi/2$.

Half-Integer *q*

In evaluating (2.5.27) in terms of Shkarofsky functions, the function and its derivative with $q = 5/2$ appear. The expansion (2.5.38) then leads to Dnestrovskii functions with half-integer q. For q a positive half-integer, the Dnestrovskii functions are expressible in terms of the plasma dispersion function

$$Z(y) = \pi^{-1/2} \int_{-\infty}^{\infty} dt \, \frac{e^{-t^2}}{t-y} = -\frac{\phi(y)}{y} + i\pi^{1/2} e^{-z^2}, \tag{A.2.25}$$

The relevant form is

$$\Gamma(q) F_q(z) = \sum_{j=0}^{q-3/2} (-z)^j \Gamma(q-1-j) + \pi^{1/2} (-z)^{q-3/2} [i z^{1/2} e^z Z(i z^{1/2})]. \tag{A.2.26}$$

Expansions for small and large arguments are

$$\Gamma(q) F_q(z) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} (-z)^j \Gamma(q-1-j) - i\pi(-z)^{q-1} e^z & \text{for } |z|^2 \ll 1, \\ \displaystyle -\sum_{j=0}^{\infty} \Gamma(q+j)(-z)^{-1-j} - i\sigma\pi(-z)^{q-1} e^z & \text{for } |z| \gg 1, \end{cases} \tag{A.2.27}$$

with $\sigma = 0$ for $\arg z < \pi$, $\sigma = 1$ for $\arg z = \pi$ and $\sigma = 2$ for $\pi < \arg z < 2\pi$.

A.3 Dirac Algebra

In this section some results associated with the properties of Dirac matrices are summarized.

A.3.1 Definitions and the Standard Representation

The Dirac matrices are defined to satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \tag{A.3.1}$$

where the unit Dirac matrix is implicit on the right hand side. The Dirac Hamitonian is

$$\hat{H} = \boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}} + \beta m, \qquad \boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}, \quad \beta = \gamma^0. \tag{A.3.2}$$

The requirement that the Dirac Hamiltonian be self-adjoint implies

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \tag{A.3.3}$$

Standard Representation

The specific choice for the Dirac matrices used here is referred to as the standard representation. It corresponds to

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{A.3.4}$$

A convenient way of writing these and other 4×4 matrices is in terms of block matrices. Let $\mathbf{0}$ and $\mathbf{1}$ be the null and unit 2×2 matrices. One writes

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}, \qquad \rho_x = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

$$\rho_y = \begin{pmatrix} \mathbf{0} & -i\mathbf{1} \\ i\mathbf{1} & \mathbf{0} \end{pmatrix}, \qquad \rho_z = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \tag{A.3.5}$$

where the 2×2 matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.3.6}$$

are the usual Pauli matrices. In this representation one has

$$\gamma^\mu = [\rho_z, i\rho_y \boldsymbol{\Sigma}], \qquad \boldsymbol{\alpha} = \rho_x \boldsymbol{\sigma}, \qquad \beta = \rho_z. \tag{A.3.7}$$

Dirac Matrices $\sigma^{\mu\nu}$ and γ^5

Two additional Dirac matrices that play an important role in the theory are

$$\sigma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu, \gamma^\nu], \tag{A.3.8}$$

which plays the role of a spin angular momentum, and

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{A.3.9}$$

which satisfies the relations

$$\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0, \qquad (\gamma^5)^2 = 1, \qquad (\gamma^5)^\dagger = \gamma^5. \tag{A.3.10}$$

One also has

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = -i\epsilon^{\mu\nu\rho\sigma}. \tag{A.3.11}$$

In the standard representation one has $\gamma^5 = -\rho_x$. The spin 4-tensor $\sigma^{\mu\nu}$, defined by (A.3.8), has components

$$\sigma^{\mu\nu} = \begin{pmatrix} 0 & \alpha_x & \alpha_y & \alpha_z \\ -\alpha_x & 0 & -i\sigma_z & i\sigma_y \\ -\alpha_y & i\sigma_z & 0 & -i\sigma_x \\ -\alpha_z & -i\sigma_y & i\sigma_x & 0 \end{pmatrix}. \tag{A.3.12}$$

A.3.2 Basic Set of Dirac Matrices

There are 16 independent 4×4 matrices and for the Dirac matrices it is sometimes convenient to choose a set of 16 basis vectors. A specific choice of 16 independent matrices is the set

$$\gamma^A = \Big[1,\ \gamma^\mu,\ i\sigma^{\mu\nu},\ i\gamma^\mu\gamma^5,\ \gamma^5\Big]. \tag{A.3.13}$$

This choice involves a scalar and a pseudo scalar $(1,\gamma^5)$, a 4-vector and a pseudo 4-vector $(\gamma^\mu, i\gamma^\mu\gamma^5)$ and an antisymmetric second rank 4-tensor $(\sigma^{\mu\nu})$. These have 1, 1, 4, 4, and 6 components, respectively. This set is chosen such that the analogous set, γ_A with indices down, $\gamma_A = [1,\ \gamma_\mu, i\sigma_{\mu\nu}, i\gamma_\mu\gamma^5, \gamma^5]$ satisfy

$$\gamma^A\gamma_A = 1 \quad \text{(no sum)}, \qquad \gamma^A\gamma_B = \delta^A_B. \tag{A.3.14}$$

The expansion of an arbitrary Dirac matrix, O say, in this basis gives

$$O = \sum_A c_A\gamma^A, \qquad c_A = \tfrac{1}{4}\mathrm{Tr}\,[\gamma_A O]. \tag{A.3.15}$$

Traces of Products of γ-Matrices

The traces of products of γ-matrices are important in detailed calculations in QED. Consider

$$T^{\alpha_1\alpha_2\ldots\alpha_n} = \mathrm{Tr}\left(\gamma^{\alpha_1}\gamma^{\alpha_2}\ldots\gamma^{\alpha_n}\right). \tag{A.3.16}$$

The trace of γ^μ is zero, as are the traces of $\sigma^{\mu\nu}$, $\gamma^\mu\gamma^5$ and γ^5. The trace of a product of an odd number of γ-matrices is also zero: $T^{\alpha_1\alpha_2\ldots\alpha_n} = 0$ for n odd. The trace of a product of two γ-matrices is nonzero. This trace is evaluated as follows. First the invariance of the trace of a product of matrices under cyclic permutations of the matrices implies $T^{\mu\nu} = T^{\nu\mu}$. The trace of (5.1.1) implies $T^{\mu\nu} = 4g^{\mu\nu}$, where the factor of 4 arising from the trace of the unit 4×4 matrix. Using the invariance of the trace under cyclic permutations and (5.1.1) allows one to evaluate the traces (A.3.16) for all even n. One finds

$$T^{\mu\nu} = 4g^{\mu\nu}, \qquad T^{\mu\nu\rho\sigma} = 4\Big[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\Big], \tag{A.3.17}$$

$$T^{\mu\nu\rho\sigma\alpha\beta} = 4\left[g^{\mu\nu}T^{\rho\sigma\alpha\beta} - g^{\mu\rho}T^{\nu\sigma\alpha\beta} + g^{\mu\sigma}T^{\nu\rho\alpha\beta} - g^{\mu\alpha}T^{\nu\rho\sigma\alpha} + g^{\mu\beta}T^{\nu\rho\sigma\alpha}\right], \tag{A.3.18}$$

and so on.

References

1. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965)
2. B.B. Godfrey, B.S. Newberger, K.A. Taggart, IEEE Trans. Plasma Phys. **PS-3**, 60, 68 (1975)
3. I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965)

4. J.J. Quinn, S. Rodriguez, Phys. Rev. **128**, 2487 (1962)
5. M. Rotham, Q. J. Mech. Appl. Math. **7**, 379 (1954)
6. A.A. Sokolov, I.M. Ternov, *Synchrotron Radiation* (Pergamon Press, Oxford, 1968)
7. A.A. Sokolov, I.M. Ternov, *Radiation from Relativistic Electrons* (AIP, New York, 1986)
8. G.N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944)

Index

D. Melrose, *Quantum Plasmadynamics: Magnetized Plasmas*, Lecture Notes in Physics 854, DOI 10.1007/978-1-4614-4045-1,

Q

R

S

T

U

V