Appendix A
Calculating the Mean and Variance of the Error Signal in Mid-Tread and 1-Bit Quantizers

A.1 Mid-Tread Quantizer

In the digital mid-tread quantizer considered in Section 3.2.1 with the characteristic shown in Fig. A.1, the error signal in the no overload range is in the range $[-\frac{M}{2} + 1, \frac{M}{2}]$, as shown in Fig. A.2. If the error signal is uniformly distributed within this range, as shown in Fig. A.3, we can calculate its mean and variance as follows.

\[ e_q[n] \]

Fig. A.1 The input–output characteristic of a multi-level digital mid-tread quantizer with 5 output levels

Fig. A.2 Illustration of the error signal $e_q$ for a mid-tread quantizer like that shown in Fig. A.1
The mean is defined by

\[ M_e = E[e_q] = \sum_{k=\frac{-M}{2}+1}^{\frac{M}{2}} k P(e_q = k) \]

\[ = \frac{1}{M} \sum_{k=\frac{-M}{2}+1}^{\frac{M}{2}} k \]

\[ = \frac{1}{2}, \quad (A.1) \]

where \( P(e_q = k) \) denotes the probability that \( e_q = k \).

In order to find the variance, we first calculate \( E[e_q^2] \):

\[ E[e_q^2] = \sum_{k=\frac{-M}{2}+1}^{\frac{M}{2}} k^2 P(e_q = k) \]

\[ = \frac{1}{M} \sum_{k=\frac{-M}{2}+1}^{\frac{M}{2}} k^2 \]

\[ = \frac{1}{M} \left( \frac{M}{2} \right)^2 + 2 \frac{1}{M} \sum_{k=0}^{M-1} k^2 \]

\[ = \frac{M}{4} + 2 \frac{1}{M} \left( \frac{M}{2} - 1 \right) \left( \frac{M}{2} \right) (M - 1) \frac{6}{6} \]

\[ = \frac{M^2}{12} + 2 \frac{1}{12}. \quad (A.2) \]

In deriving (A.2), we have used the identity:
\[
\sum_{k=0}^{M-1} k^2 = \frac{(M-1)(M)(2M-1)}{6}.
\]

We can use Eqs. (A.1) and (A.2) to find the variance, which is given by
\[
\sigma_{ee}^2 = E[e_q^2] - (E[e_q])^2 = \frac{M^2 - 1}{12}.
\]

\section*{A.2 1-Bit Quantizer}

We consider the quantizer shown in Fig. 2.10c. The quantization error signal \(e_q\) is in the range \(\{0, -\frac{1}{M}, \ldots, -\frac{M-1}{M}\}\). Assuming a uniform distribution for \(e_q\) one can obtain:
\[
E[e_q] = -\frac{1}{2} \left( \frac{M-1}{M} \right).
\]

For large \(M\), (A.5) is approximately \(-0.5\).

In order to determine the variance, we calculate \(E[e_q^2]\):
\[
E[e_q^2] = \frac{1}{M^2} \sum_{k=0}^{M-1} k^2 P(e_q = k)
= \frac{1}{M^3} \sum_{k=0}^{M-1} k^2
= \frac{(M-1)(2M-1)}{6M^2}.
\]

Using (A.6) and (A.5), we obtain the variance:
\[
\sigma_{e_q}^2 = E[e_q^2] - (E[e_q])^2 = \frac{M^2 - 1}{12M^2}.
\]

For large \(M\), (A.7) is approximately \(\frac{1}{12}\).
Appendix B  
Mathematical Analysis of the HK-MASH DDSM

B.1 Proof of the Cycle Length for the Modified First Order Delta Sigma Modulator

Consider the modified first order EFM in Fig. 4.1a which has the maximum cycle length for all constant digital inputs and for all initial conditions. For this structure, we have that

$$e[n] = v[n] \mod M, \quad \text{(B.1)}$$

where $M = 2^{n_0}$ and $n_0$ is the accumulator word length. Considering Fig. 4.1a, note that we can write (B.1) as:

$$e[n] = (x[n] + e[n-1] + ay[n-1]) \mod M. \quad \text{(B.2)}$$

Expanding (B.2) with its indices, we have:

$$e[0] = (x[0] + s[0]) \mod M, \quad \text{(B.3)}$$

where $s[0]$ is the initial condition. Continuing with the other indices, we obtain:

$$e[1] = (x[1] + e[0] + ay[0]) \mod M, \quad \text{(B.4)}$$

and for the indices $(n-1)$ and $n$ we have:

$$e[n-1] = (x[n-1] + e[n-2] + ay[n-2]) \mod M, \quad \text{(B.6)}$$
$$e[n] = (x[n] + e[n-1] + ay[n-1]) \mod M. \quad \text{(B.7)}$$

If we replace the value of each $e[i]$ with its value from the previous equation and recall that:

$$(a + b \mod M) \mod M = (a + b) \mod M, \quad \text{(B.8)}$$
for positive integers $a$ and $b$ [5], we can write $e[n]$ as:

$$e[n] = \left( s[0] + \sum_{k=0}^{n} x[k] + a \sum_{k=0}^{n-1} y[k] \right) \mod M,$$

(B.9)

which, in the case of a constant dc input $X$, gives:

$$e[n] = \left( s[0] + (n + 1)X + a \sum_{k=0}^{n-1} y[k] \right) \mod M.$$

(B.10)

Before starting the procedure of determining the period of (B.9), we find a relationship between $x[n]$ and $y[n]$. By looking at Fig. 4.1a, we write:

$$e[n] = v[n] - M y[n],$$

$$= ay[n - 1] + x[n] + e[n - 1] - M y[n],$$

(B.11)

which we can rewrite as:


(B.12)

Expanding (B.12) with its indices yields:


(B.13)


(B.14)

$$\vdots$$


(B.15)

If we add the terms on the left side together and do the same for the terms on the right side of the above equations, we obtain:

$$\sum_{k=1}^{n} M y[k] - a \sum_{k=0}^{n-1} y[k] = \sum_{k=1}^{n} x[k] + e[0] - e[n].$$

(B.16)

If the system is periodic in the steady state with period $N$, we have:

$$\sum_{k=1}^{N} M y[k] - a \sum_{k=0}^{N-1} y[k] = \sum_{k=1}^{N} x[k] + e[0] - e[N],$$

(B.17)

where $e[0] = e[N]$, and
\[
\sum_{k=1}^{N} y[k] = \sum_{k=0}^{N-1} y[k]. \tag{B.18}
\]

Therefore, in the steady state, we obtain:

\[
(M - a) \sum_{k=1}^{N} y[k] = \sum_{k=1}^{N} x[k]. \tag{B.19}
\]

Rearranging (B.19) gives

\[
\sum_{k=1}^{N} y[k] = \frac{1}{M - a} \sum_{k=1}^{N} x[k]. \tag{B.20}
\]

For a constant DC input \(X\), we have:

\[
\sum_{k=1}^{N} y[k] = \frac{NX}{M - a}. \tag{B.21}
\]

Now we return to find \(N\), the period of the system. In order for (B.10) to be periodic with period \(N\), we require that:

\[
e[n] = e[n + N]. \tag{B.22}
\]

Substituting from (B.10) for \(e[n]\) and \(e[n + N]\) in (B.22), we obtain:

\[
\left( NX + a \sum_{k=1}^{N} y[k] \right) \mod M = 0, \tag{B.23}
\]

which is independent of the initial condition, \(s[0]\). Using (B.21) in (B.23), we obtain:

\[
\left( NX + a \frac{NX}{M - a} \right) \mod M = 0,
\]

\[
\left( \frac{NX}{M - a} \right) M \mod M = 0. \tag{B.24}
\]

If we choose \(M - a\) to be prime and if \(0 < X < (M - a)\), then the minimum non-zero solution for \(N\) such that the above equation is valid is:

\[
N = M - a, \text{ where } N \text{ is prime}. \tag{B.25}
\]

This solution is correct, regardless of the input \(X\) and the initial condition.
The difference between $M$ and the closest prime number to it is $a$, the values of which, for different modulator word lengths $n_0$ ($5 \leq n_0 \leq 25$), are shown in Table 4.1.

**B.2 Proof of the Cycle Length for the HK-MASH Modulator**

We extend the results of the previous section to find the cycle length of the higher order HK-MASH modulator shown in Fig. 4.3. We have proven that, for the first order modulator, the cycle length for all digital inputs and for all initial conditions is $N$ [see (B.25)], where $N$ is prime. This imposes a restriction on the form of the period of the higher order modulators. If the higher order modulator is periodic with a period $N_l$, then we require that

$$N_l = KN,$$  \hspace{1cm} (B.26)

where $N$ is prime and is determined by (B.25), and $K$ is an integer.

**B.2.1 Modified Second Order Modulator**

We use Eq. (B.20) to find the cycle length $N_2$ of a second order modulator. Considering Fig. 4.3, and using (B.25) and (B.26), we write:

$$\sum_{k=1}^{N_2} y_2[k] = \left( \frac{1}{M-a} \right) \sum_{k=1}^{N_2} e_1[k],$$

$$\sum_{k=1}^{KN} y_2[k] = \left( \frac{K}{M-a} \right) \sum_{k=1}^{N} e_1[k],$$

$$= \left( \frac{K}{M-a} \right) \sum_{k=1}^{N} e_1[k],$$

$$= \left( \frac{K}{N} \right) \sum_{k=1}^{N} e_1[k].$$  \hspace{1cm} (B.27)

The only solution for $K$ so that the right side is an integer is $N$, provided that the greatest common divisor of $N$ and $\sum_{k=1}^{N} e_1[k]$ is 1. This condition is true; otherwise the solution for $K$ is not unique. With this solution for $K$, (B.26) yields:

$$N_2 = N^2,$$  \hspace{1cm} (B.28)

where $N$ is prime and is determined by (B.25).
B.2.2 **Modified Third Order and Higher Order Modulators**

The solution found in the previous section is used in this section to find the cycle length of the third stage. The solution for $N_3$ should be in the form of:

$$N_2 = KN^2,$$  \hspace{1cm} (B.29)

where $N$ is prime and $K$ is an integer. Rewriting (B.20) for the third stage and repeating the same procedure performed for the second stage, we have:

$$\sum_{k=1}^{KN^2} y_3[k] = \frac{K}{N} \sum_{k=1}^{N^2} e_2[k].$$  \hspace{1cm} (B.30)

With the same reasoning as for the second stage, the solution for $K$ is $N$ and therefore we have:

$$N_3 = N^3,$$  \hspace{1cm} (B.31)

where $N$ is prime and determined by (B.25).

With the same reasoning as for the second and third stage cases, the solution for the $l$th order modulator is:

$$N_l = N^l,$$  \hspace{1cm} (B.32)

where $N$ is prime and determined by (B.25). Adding an extra stage to this modified structure increases the cycle length by a factor of $N$, independently of the initial condition and the constant input.
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