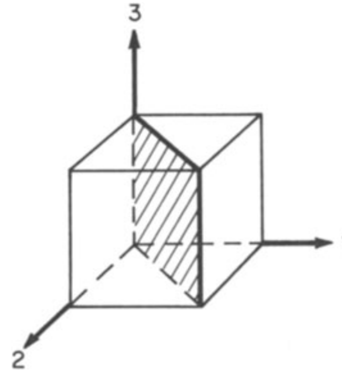
Appendix

1. The Volume Element on a Manifold

As we have argued in 3.4, the volume element on a manifold $M \subseteq \mathbb{R}^k$ of dimension $r \leq k$, given by $\vec{\psi}: \mathbb{R}^r \to \mathbb{R}^k$ say, is determined by finding the r-dimensional volume of the parallelopiped defined by the explicit basis $\vec{\psi}_j$, $j = 1, 2, \ldots, r$, of the tangent space to M. Using the law of cosines we found the formula 3.4(5) for the volume element of a surface $(r = 2)$, and we asserted the general formula in 3.4(6). We shall here establish the latter. Unfortunately the demonstration is lengthy and not entirely elementary. However, since our main applications of integration (the integral theorems of 5.3) occur in $\mathbb{R}^3$, the general formula and the ensuing long argument may be treated as optional material.

Evidently the problem may be given the following purely geometric statement: find the r-dimensional volume of the parallelopiped determined by r vectors $u_i \in \mathbb{R}^k$, $i = 1, 2, \ldots, r$, $r \leq k$; and the assertion is that the square of this volume is the sum of the squares of the $\binom{k}{r}$ minors of the r-by-k matrix whose rows are the vectors u_i. Put this way, the assertion appears as a generalization of the Pythagorean theorem. For $r = 1$ it is exactly the k-dimensional form of that theorem: the parallelopiped is a single vector, its "volume" is the length of that vector, and the square of this length is the sum of the squares of the components of



the vector, by the Pythagorean theorem. Look next at the 2-dimensional figure in $\mathbb{R}^3$ determined by $u_1 = (1,1,0)$ and $u_2 = (0,0,1)$. It is the shaded square of the figure. The perpendicular projections (or shadows) cast upon the three coordinate planes by this figure are: unit squares on (1,3) and (2,3), and a line on (1,2), as one sees in the figure. The 2-dimensional volumes (areas) of these shadows are respectively: 1, 1, and 0; the area of the figure itself is $1 \cdot \sqrt{2} = \sqrt{2}$ (since one of its sides is the diagonal of a unit square); and $1^2 + 1^2 = (\sqrt{2})^2$. That is, the sum of the squares of the areas of the shadows of the figure is the square of the area of the figure. This example displays the general scheme: there are $\binom{k}{r}$ "coordinate r-planes", so to speak, upon which the r-dimensional parallelopiped determined by r vectors in $\mathbb{R}^k$ casts shadows, and the assertion is that the r-dimensional volumes of these shadows are "Pythagorean components" of the volume of the parallelopiped.

For the demonstration we need a bit of notation. Let $[u] = (u_1, u_2, \ldots, u_r)$ denote the set of vectors in $\mathbb{R}^k$ to be considered, their components being indicated by superscripts; let $(i) = (i_1, i_2, \ldots, i_r)$ denote a set of indices, $1 \leq i_1 < \ldots < i_r \leq k$; let

$$M_r^{(i)}[u] = \det((u_m^{i_n})), \tag{1}$$

$m = 1,2,\ldots,r$ and $i_n \in (i)$, denote the r-by-r minor got by choosing the $i_1, i_2, \ldots, i_r$ columns from the r-by-k matrix of the u_i; let

$$M_r[u] = \{M_r^{(i)}[u]\} \tag{2}$$

denote the sequence of the $\binom{k}{r}$ minors (1) arranged in their lexicographic order; let

$$(M_r[u], M_r[v]) = \sum_{(i)} M_r^{(i)}[u] \cdot M_r^{(i)}[v] \tag{3}$$

denote the inner product in the space of $\binom{k}{r}$-tuples; finally, let $P[u]$ denote the parallelopiped determined by the u_i, and $V_r[u]$ its r-dimensional volume. In these terms our assertion is:

$$V_r[u] = \|M_r[u]\| . \tag{4}$$

If A is an orthogonal k-by-k matrix, we may apply it to $u_1, u_2, \ldots, u_r$, and we denote the resulting set $Au_1, Au_2, \ldots, Au_r$ by $[Au]$. We then have a new sequence $M_r[Au]$ of minors. It is plausible that the inner product (3) is preserved under this action of A,

$$(M_r[Au], M_r[Av]) = (M_r[u], M_r[v]) , \tag{5}$$

a fact we shall use without proof.

We proceed by induction. We know (4) holds for $r = 1$ and $r = 2$. Suppose it holds for all $r \leq s < k$. Given $s+1$ vectors $u_1, u_2, \ldots, u_s, w$, by a process of successive orthogonalization [1] we can express w as

(1) See Schreier-Sperner [5], pp. 140-141. In recent texts this construction is called the Gram-Schmidt process.

$w = a+b$, where $b \varepsilon P(u_1, u_2, \ldots, u_s)$ and a is orthogonal to u_i, $i = 1,2,\ldots,s$. We may (and do) assume that $u_1, u_2, \ldots, u_s$ are linearly independent. It then follows that b is dependent upon them, so that $M^{(j)}_{s+1}(u_1,\ldots,u_s,b) = 0$ for every $(j) = (j_1, j_2, \ldots, j_{s+1})$, $1 \leq j_1 < \ldots < j_{s+1} \leq k$, whence

$$M^{(j)}_{s+1}(u_1,u_2,\ldots,u_s,w) = M^{(j)}_{s+1}(u_1,u_2,\ldots,u_s,a) , \tag{6}$$

by the multilinearity of determinants. For the volumes we have the corresponding reduction

$$V_{s+1}(u_1,\ldots,u_s,w) = V_{s+1}(u_1,\ldots,u_s,a) , \tag{7}$$

by the meaning of volume. Since a is orthogonal to u_i, $i = 1,2,\ldots,s$, we have $V_{s+1}(u_1,\ldots,u_s,a) = V_s(u_1,\ldots,u_s) \cdot \|a\|_k$, where $\|a\|_k$ denotes the length of a in $\mathbb{R}^k$. By the inductive hypothesis we now have

$$V_{s+1}(u_1,\ldots,u_s,w) = \|M_s[u]\| \cdot \|a\|_k . \tag{8}$$

Let A be an orthogonal transformation such that $Aa = (\lambda,0,0,\ldots,0)$, where $\lambda = \|a\|_k$. Then, because A is orthogonal, Aa is orthogonal to $Au_1, Au_2, \ldots, Au_s$. Now $V_{s+1}(u_1,u_2,\ldots,u_s,w) = V_{s+1}(Au_1,Au_2,\ldots,Au_s,Aw)$ by the meaning of volume. Since Au_i, Aw are exactly as general as u_i, w we have, by (8),

$$V_{s+1}(Au_1,\ldots,Au_s,Aw) = \|M_s[Au]\| \cdot \|Aa\|_k . \tag{9}$$

Since all Au_i have vanishing first coefficients (by virtue of being orthogonal to $Aa = (\lambda,0,\ldots,0)$), the (non-vanishing) minors of the (s+1)-by-k matrix of the Au_i and Aa consist of the (non-vanishing) minors of the s-by-k matrix of the Au_i multiplied by $\lambda = \|Aa\|_k$. Therefore the right side of (9) equals $\|M_{s+1}(Au_1,Au_2,\ldots,Au_s,Aa)\|$. At this point we invoke the invariance (5), by which we may remove A from the last expression. Then, taking account of (6), and retracing our steps back to (7), we have shown that

$$V_{s+1}(u_1,\ldots,u_s,w) = \|M_{s+1}(u_1,\ldots,u_s,w)\|,$$

which establishes the assertion QED.

2. The Algebra of Forms

The algebra of forms rests upon the axioms (1), (5), and (6) of 2.3. These assert respectively the anti-commutation of the wedge product, its homogeneity with respect to $\mathbb{S}$ multiplication, and its distributivity with respect to form addition. To show these axioms consistent it suffices to establish this in the particular case of $\mathbb{R}^3$. In 5.1(6) we have shown how to put forms and vector fields into one-to-one correspondence in such way that the wedge and vector products correspond. Since the vector product has the properties corresponding to the axioms in question, the latter are consistent.

3. A Remark on Curl^2

Among the higher differentiations (5.2) the only ones which carry scalar fields to scalar fields are the powers Δ^n of Δ (5.2(7)). These may be applied componentwise to a vector field, and this yields new differentiations carrying vector fields to vector fields. These new differentiations may then be combined with the standard vector differentiations Curl^k, $\mathrm{Grad}\,\Delta^\ell\,\mathrm{Div}$ (5.2(7)), with k,ℓ so that the orders match. The lowest-order such combination is noteworthy:

$$\mathrm{Curl}^2\vec{f} = \mathrm{Grad}\,\mathrm{Div}\,\vec{f} - \Delta\vec{f}. \tag{1}$$

The verification is straightforward: we interprete $\vec{f}$ formwise as needed to carry out the operations; then interprete the results as vector fields; and compare components. For the left side we start with $\vec{f} = \sum a^i dx_i$. Then

$$\begin{gathered} \mathrm{Curl}\,\vec{f} = \sum A^i dx_i , \\ A^1 = a^3_2 - a^2_3, \qquad A^2 = a^1_3 - a^3_1, \qquad A^3 = a^2_1 - a^1_2; \end{gathered} \tag{2}$$

and the first field-component of $\mathrm{Curl}^2\vec{f}$ is

$$\begin{aligned} (\mathrm{Curl}^2\vec{f})^1 &= A^3_2 - A^2_3 \\ &= a^2_{12} - a^1_{22} - a^1_{33} + a^3_{13} . \end{aligned} \tag{3}$$

For the right side we have

$$(\text{Grad Div } \vec{f})^1 = a^1_{11} + a^2_{21} + a^3_{31} , \tag{4}$$

$$(\Delta \vec{f})^1 = a^1_{11} + a^1_{22} + a^1_{33} . \tag{5}$$

By the equality of mixed partials the difference (4) less (5) is (3), QED.

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