APPENDIX

SOLUTION OF QUASILINEAR FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

In Part 2 of this book the invariance of a partial differential equation under a one-parameter Lie group of transformations leads to the consideration of an invariant surface condition for the construction of a similarity (invariant) solution. In the case of two independent variables (x,t) the invariant surface condition corresponds to a quasilinear first order partial differential equation of the form

$$\xi(x,t,u)\,\frac{\partial u}{\partial x} + \tau(x,t,u)\,\frac{\partial u}{\partial t} = \eta(x,t,u) \qquad \text{(A-1)}$$

where $\{\xi,\tau,\eta\}$ are given functions (the infinitesimals of the Lie group) of $\{x,t,u\}$. The solution of (A-1) for the unknown $u(x,t)$ leads to a functional form for u. We now briefly discuss how to solve formally equations of the form (A-1).

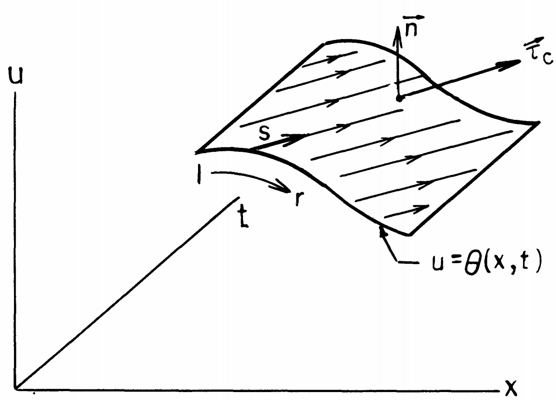

Figure A-1.

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(1) R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II, Interscience, 1962, Chapter II.

Consider, at first, an arbitrary surface in space given by

$$u = \theta(x,t) . \tag{A-2}$$

This will later be identified as a solution surface connected to (A-1). We can write the condition that a curve in (u,x,t) space lies in the surface $u = \theta(x,t)$ (see Fig. A-1) as follows:

Let s = parameter along the curve in space so that

$$x = x(s),\ t = t(s),\ u = u(s) \quad \text{defines the curve.} \tag{A-3}$$

Then, the tangent vector to this curve is given by

$$\vec{\tau}_c = \vec{i}_x \frac{dx}{ds} + \vec{i}_t \frac{dt}{ds} + \vec{i}_u \frac{du}{ds} , \quad \vec{i}_x, \vec{i}_t, \vec{i}_u = \text{unit vectors.} \tag{A-4}$$

The normal direction to the surface $u = \theta(x,t)$ can be written easily if we imagine the surface to lie in the family of surfaces

$$S(x,t,u) = u - \theta(x,t) = \text{const.} \tag{A-5}$$

Then the unit normal to $S = 0$ is given by

$$\vec{n} = \frac{\nabla S}{|\nabla S|} = \frac{1}{|\nabla S|} \{-\vec{i}_x\theta_x - \vec{i}_t\theta_t + \vec{i}_u\}. \tag{A-6}$$

The condition that the curve $x(s)$, $t(s)$, $u(s)$ lie in the surface is thus $\vec{n}\cdot\vec{\tau}_c = 0$ or

$$\theta_x \frac{dx}{ds} + \theta_t \frac{dt}{ds} = \frac{du}{ds} . \tag{A-7}$$

Thus, if we define a family of curves (characteristics) locally by the differential equations

$$\left.\begin{aligned} \frac{dx}{ds} &= \xi(x,t,u) \\ \frac{dt}{ds} &= \tau(x,t,u) \\ \frac{du}{ds} &= \eta(x,t,u) \end{aligned}\right\} \text{(characteristic differential equations)} \tag{A-8}$$

then (i) each one-parameter family of such characteristic curves generates a surface which is an integral surface $(u = \theta(x,t))$ of

$$\xi(x,t,u)\,\frac{\partial u}{\partial x} + \tau(x,t,u)\,\frac{\partial u}{\partial t} = \eta(x,t,u) \tag{A-9}$$

and

(ii) conversely, each such integral surface is generated by a one-parameter family of characteristic curves.

The family of solutions of the characteristic differential equations can be represented parametrically by

$$\begin{aligned} x &= x(s,r) \quad \text{where } s = \text{parameter along a characteristic curve,} \\ t &= t(s,r) \quad r = \text{parameter identifying a characteristic} \\ u &= u(s,r) \quad \text{curve} = \text{const. on a characteristic.} \end{aligned} \tag{A-10}$$

The condition that the parameters (r,s) can be eliminated so that we can obtain a solution surface, one value of u for each (x,t), is the condition that the Jacobian

$$J = \frac{\partial(x,t)}{\partial(s,r)} = \frac{\partial x}{\partial s}\frac{\partial t}{\partial r} - \frac{\partial t}{\partial s}\frac{\partial x}{\partial r} \neq 0. \tag{A-11}$$

For our applications we are interested in functional forms or in the so-called general solution of the basic equation (A-1), containing one arbitrary function. One way to approach the general solution is to consider the general initial value problem, that is, to look for the solution surface passing through an arbitrary initial curve given by

$$s = 0 \quad \text{and} \quad x = x_o(r), \quad t = t_o(r), \quad u = u_o(r). \tag{A-12}$$

We assume that the initial curve has nowhere the characteristic direction so that

$$J(0,r) = \xi(0,r)\,\frac{\partial t_o}{\partial r} - \tau(0,r)\,\frac{\partial x_o}{\partial r} \neq 0. \qquad \text{(A-13)}$$

Then a unique solution exists, at least in the neighborhood of the initial curve. Further, it can be shown that for the linear case $\{\xi(x,t),\ \tau(x,t)\}$, $J(s,r) \neq 0$. However, for the non-linear case the solution may fail to exist when $J(s,r) = 0$. This can, for example, correspond to a folding or turning back of the solution surface in (x,t,u). Such cases have not arisen in the problems considered here.

Example (i). Find the general solution of

$$xu_x + tu_t = u. \qquad \text{(A-14)}$$

The characteristic differential equations are

$$\frac{dx}{ds} = x,\ \frac{dt}{ds} = t,\ \frac{du}{ds} = u. \qquad \text{(A-15)}$$

The one-parameter family of integral curves passing through $x_o(r)$, $t_o(r)$, $u_o(r)$ at $s = 0$ is thus

$$x(s,r) = x_o(r)e^s,\ t(s,r) = t_o(r)e^s,\ u(s,r) = u_o(r)e^s. \qquad \text{(A-16)}$$

The parameters may be eliminated by

$$\frac{x}{t} = \frac{x_o(r)}{t_o(r)}\,, \quad e^s = t.$$

Since $r = \text{const.}$ on a characteristic, $\frac{x}{t} = \text{const.}$ on the projections of the characteristics on the plane $u = 0$. r can be eliminated in favor of (x,t). A simple way is to choose $x_o(r) = r$, $t_o(r) = 1$. Then

$$r = \frac{x}{t}\,, \quad e^s = t \qquad \text{(A-17)}$$

and

$$u = tu_o\left(\frac{x}{t}\right), \text{ the general solution,} \qquad \text{(A-18)}$$

where $u_o(\frac{x}{t})$ is an arbitrary (differentiable) function.

A short-hand version of the idea above is used in the calculations of this book. The characteristic differential equations are written free of parameters as

$$\frac{dx}{x} = \frac{dt}{t} = \frac{du}{u} . \tag{A-19}$$

Integration of the first two of (A-19) defines the projections of the characteristic curves on (x,t)

$$\frac{x}{t} = \text{const.} = r. \tag{A-20}$$

Then, the variation of u along the characteristics can be found by integrating either

$$\frac{dx}{x} = \frac{du}{u} \quad \text{or} \quad \frac{dt}{t} = \frac{du}{u} \tag{A-21}$$

along $r = \text{const.}$, remembering that the "constant" of integration is thus an arbitrary function of r. For example,

$$\log t = \log u - \log u_o(r) \tag{A-22}$$

or

$$u(x,t) = tu_o(\frac{x}{t}) \tag{A-23}$$

as before. (Note that the form obtained by using the other equation $u(x,t) = xu_o^*(\frac{x}{t})$ is equivalent.)

Exactly the same idea can be used for any case with linear left hand side. It is sufficient to consider

$$\frac{dx}{\xi(x,t)} = \frac{dt}{\tau(x,t)} = \frac{du}{\eta(x,t,u)} \tag{A-24}$$

and to follow the procedure above.

For the general non-linear case, however, the system of equations

$$\frac{dx}{\xi(x,t,u)} = \frac{dt}{\tau(x,t,u)} = \frac{du}{\eta(x,t,u)} \tag{A-25}$$

must be solved simultaneously. The solution may take an implicit form.

Example (ii). Find the form of the general solution of

$$uu_x + u_t = t. \tag{A-26}$$

The characteristic differential equations are

$$\frac{dx}{u} = \frac{dt}{1} = \frac{du}{t} . \tag{A-27}$$

Integrating the last two shows that

$$u - \frac{t^2}{2} = \text{const.} = r \text{ (say) on a characteristic.} \tag{A-28}$$

Thus, the first two become

$$\frac{dx}{r + \frac{t^2}{2}} = \frac{dt}{1} \tag{A-29}$$

which integrates to

$$x = rt + \frac{t^3}{6} + F(r) \tag{A-30}$$

where F is an arbitrary (differentiable) function of r. (A-30) may be written

$$u = \frac{x}{t} + \frac{t^2}{3} - \frac{1}{t} F(u - \frac{t^2}{2}) \tag{A-31}$$

or equivalently

$$u = \frac{t^2}{2} + G(ut - x - \frac{t^3}{3}). \tag{A-32}$$

In this case the general solution can only be obtained in implicit form. This result is connected with the fact that the solution to the initial value problem may cease to exist when $J = 0$ as s departs from its initial value.

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