

## Appendix 1

In Exercise 8 of Chapter VI the concept of limit point is defined and it is proved that  $C$  is closed  $\Leftrightarrow (x \text{ lp } C \Rightarrow x \in C)$ ; i.e., a set is closed if and only if it contains all of its limit points.

Lemma: If  $C \subset \mathbb{R}^n$  is bounded and infinite then  $C$  has a limit point.

Proof: We must find  $x \in \mathbb{R}^n$  such that each  $B(x, \epsilon) \cap C$  is an infinite set.

Since  $C$  is bounded we can find an  $n$ -dimensional cube  $K_1$  containing  $C$ . Let  $\lambda$  be the length of the sides of  $K_1$ . Divide each edge into two equal parts and cut  $K_1$  into  $2^n$  equal cubes of side length  $\frac{\lambda}{2}$ . Call those cubes  $K_{11}, K_{12}, \dots, K_{12^n}$ . At least one of those must contain infinitely-many points of  $C$ . Choose one such and call it  $K_2$ . Subdivide  $K_2$  into  $2^n$  cubes of side  $\frac{\lambda}{4}$  and choose one of these,  $K_3$ , containing infinitely-many points of  $C$ . Continue this process to get  $K_1 \supset K_2 \supset K_3 \supset \dots$ . Clearly  $K_1 \cap K_2 \cap K_3 \cap \dots$  is a single point  $x \in \mathbb{R}^n$ .

Then  $x \text{ lp } C$ . For, consider any  $B(x, \epsilon)$ . Take  $m$  such that  $\frac{\lambda}{2^m} < \frac{\epsilon}{2}$ . Then

$$K_m \subset B(x, \epsilon)$$

and  $K_m$  contains infinitely-many points of  $C$ . q.e.d.

Theorem: (Proposition 5 of Chapter VI) If  $C \subset \mathbb{R}^n$  is closed and bounded and

$$f: C \rightarrow \mathbb{R}^m$$

is continuous, then  $f(C)$  is closed and bounded.

Proof: Suppose  $f(C)$  is not bounded. Choose  $x_1 \in C$  such that  $y_1 = f(x_1) \notin B(0,1)$ .

Choose  $x_2 \in C$  such that  $y_2 = f(x_2) \notin B(0,2)$ .

$\vdots$

Choose  $x_k \in C$  such that  $y_k = f(x_k) \notin B(0,k)$ . It is easy to prove that  $Y = \{y_1, y_2, \dots\}$  has no limit point and that  $X = \{x_1, x_2, \dots\}$  is an infinite set. Also  $X \subset C$  is bounded. So by the lemma  $X$  has some limit point  $x$ . Then also  $x \in C$ . Since  $C$  is closed  $x \in C$ . But then  $y = f(x)$  must be a limit point of  $Y = f(X)$  by continuity. Thus  $f(C)$  is bounded.

Suppose  $f(C)$  is not closed. Then there exists  $y \in \mathbb{R}^m$  such that  $y \in \text{lp } f(C)$  but  $y \notin f(C)$ . Choose  $x_1 \in C$  such that  $y_1 = f(x_1) \in B(y, 1)$ .

Choose  $x_2 \in C$  such that  $y_2 = f(x_2) \in B(y, \frac{1}{2})$ .

$\vdots$

Choose  $x_k \in C$  such that  $y_k = f(x_k) \in B(y, \frac{1}{k})$ . Clearly  $y$  is the only limit point of  $Y = \{y_1, \dots, y_k, \dots\}$ . Just as before  $X = \{x_1, \dots, x_k, \dots\}$  is infinite and bounded. Let  $x \in \text{lp } X$ . Then  $x \in C$  so  $x \in C$  and  $f(x) \in \text{lp } Y = f(X)$ . This implies  $f(x) = y$ , but  $f(x) \notin f(C)$ . So  $f(C)$  is closed. q.e.d.

## Appendix 2

Proof of Proposition 10 in Chapter I.

Let  $A$  be a finite-dimensional algebra (over some field  $k$ ) and let  $a \in A$ . Then

$$\{a^0 = 1, a, a^2, a^3, \dots\}$$

cannot all be linearly independent, so there is some polynomial  $g(x) \in k[x]$  such that  $g(a) = 0$ . The monic polynomial with minimal degree having  $a$  as a zero is the minimal polynomial for  $a$ .

Lemma: If  $a \in A$  and  $p(x)$  is the minimal polynomial for  $a$ , then  $a$  is a unit in  $A \iff p(x)$  has non-zero constant term.

Proof:

Let 
$$p(x) = x^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0$$

$\Leftarrow$  If  $c_0 \neq 0$  we can easily calculate that

$$(-1/c_0)(a^{n-1} + c_{n-1}a^{n-2} + \dots + c_1)a = 1$$

so that  $(1/c_0)(a^{n-1} + c_{n-1}a^{n-2} + \dots + c_1)$  is an inverse for  $a$ ,

and thus  $a$  is a unit.

$\Rightarrow$  If  $c_0 = 0$  we get

$$(a^{n-1} + c_{n-1}a^{n-2} + \dots + c_1)a = 0$$

and, since  $p(x)$  is minimal for  $a$ ,  $a^{n-1} + \dots + c_1 \neq 0$

and  $a$  is a divisor of zero and thus not a unit. This proves the lemma.

Proof of Proposition 10 (Chapter I)

Clearly  $U(A) \subset A \cap U(B)$ . Suppose  $a \in A \cap U(B)$  and we will show  $a \in U(A)$ . Since  $a$  is a unit in  $B$  there exists  $b \in B$  such that

$$ab = 1 = ba$$

It remains to show  $b \in A$  (so that  $a \in U(A)$ ).

Let 
$$p(x) = x^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0$$

be the minimal polynomial for  $a$ .

Since  $a$  is a unit,  $c_0 \neq 0$ .

From 
$$p(a) = (a^{n-1} + \dots + c_1)a - c_0$$
 we get

$$(a^{n-1} + \dots + c_1)ab = -c_0b \quad \text{and } ab = 1$$

so 
$$b = -(1/c_0)(a^{n-1} + \dots + c_1)$$

so that  $b$  is a polynomial in  $a$  and hence  $b \in A$ . QED.

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# Index

Abelian group	4
Algebra	15
Algebra of matrices	15
Atlas	163
Basis for open sets	82
Binary operation	1
Bounded set	81
Cartesian product of sets	1
Cartesian product of groups	92
Category	173
Center of a group	20
Centers of $Sp(1)$ and $SO(3)$	64
Center of $Sp(n)$	100
Centers of $U(n)$ and $SU(n)$	101
Center of $Spin(n)$	141
Centralizer (of a set in a group)	20
Chart	163
Clifford algebra	134
Closed manifold	87
Closed set	79
Commutator	69
Commutator subgroup	69
Compact set	81
Complex numbers	9
Conjugation in $\mathbb{R}$ , $\mathbb{C}$ , $\mathbb{H}$	23

Conjugate of a matrix	24
Conjugacy of maximal tori	126
Conjugate of a reflection	120
Conjugates of $\max T$ cover $U(n)$ , $SU(n)$	110
Conjugates of $\max T$ cover $SO(n)$	114
Conjugates of $\max T$ cover $Sp(n)$	118
Connected set	79
Continuity of a function	76
Cosets of a subgroup	67
Countability	83
Countable basis for open sets	85
Cross section	148
Curve in a vector space	35
Curve in a matrix group	36
Dense subset	124
Diffeomorphism	163
Differentiable curve	164
Differentiable manifold	164
Differentiable structure	164
Differential of a smooth homomorphism	42, 171
Dimension of a matrix group	37
Dimension of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$	38, 39
Dimensions of some matrix groups	41
Direct sum of algebras	135
Discrete subgroup	94
Divisors of zero	8, 135
Eigenvector, eigenvalue	107

Eigen space	108
Exponential of a matrix	45
Field	7
Functor	174
Fundamental group of a matrix group	131
General linear groups	16
Generator (of a monogenic group)	125
Group	2
Group extensions	148
Groups of rank 1,2,3	128
Homeomorphism (of spaces)	86
Homomorphism (of groups)	4
Homomorphism of $Sp(1)$ onto $SO(3)$	61
Homomorphism of $Pin(k)$ onto $O(k)$	138
Idempotent matrix	33
Identity component of a group	132
Injective homomorphism	5
Inner product	23
Isomorphism (linear)	14
Isomorphism of groups	6
Isomorphism of $Sp(1)$ and $SU(2)$	30
Isomorphism of $Sp(2)$ and $Spin(5)$	143
Isomorphism of $SU(4)$ and $Spin(6)$	143
Jacobi identity	57
Kernel of a homomorphism	19
Lattice subgroup of $\mathbb{R}^n$	104
Left translation	60, 71

Length of a vector	24
Lie algebra	57, 171
Lie group	172
Lie algebras of $Sp(1)$ and $SO(3)$	58
Linear map	12
Logarithm of a matrix (near $I$ )	49
Loop group $\Omega(G)$	131
Manifold	87
Maximal torus	95
Maximal torus in $SO(n)$	97
Maximal tori in $U(n)$ and $SU(n)$	97, 98
Maximal torus in $Sp(n)$	99
Metric	74
Monogenic group	124
Nilpotent matrix	34
Normal subgroup	20
Normalizer ( of a set in a group )	20
Normalizers (of max. tori) in $Sp(1)$ and $SO(3)$	147
One-parameter subgroup	51
Open ball	75
Open set	78
Orthogonal groups	27
Path	80
Path in a group	130
$Pin(k)$	137
Primitive root of unity	22
Projection	33



Quaternions $\mathbb{H}$	11
Quaternions have square roots	116
Quotient group	68
Rank of a matrix group	127
Reflections	31, 119
Reflections generate $\mathfrak{o}(n)$	121
Rotation group	27
Schwarz inequality	75
Simple group	129
Simply-connected group	131
Skew-Hermetian matrix	40
Skew-Symmetric matrix	39
Skew-symplectic matrix	40
Smooth homomorphism	41
Split group extension	148
Spin(k)	140
Special orthogonal group	29
Special unitary group	29
Stable subspace for a linear map	107
Subgroup	16
Subspace topology	82
Surjective homomorphism	5
Symmetric group	4
Symmetric linear map	112
Sumplectic group	27
Tangent space	37
Tangent vector	35, 165

Table of dimensions, centers, maximal tori	103
Torus	93
Trace of a matrix	54
Transpose of a matrix	24
Triangle inequality	74
Unipotent matrix	34
Uniqueness of one-parameter subgroups	53
Unit (in an algebra)	15
Unitary group	27
Universal covering group	131
Vector field	168
Weyl group	149

## Supplementary Index (for Chapter 13)

Adjoint representation	190, 192
Coroot	188
Divisible Group	182
Reflection	187
Regular element	185
Root kernel	189
Roots	194
Singular element	185