Appendix 1

In Exercise 8 of Chapter VI the concept of limit point is defined and it is proved that C is closed $\Leftrightarrow$ (x lp $C \Rightarrow x \in C$) ; i.e., a set is closed if and only if it contains all of its limit points.

Lemma: If $C \subset R^n$ is bounded and infinite then C has a limit point.

Proof: We must find $x \in R^n$ such that each $B(x,\varepsilon) \cap C$ is an infinite set.

Since C is bounded we can find an n - dimensional cube K_1 containing C. Let λ be the length of the sides of K_1. Divide each edge into two equal parts and cut K_1 into 2^n equal cubes of side length $\frac{\lambda}{2}$. Call those cubes $K_{11}, K_{12}, \ldots, K_{12^n}$. At least one of those must contain infinitely-many points of C. Choose one such and call it K_2. Subdivide K_2 into 2^n cubes of side $\frac{\lambda}{4}$ and choose one of these, K_3, containing infinitely-many points of C. Continue this process to get $K_1 \supset K_2 \supset K_3 \supset \ldots$. Clearly $K_1 \cap K_2 \cap K_3 \cap \ldots$ is a single point $x \in R^n$.

Then x lp C. For, consider any $B(x,\varepsilon)$. Take m such that $\frac{\lambda}{2^m} < \frac{\varepsilon}{2}$. Then

$$K_m \subset B(x,\varepsilon)$$

and K_m contains infinitely-many points of C. q.e.d.

Theorem: (Proposition 5 of Chapter VI) If $C \subset R^n$ is closed and bounded and

$$f\colon C \to R^m$$

<u>is continuous, then</u> $f(C)$ <u>is closed and bounded.</u>

<u>Proof</u>: Suppose $f(C)$ is not bounded. Choose $x_1 \in C$ such that $y_1 = f(x_1) \notin B(o,1)$.

Choose $x_2 \in C$ such that $y_2 = f(x_2) \notin B(o,2)$.

⋮

Choose $x_k \in C$ such that $y_k = f(x_k) \notin B(o,k)$. It is easy to prove that $Y = \{y_1, y_2, \ldots\}$ has no limit point and that $X = \{x_1, x_2, \ldots\}$ is an ifinite set. Also $X \subset C$ is bounded. So by the lemma X has some limit point x . Then also x lp C . Since C is closed $x \in C$. But then $y = f(x)$ must be a limit point of $Y = f(X)$ by continuity. Thus $f(C)$ is bounded.

Suppose $f(C)$ is not closed. Then there exists $y \in R^m$ such that y lp $f(C)$ but $y \notin f(C)$. Choose $x_1 \in C$ such that $y_1 = f(x_1) \in B(y,1)$.

Choose $x_2 \in C$ such that $y_2 = f(x_2) \in B(y,\frac{1}{2})$.

⋮

Choose $x_k \in C$ such that $y_k = f(x_k) \in B(y,\frac{1}{2})$. Clearly y is the only limit point of $Y = \{y_1, \ldots, y_k, \ldots\}$. Just as before $X = \{x_1, \ldots, x_k, \ldots\}$ is infinite and bounded. Let x lp X . Then x lp C so $x \in C$ and $f(x)$ lp $Y = f(X)$. This implies $f(x) = y$, but $f(x) \in f(C)$. So $f(C)$ is closed. q.e.d.

Appendix 2

Proof of Proposition 10 in Chapter I.

Let A be a finite-dimensional algebra (over some field k) and let $a \in A$. Then

$$\{a^0 = 1, a, a^2, a^3, \ldots\}$$

cannot all be linearly independent, so there is some polynomial $g(x) \in k[x]$ such that $g(x) = 0$. The monic polynomial with minimal degree having a as a zero is the minimal polynomial for a.

Lemma: If $a \in A$ and $p(x)$ is the minimal polynomial for a, then a is a unit in A <=> $p(x)$ has non-zero constant term.

Proof:

Let $p(x) = x^n + c_{n-1}x^{n-1} + \ldots c_1 x + c_0$

<== If $c_0 \neq 0$ we can easily calculate that

$$(-1/c_0)(a^{n-1} + c_{n-1}a^{n-2} + \ldots + c_1)a = 1$$

so that $(1/c_0)(a^{n-1} + c_{n-1}a^{n-2} + \ldots + c_1)$ is an inverse for a, and thus is a unit.

==> If $c_0 = 0$ we get

$$(a^{n-1} + c_{n-1}a^{n-2} + \ldots + c_1)a = 0$$

and, since $p(x)$ is minimal for a, $a^{n-1} + \ldots + c_1 \neq 0$ and a is a divisor of zero and thus not a unit. This proves the lemma.

Proof of Proposition 10 (Chapter I)

Clearly $U(A) \subset A \cap U(B)$. Suppose $a \in A \cap U(B)$ and we will show $a \in U(A)$. Since a is a unit in B there exists $b \in B$ such that

$$ab = 1 = ba$$

It remains to show $b \in A$ (so that $a \in U(A)$).

Let $$p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$$

be the minimal polynomial for a.

Since a is a unit, $c_0 \neq 0$.

From $p(a) = (a^{n-1} + \ldots + c_1)a - c_0$ we get

$$(a^{n-1} + \ldots + c_1)ab = -c_0 b \qquad \text{and } ab = 1$$

so $$b = -(1/c_0)(a^{n-1} + \ldots + c_1)$$

so that b is a polynomial in a and hence $b \in A$. QED.

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Index

Supplementary Index (for Chapter 13)