Samples of Examination Problems

In the four-hour written examination, 15 interrelated problems are given. Within square brackets, we indicate the point value of each problem. These values are revealed to the students beforehand.

Variant 1

$$\ddot{x} = -\sin x + \varepsilon \cos t. \tag{1}$$

I. Let $\varepsilon = 0$.

(1) Linearize at the point $x = \pi$, $\dot{x} = 0$, [1].
(2) Is this equilibrium position stable [1]?
(3) Find the Jacobian of the mapping of the phase flow at the point $x = \pi$, $\dot{x} = 0$ at time $t = 2\pi$ [3].
(4) Find the derivative of the solution with initial condition $x = \pi$, $\dot{x} = 0$ with respect to the parameter ε at $\varepsilon = 0$ [5].
(5) Draw the graph of the solution and its derivative with respect to t under the initial condition $x = 0$, $\dot{x} = 2$ [3].
(6) Find this solution [3].

II. Let Eq. (2) be the linearized equation along the solution indicated in problem (5).

(7) Does Eq. (2) have unbounded solutions [8]?
(8) Does Eq. (2) have nonzero bounded solutions [8]?
(9) Find the Wronskian of a fundamental system of solutions of Eq. (2), given that $W(0) = 1$ [5].
(10) Write out Eq. (2) explicitly and solve it [10].
(11) Find the eigenvalues and eigenvectors of the monodromy operator for the linearized equation along the solution with initial condition $x = \pi/2$, $\dot{x} = 0$ [16].
(12) Prove that Eq. (1) has a 2π-periodic solution depending smoothly on ε and vanishing at $x = \pi$ for $\varepsilon = 0$ [6].
(13) Find the derivative of this solution with respect to ε at $\varepsilon = 0$ [6].

III. Consider the equation $u_t + uu_x = -\sin x$.

(14) Write out the equation of characteristics [2].
(15) Find the largest value of t for which the solution of the Cauchy problem with $u|_{t=0} = 0$ can be extended to $[0, t)$ [8].

Variant 2

I. Let a vector field in three-dimensions have the origin as its singular point and let one of the eigenvalues of this singular point be equal to zero and the other two purely imaginary.

(1) Reduce to normal form the terms of degree 1 of the Taylor series expansion at zero of the components of the field [1].
(2) Do the same for terms of degree 2 [3].
(3) Do the same for terms of any degree [8].
(4) Average the system with respect to the fast rotation given by the linear part of the field [12].

II. Let us have a family of fields depending on a parameter and containing the field in part I for the zero value of the parameter.

(5) Reduce a beginning segment of the Taylor series at zero of the fields of the family to the simplest possible form by a diffeomorphism depending smoothly on parameters varying in the neighborhood of zero [10].
(6) Average the same system with respect to the fast rotation given by the linear part of the initial field [20].

III. In the space of 1-jets of vector fields in three-dimensions we consider the manifold of jets with one vanishing and two purely imaginary eigenvalues at the singular points.

(7) Find the codimension of the indicated manifold [2].
(8) Write the condition of transversality of the family, given in the form found in problem (5), to the indicated manifold [8].
(9) Analyze the bifurcations of singular points in generic two-parameter families transversal to the indicated manifold [10].
(10) Analyze the bifurcations of cycles from these singular points [15].
(11) Study the existence and smoothness of a phase curve connecting these singular points [15].

IV. Let a straight line going through zero be distinguished in the plane. A diffeomorphism of the plane is said to be *distinguished* if it transforms the distinguished line into itself. A vector field is said to be distinguished if it is tangent to the distinguished line at all of its points. Let a distinguished field be given which has a singular point at zero with two vanishing eigenvalues.

(12) Reduce a segment of the Taylor series of the field at zero to the simplest possible form by means of a distinguished diffeomorphism [12].

(13) Reduce a family of distinguished fields, which is a deformation of the given field, to formal normal form by means of distinguished formal diffeomorphisms depending formally smoothly on the parameters varying in the neighborhood of zero [16].
(14) Analyze the bifurcations of singular points in generic families obtained from the normal forms of problem (13) by omitting terms of high degree [18].
(15) Apply the results of problems (12)–(14) to the study of bifurcations of the phase portrait of a field with one vanishing and two purely imaginary eigenvalues [25].

Additional Problems

(1) Let $\dot{z} = \varepsilon z + Az|z|^2 + \bar{z}^3$. Prove that the number of limit cycles is not greater than 1 if $|\operatorname{Re} A| > 1$. *Hint*: Divide the field by $z\bar{z}$ and use the formula $\operatorname{div} P(z, \bar{z}) = 2 \operatorname{Re}(\partial P/\partial z)$.
(2) Let $A = (3 + i)/\sqrt{2}$. Then, for $\arg \varepsilon = 5\pi/4$, the singular separatrix of every saddle-node coincides with the nonsingular separatrix of the next saddle-node. *Hint*: At the moment of saddle-node coalescence, the equation can be reduced to the form $\dot{w} = e^{i\theta}\,[Rw(|w|^2 - 1) + i(\bar{w}^2 - w^2)]$, where $A = (R - i)e^{i\theta}$. If $R = 2$ and $\theta = \pi/4$, the separatrices are straight lines.
(3) Analyze the curves in the complex A-plane which divide domains where, when $\arg \varepsilon$ varies, singular points coalesce on the cycle, inside the cycle, and outside the cycle. *Hint*: If one varies θ, the field vectors rotate. The curves are located approximately like the four parabolas $a^2 = 2(\pm b \pm 1)$, $A = a + ib$.
(4) For small $|\operatorname{Re} A|$ and $1 < |\operatorname{Im} A| < c \approx 4.11$, the equation of problem (1) has (for appropriate ε) two limit cycles, with nine singular point inside the inner cycle. For $|\operatorname{Im} A| > c$, only one cycle exists (Neĭštadt). The boundary between domains of existence of one or two cycles looks like an ellipse with the major axis $1 \leqslant |\operatorname{Im} A| \leqslant 4.11$ and the minor axis of length 2.
(5) Prove that there are no limit cycles in the generalized system of Lotka–Volterra, $\dot{x} = x\,(\alpha + ax + by)$, $\dot{y} = y\,(\beta + cx + dy)$. *Hint*: At stability loss, the system has a first integral: a product of degrees of three linear functions (Bautin).

Index

Grundlehren der mathematischen Wissenschaften
Continued from page ii

258. Smoller: Shock Waves and Reaction–Diffusion Equations, 2nd Edition
259. Duren: Univalent Functions
260. Freidlin/Wentzell: Random Perturbations of Dynamical Systems
261. Bosch/Güntzer/Remmert: Non Archimedian Analysis—A System Approach to Rigid Analytic Geometry
262. Doob: Classical Potential Theory and Its Probabilistic Counterpart
263. Krasnosel'skiĭ/Zabreĭko: Geometrical Methods of Nonlinear Analysis
264. Aubin/Cellina: Differential Inclusions
265. Grauert/Remmert: Coherent Analytic Sheaves
266. de Rham: Differentiable Manifolds
267. Arbarello/Cornalba/Griffiths/Harris: Geometry of Algebraic Curves, Vol. I
268. Arbarello/Cornalba/Griffiths/Harris: Geometry of Algebraic Curves, Vol. II
269. Schapira: Microdifferential Systems in the Complex Domain
270. Scharlau: Quadratic and Hermitian Forms
271. Ellis: Entropy, Large Deviations, and Statistical Mechanics
272. Elliott: Arithmetic Functions and Integer Products
273. Nikol'skiĭ: Treatise on the Shift Operator
274. Hörmander: The Analysis of Linear Partial Differential Operators III
275. Hörmander: The Analysis of Linear Partial Differential Operators IV
276. Liggett: Interacting Particle Systems
277. Fulton/Lang: Riemann-Roch Algebra
278. Barr/Wells: Toposes, Triples and Theories
279. Bishop/Bridges: Constructive Analysis
280. Neukirch: Class Field Theory
281. Chandrasekharan: Elliptic Functions
282. Lelong/Gruman: Entire Functions of Several Complex Variables
283. Kodaira: Complex Manifolds and Deformation of Complex Structures
284. Finn: Equilibrium Capillary Surfaces
285. Burago/Zalgaller: Geometric Inequalities
286. Andrianov: Quadratic Forms and Hecke Operators
287. Maskit: Kleinian Groups
288. Jacod/Shiryaev: Limit Theorems for Stochastic Processes
289. Manin: Gauge Field Theory and Complex Geometry
290. Conway/Sloane: Sphere Packings, Lattices and Groups
291. Hahn/O'Meara: The Classical Groups and K-Theory
292. Kashiwara/Schapira: Sheaves on Manifolds
293. Revuz/Yor: Continuous Martingales and Brownian Motion
294. Knus: Quadratic and Hermitian Forms over Rings
295. Dierkes/Hildebrandt/Küster/Wohlrab: Minimal Surfaces I
296. Dierkes/Hildebrandt/Küster/Wohlrab: Minimal Surfaces II
297. Pastur/Figotin: Spectra of Random and Almost-Periodic Operators
298. Berline/Getzler/Vergne: Heat Kernels and Dirac Operators
299. Pommerenke: Boundary Behaviour of Conformal Maps
300. Orlik/Terao: Arrangements of Hyperplanes
301. Loday: Cyclic Homology
302. Lange/Birkenhake: Complex Abelian Varieties
303. DeVore/Lorentz: Constructive Approximation
304. Lorentz/v. Golitschek/Makovoz: Constructive Approximation. Advanced Problems
305. Hiriart-Urruty/Lemaréchal: Convex Analysis and Minimization Algorithms I. Fundamentals
306. Hiriart-Urruty/Lemaréchal: Convex Analysis and Minimization Algorithms II. Advanced Theory and Bundle Methods
307. Schwarz: Quantum Field Theory and Topology