Appendix: Mathematical Induction

The construction of the real numbers usually starts with Peano's axioms for the integers. Next, one develops the arithmetic of the integers; then the rational numbers are introduced as ordered pairs of integers; and then, finally, one constructs the real numbers, either as suitable sequences of rational numbers or by the so-called Dedekind cuts of rational numbers. In any event, the end product is a set of objects that is an ordered field in which the Least Upper Bound Axiom holds and that is denoted by $\mathbb{R}$. In this construction (and this is the usual one) Mathematical Induction appears as one of Peano's axioms.

In this text (as in many others) we have chosen to avoid these foundational matters and start with the assumption that we have a set, $\mathbb{R}$, which we call the set of real numbers, which is an ordered field in which the Least Upper Bound Axiom holds. The question now is whether or not we have implicitly included Mathematical Induction in our assumptions. In short, do we still have to incorporate it as an *axiom*, or is it now a *theorem*? The answer is that it is a theorem, and our aim here is to sketch its proof.

We say that a set A of real numbers is **inductive** if $1 \in A$ and if $x+1 \in A$ whenever $x \in A$. Such sets exist, for example, the set of positive real numbers. It is clear that the intersection Ω of all inductive sets is itself an inductive set; thus Ω is inductive, $1 \in \Omega$, and if A is inductive, then $\Omega \subset A$. The Principle of Induction *for the set* Ω is now a triviality.

The Principle of Mathematical Induction.

Suppose that $A \subset \Omega$, $1 \in A$, and that $x + 1 \in A$ whenever $x \in A$. Then $A = \Omega$.

Proof

Let A satisfy the given hypotheses. Then A is inductive, so that $\Omega \subset A$. But by assumption, $A \subset \Omega$; thus $A = \Omega$. ■

Of course, this is not yet the familiar Principle of Induction, but our problem has shifted. We have *proved* the Principle of Induction for Ω, and it now only remains to identify Ω as the set of positive integers. Note that because $\{x \in \mathbb{R} : x \geq 1\}$ is inductive, it contains Ω, so that every element y in Ω satisfies $y \geq 1$. In particular, this already shows that 1 is the smallest member of Ω.

The set $\mathbb{Z}$ of integers is, by definition, the additive subgroup of the real numbers that is generated by 1 or, more precisely, the intersection of all additive subgroups of $\mathbb{R}$ that contain the element 1. To derive the Principle of Mathematical Induction for $\mathbb{N}$, we have to show that $\Omega = \mathbb{N}$, and this is a consequence of the following result (where $-\Omega$ is defined by $x \in -\Omega$ if and only if $-x \in \Omega$).

Theorem A.

$\mathbb{Z} = \Omega \cup \{0\} \cup (-\Omega)$. *In particular,* 1 *is the smallest positive integer.*

For this, we need two preliminary results.

Lemma 1.

$\Omega = \{1\} \cup \{x + 1 : x \in \Omega\}$.

Proof

Let $A = \{1\} \cup \{x + 1 : x \in \Omega\}$. We begin by showing that A is inductive. First, $1 \in A$. Now suppose that $a \in A$. Then either $a = 1$ or $a = x + 1$, where $x \in \Omega$. In both cases, $a \in \Omega$, so that

(i) $A \subset \Omega$, and
(ii) $a + 1 \in A$.

Now, (ii) shows that A is inductive, and hence $\Omega \subset A$. We have now proved that $A = \Omega$. ■

Lemma 2.

Let $\Gamma = \Omega \cup \{0\} \cup (-\Omega)$. *Then*

(a) Γ *is inductive;*
(b) *if* $x \in \Gamma$ *and* $y \in \Omega$, *then* $x + y \in \Gamma$;
(c) *if* $x \in \Gamma$ *and* $y \in \Omega$, *then* $x - y \in \Gamma$.

Proof

We prove (a). First, by definition, $1 \in \Gamma$. Now take any x in Γ; then either $x \in \Omega$ or $x = 0$ or $-x \in \Omega$. In the first case, $x + 1 \in \Omega$ (because Ω

is inductive). In the second case, $x + 1 = 1 \in \Omega$. By Lemma 1, the third case implies that either $-x = 1$ (in which case $x + 1 = 0 \in \Gamma$) or $-x = r + 1$ for some r in Ω (in which case $x + 1 = -r \in \Gamma$). Thus in all cases, $x + 1 \in \Omega \cup \Gamma = \Gamma$, so that (a) holds.

We now prove (b). Take x in Γ and let $A = \{y \in \Omega : x + y \in \Gamma\}$. We want to show that $A = \Gamma$. By Lemma 1, $1 \in A$. Now take y in A; then $x + y \in \Gamma$, so that $x + y + 1 \in \Gamma$, whence $y + 1 \in A$. Thus A is inductive, so that $\Omega \subset A$; whence $A = \Omega$. The proof of (c) is similar. ■

Proof of Theorem 1.

As $\mathbb{Z}$ is inductive, $\Omega \subset \mathbb{Z}$. Thus (because $\mathbb{Z}$ is an additive group) $\Gamma \subset \mathbb{Z}$. It remains to prove that Γ is an additive group, for then, as $1 \in \Gamma$, we must have $\mathbb{Z} \subset \Gamma$, and hence $\mathbb{Z} = \Gamma$, which is the desired conclusion.

As $0 \in \Gamma$ and as $-x \in \Gamma$ whenever $x \in \Gamma$, in order to show that Γ is a group we have only to show that it is closed under addition. Take any x and y in Γ. Now, either $y \in \Omega$, $y = 0$, or $-y \in \Omega$. If $y \in \Omega$, then $x + y \in \Gamma$ by Lemma 2(b). If $y = 0$, then $x + y = x \in \Gamma$. If $-y \in \Omega$, then by Lemma 2(c), $x - (-y) \in \Gamma$. The proof of Theorem 1 is complete. ■

Finally, we have

Theorem B.

Every nonempty set of positive integers has a smallest member.

Proof

Let A be a nonempty subset of Ω, and define

$$B = \{x \in \Omega : \text{ for all } a \text{ in } A, x \leq a\}.$$

Choose any element a_1 of A. Then $a_1 + 1 \notin B$ (for $a_1 + 1 \leq a_1$ is false). However, as a_1 is in A, it is in Ω, so that $a_1 + 1 \in \Omega$. It follows that $B \neq \Omega$, and hence (as $B \subset \Omega$) that B is not inductive.

As B is not inductive and as $1 \in B$, there is some b in B with $b + 1 \notin B$. Now, $b + 1 \notin B$ implies that there is some α in A with $\alpha < b + 1$, and because $b \in B$, $b \leq \alpha$; thus $b \leq \alpha < b+1$. However, $\mathbb{Z}$ is an additive group that respects addition, and as $b \in B \subset \Omega \subset \mathbb{Z}$, we see that $b - \alpha$ is an integer in the range $[0, 1)$. By (1), $b = \alpha$, whence $\alpha \in B$, and this implies that α is the smallest member of A. This completes the proof. ■

References

We recommend the following references to readers interested in directed sets.

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