Appendix A
Answers to Exercises

Solutions

Exercises 1.4

1. (a) $\ln t$ is singular at $t = 0$, hence one might assume that the Laplace transform does not exist. However, for those familiar with the Gamma function consider the result

$$\int_0^\infty e^{-st} t^k dt = \frac{\Gamma(k+1)}{s^{k+1}}$$

which is standard and valid for non integer k. Differentiate this result with respect to k to obtain

$$\int_0^\infty e^{-st} t^k \ln t dt = \frac{\Gamma'(k+1) - \Gamma(k+1)\ln s}{s^{k+1}}.$$

Let $k = 0$ in this result to obtain

$$\int_0^\infty e^{-st} \ln t dt = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$

where γ is the Euler-Mascheroni constant ($= 0.5772156649\ldots$). The right hand side is the Laplace transform of $\ln t$, so it does exist. The apparent singularity is in fact removable. (c.f. The Laplace transform of $t^{-\frac{1}{2}}$ also exists and is a finite quantity.)

(b)

$$\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{3t} e^{-st} dt = \left[\frac{1}{3-s} e^{(3-s)t}\right]_0^\infty = \frac{1}{s-3}.$$

(c) $e^{t^2} > |e^{Mt}|$ for any M for large enough t, hence the Laplace transform does not exist (not of exponential order).

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4,

(d) the Laplace transform does not exist (singular at $t = 0$).
(e) the Laplace transform does not exist (singular at $t = 0$).
(f) does not exist (infinite number of (finite) jumps), also not defined unless t is an integer.

2. Using the definition of Laplace transform in each case, the integration is reasonably straightforward:

$$\text{(a)} \quad \int_0^\infty e^{kt}e^{-st}dt = \frac{1}{s-k}$$

as in part (b) of the previous question.
(b) Integrating by parts gives,

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}dt = \left[-\frac{t^2}{s}e^{-st}\right]_0^\infty + \int_0^\infty \frac{2t}{s}e^{-st}dt = \frac{2}{s}\int_0^\infty te^{-st}dt.$$

Integrating by parts again gives the result $\frac{2}{s^3}$.
(c) Using the definition of $\cosh t$ gives

$$\mathcal{L}\{\cosh t\} = \frac{1}{2}\left\{\int_0^\infty e^t e^{-st}dt + \int_0^\infty e^{-t}e^{-st}dt\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-1} + \frac{1}{s+1}\right\} = \frac{s}{s^2-1}.$$

3. (a) This demands use of the first shift theorem, Theorem 1.2, which with $b = 3$ is

$$\mathcal{L}\{e^{-3t}F(t)\} = f(s+3)$$

and with $F(t) = t^2$, using part (b) of the last question gives the answer $\frac{2}{(s+3)^3}$.
(b) For this part, we use Theorem 1.1 (linearity) from which the answer

$$\frac{4}{s^2} + \frac{6}{s-4}$$

follows at once.
(c) The first shift theorem with $b = 4$ and $F(t) = \sin(5t)$ gives

$$\frac{5}{(s+4)^2+25} = \frac{5}{s^2+8s+41}.$$

4. When functions are defined in a piecewise fashion, the definition integral for the Laplace transform is used and evaluated directly. For this problem we get

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)dt = \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt$$

which after integration by parts gives

$$\frac{1}{s^2}(1-e^{-s})^2.$$

5. Using Theorem 1.3 we get

$$\text{(a)}\quad \mathcal{L}\{te^{2t}\} = -\frac{d}{ds}\frac{1}{(s-2)} = \frac{1}{(s-2)^2}$$

$$\text{(b)}\quad \mathcal{L}\{t\cos(t)\} = -\frac{d}{ds}\frac{s}{1+s^2} = \frac{-1+s^2}{(1+s^2)^2}$$

The last part demands differentiating twice,

$$\text{(c)}\quad \mathcal{L}\{t^2\cos(t)\} = \frac{d^2}{ds^2}\frac{s}{1+s^2} = \frac{2s^3-6s}{(1+s^2)^3}.$$

6. These two examples are not difficult: the first has application to oscillating systems and is evaluated directly, the second needs the first shift theorem with $b = 5$.

$$\text{(a)}\quad \mathcal{L}\{\sin(\omega t+\phi)\} = \int_0^\infty e^{-st}\sin(\omega t+\phi)dt$$

and this integral is evaluated by integrating by parts twice using the following trick. Let

$$I = \int_0^\infty e^{-st}\sin(\omega t+\phi)dt$$

then derive the formula

$$I = \left[-\frac{1}{s}e^{-st}\sin(\omega t+\phi) - \frac{\omega}{s^2}e^{-st}\cos(\omega t+\phi)\right]_0^\infty - \frac{\omega^2}{s^2}I$$

from which

$$I = \frac{s\sin(\phi)+\omega\cos(\phi)}{s^2+\omega^2}.$$

$$\text{(b)}\quad \mathcal{L}\{e^{5t}\cosh(6t)\} = \frac{s-5}{(s-5)^2-36} = \frac{s-5}{s^2-10s-11}.$$

7. This problem illustrates the difficulty in deriving a linear translation plus scaling property for Laplace transforms. The zero in the bottom limit is the culprit. Direct integration yields:

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} G(t)dt = \int_{-b/a}^\infty ae^{(u-b)s/a} F(u)du$$

where we have made the substitution $t = au + b$ so that $G(t) = F(u)$. In terms of $\bar{f}(as)$ this is

$$ae^{-sb}\bar{f}(as) + ae^{-sb}\int_{-b/a}^0 e^{-ast}F(t)dt.$$

8. The proof proceeds by using the definition as follows:

$$\mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at)dt = \int_0^\infty e^{-su/a} F(u)du/a$$

which gives the result. Evaluation of the two Laplace transforms follows from using the results of Exercise 5 alongside the change of scale result just derived with, for (a) $a = 6$ and for (b) $a = 7$. The answers are

$$\text{(a)}\frac{-36 + s^2}{(36 + s^2)^2}, \ \text{(b)}\frac{2s(s^2 - 147)}{(s^2 + 49)^3}.$$

Exercises 2.8

1. If $F(t) = \cos(at)$ then $F'(t) = -a\sin(at)$. The derivative formula thus gives

$$\mathcal{L}\{-a\sin(at)\} = s\mathcal{L}\{\cos(at)\} - F(0).$$

Assuming we know that $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2 + a^2}$ then, straightforwardly

$$\mathcal{L}\{-a\sin(at)\} = s\frac{s}{s^2 + a^2} - 1 = -\frac{a^2}{s^2 + a^2}$$

i.e $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2 + a^2}$ as expected.

2. Using Theorem 2.1 gives

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = s\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}du\right\}$$

In the text (after Theorem 2.3) we have derived that

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{s}\tan^{-1}\left\{\frac{1}{s}\right\},$$

in fact this calculation is that one in reverse. The result

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$$

is immediate. In order to derive the required result, the following manipulations need to take place:

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-st}\frac{\sin t}{t}dt$$

and if we substitute $ua = t$ the integral becomes

$$\int_0^\infty e^{-asu}\frac{\sin(au)}{u}du.$$

This is still equal to $\tan^{-1}\left\{\frac{1}{s}\right\}$. Writing $p = as$ then gives the result. (p is a dummy variable of course that can be re-labelled s.)

3. The calculation is as follows:

$$\mathcal{L}\left\{\int_0^t p(v)dv\right\} = \frac{1}{s}\mathcal{L}\{p(v)\}$$

so

$$\mathcal{L}\left\{\int_0^t\int_0^v F(u)du dv\right\} = \frac{1}{s}\mathcal{L}\left\{\int_0^v F(u)du\right\} = \frac{1}{s^2}f(s)$$

as required.

4. Using Theorem 2.3 we get

$$\mathcal{L}\left\{\int_0^t \frac{\cos(au)-\cos(bu)}{u}du\right\} = \frac{1}{s}\int_s^\infty \frac{u}{a^2+u^2} - \frac{u}{b^2+u^2}du.$$

These integrals are standard "ln" and the result $\frac{1}{s}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)$ follows at once.

5. This transform is computed directly as follows

$$\mathcal{L}\left\{\frac{2\sin t\sinh t}{t}\right\} = \mathcal{L}\left\{\frac{e^t\sin t}{t}\right\} - \mathcal{L}\left\{\frac{e^{-t}\sin t}{t}\right\}.$$

Using the first shift theorem (Theorem 1.2) and the result of Exercise 2 above yields the result that the required Laplace transform is equal to

$$\tan^{-1}\left(\frac{1}{s-1}\right) - \tan^{-1}\left(\frac{1}{s+1}\right) = \tan^{-1}\left(\frac{2}{s^2}\right).$$

(The identity $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ has been used.)

6. This follows straight from the definition of Laplace transform:

$$\lim_{s\to\infty} \bar{f}(s) = \lim_{s\to\infty} \int_0^\infty e^{-st}F(t)dt = \int_0^\infty \lim_{s\to\infty} e^{-st}F(t)dt = 0.$$

It also follows from the final value theorem (Theorem 2.7) in that if $\lim_{s\to\infty} s\bar{f}(s)$ is finite then by necessity $\lim_{s\to\infty} \bar{f}(s) = 0$.

7. These problems are all reasonably straightforward

$$\text{(a)}\ \frac{2(2s+7)}{(s+4)(s+2)} = \frac{3}{s+2} + \frac{1}{s+4}$$

and inverting each Laplace transform term by term gives the result $3e^{-2t} + e^{-4t}$

$$\text{(b) Similarly}\frac{s+9}{s^2-9} = \frac{2}{s-3} - \frac{1}{s+3}$$

and the result of inverting each term gives $2e^{3t} - e^{-3t}$

$$\text{(c)}\frac{s^2+2k^2}{s(s^2+4k^2)} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4k^2}\right)$$

and inverting gives the result

$$\frac{1}{2} + \frac{1}{2}\cos(2kt) = \cos^2(kt).$$

$$\text{(d)}\ \frac{1}{s(s+3)^2} = \frac{1}{9s} - \frac{1}{9(s+3)} - \frac{1}{3(s+3)^2}$$

which inverts to

$$\frac{1}{9} - \frac{1}{9}(3t+1)e^{-3t}.$$

(d) This last part is longer than the others. The partial fraction decomposition is best done by computer algebra, although hand computation is possible. The result is

$$\frac{1}{(s-2)^2(s+3)^3} = \frac{1}{125(s-2)^2} - \frac{3}{625(s-2)} + \frac{1}{25(s+3)^3} + \frac{2}{125(s+3)^2} + \frac{3}{625(s+3)}$$

and the inversion gives $\frac{e^{2t}}{625}(5t-3) + \frac{e^{-3t}}{1250}(25t^2+20t+6)$.

8. (a) $F(t) = 2 + \cos(t) \to 3$ as $t \to 0$, and as $\frac{2}{s} + \frac{s}{s^2+1}$ we also have that $sf(s) \to 2+1 = 3$ as $s \to \infty$ hence verifying the initial value theorem.
(b) $F(t) = (4+t)^2 \to 16$ as $t \to 0$. In order to find the Laplace transform, we expand and evaluate term by term so that $sf(s) = 16 + 8/s + 2/s^2$ which obviously also tends to 16 as $s \to \infty$ hence verifying the theorem once more.
9. (a) $F(t) = 3 + e^{-t} \to 3$ as $t \to \infty$. $f(s) = \frac{3}{s} + \frac{1}{s+1}$ so that $sf(s) \to 3$ as $s \to 0$ as required by the final value theorem.
(b) With $F(t) = t^3e^{-t}$, we have $f(s) = 6/(s+1)^4$ and as $F(t) \to 0$ as $t \to \infty$ and $sf(s)$ also tends to the limit 0 as $s \to 0$ the final value theorem is verified.
10. For small enough t, we have that

$$\sin(\sqrt{t}) = \sqrt{t} + O(t^{3/2})$$

and using the standard form (Appendix B):

$$\mathcal{L}\{t^{x-1}\} = \frac{\Gamma\{x\}}{s^x}$$

with $x = 3/2$ gives

$$\mathcal{L}\{\sin(\sqrt{t})\} = \mathcal{L}\{\sqrt{t}\} + \cdots = \frac{\Gamma\{3/2\}}{s^{3/2}} + \cdots$$

and using that $\Gamma\{3/2\} = (1/2)\Gamma\{1/2\} = \sqrt{\pi}/2$ we deduce that

$$\mathcal{L}\{\sin(\sqrt{t})\} = \frac{\sqrt{\pi}}{2s^{3/2}} + \cdots.$$

Also, using the formula given,

$$\frac{k}{s^{3/2}}e^{-\frac{1}{4s}} = \frac{k}{s^{3/2}} + \cdots.$$

Comparing these series for large values of s, equating coefficients of $s^{-3/2}$ gives

$$k = \frac{\sqrt{\pi}}{2}.$$

11. Using the power series expansions for sin and cos gives

$$\sin(t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n+2}}{(2n+1)!}$$

and

$$\cos(t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n}}{2n!}.$$

Taking the Laplace transform term by term gives

$$\mathcal{L}\{\sin(t^2)\} = \sum_{n=0}^{\infty}(-1)^n \frac{(4n+2)!}{(2n+1)!s^{4n+3}}$$

and

$$\mathcal{L}\{\cos(t^2)\} = \sum_{n=0}^{\infty}(-1)^n \frac{(4n)!}{(2n)!s^{4n+1}}.$$

12. Given that $Q(s)$ is a polynomial with n distinct zeros, we may write

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_k}{s-a_k} + \cdots + \frac{A_n}{s-a_n}$$

where the A_ks are some real constants to be determined. Multiplying both sides by $s - a_k$ then letting $s \to a_k$ gives

$$A_k = \lim_{s\to a_k} \frac{P(s)}{Q(s)}(s-a_k) = P(a_k) \lim_{s\to a_k} \frac{(s-a_k)}{Q(s)}.$$

Using l'Hôpital's rule now gives

$$A_k = \frac{P(a_k)}{Q'(a_k)}$$

for all $k = 1, 2, \ldots, n$. This is true for all k, thus we have established that

$$\frac{P(s)}{Q(s)} = \frac{P(a_1)}{Q'(a_1)} \frac{1}{(s-a_1)} + \cdots + \frac{P(a_k)}{Q'(a_k)} \frac{1}{(s-a_k)} + \cdots \frac{P(a_n)}{Q'(a_n)} \frac{1}{(s-a_n)}.$$

Taking the inverse Laplace transform gives the result

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(a_k)}{Q'(a_k)} e^{a_k t}$$

sometimes known as Heaviside's expansion formula.

13. All of the problems in this question are solved by evaluating the Laplace transform explicitly.

$$\text{(a) } \mathcal{L}\{H(t-a)\} = \int_a^\infty e^{-st}dt = \frac{e^{-as}}{s}.$$

$$\text{(b) } \mathcal{L}\{f_1(t)\} = \int_0^2 (t+1)e^{-st}dt + \int_2^\infty 3e^{-st}dt.$$

Evaluating the right-hand integrals gives the solution

$$\frac{1}{s} + \frac{1}{s^2}(e^{-2s} - 1).$$

$$\text{(c) } \mathcal{L}\{f_2(t)\} = \int_0^2 (t+1)e^{-st}dt + \int_2^\infty 6e^{-st}dt.$$

Once again, evaluating gives

$$\frac{1}{s} + \frac{3}{s}e^{-2s} + \frac{1}{s^2}(e^{-2s} - 1)$$

(d) As the function $f_1(t)$ is in fact continuous in the interval $[0, \infty)$ the formula for the derivative of the Laplace transform (Theorem 2.1) can be used to give the result $\frac{1}{s}(e^{-2s} - 1)$ at once. Alternatively, f_1 can be differentiated (it is $1 - H(t-2)$) and evaluated directly.

14. We use the formula for the Laplace transform of a periodic function Theorem 2.8 to give

$$\mathcal{L}\{F(t)\} = \frac{\int_0^{2c} e^{-st}F(t)dt}{(1 - e^{-2sc})}.$$

The numerator is evaluated directly:

$$\int_0^{2c} e^{-st}F(t)dt = \int_0^c te^{-st}dt + \int_c^{2c} (2c - t)e^{-st}dt$$

which after routine integration by parts simplifies to

$$\frac{1}{s^2}(e^{-sc} - 1)^2.$$

The Laplace transform is thus

$$\mathcal{L}\{F(t)\} = \frac{1}{1 - e^{-2sc}} \frac{1}{s^2}(e^{-sc} - 1)^2 = \frac{1}{s^2} \frac{1 - e^{-sc}}{1 + e^{-sc}}$$

which simplifies to

$$\frac{1}{s^2}\tanh\left(\frac{1}{2}sc\right).$$

15. Evaluating the integral gives:

$$f(s) = \int_0^\infty e^{-st}F(t)dt = \int_a^{a+h}\frac{e^{-st}}{h}dt = \frac{e^{-as}}{sh}(1-e^{-sh})$$

so

$$f(s) = \frac{e^{-as}}{sh}(1-e^{-sh}).$$

If we let $h \to 0$ then $F(t) \to \delta(t-a)$ and we get

$$f(s) = \frac{e^{-sa}}{s}\lim_{h\to 0}\left\{\frac{1-e^{-sh}}{h}\right\} = \frac{e^{-sa}}{s}s = e^{-as}$$

using L'Hôpital's Rule, so

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}.$$

Exercises 3.6

1. (a) If we substitute $u = t - \tau$ into the definition of convolution then

$$g * f = \int_0^t g(\tau)f(t-\tau)d\tau$$

becomes

$$-\int_t^0 g(u-\tau)f(u)du = g * f.$$

(b) Associativity is proved by effecting the transformation $(u, \tau) \to (x, y)$ where $u = t - x - y$, and $\tau = y$ on the expression

$$f * (g * h) = \int_0^t \int_0^{t-\tau} f(\tau)g(u)h(t-\tau-u)dud\tau.$$

The area covered by the double integral does not change under this transformation, it remains the right-angled triangle with vertices $(0, t)$, $(0, 0)$ and $(t, 0)$. The calculation proceeds as follows:

$$dud\tau = \frac{\partial(u,\tau)}{\partial(x,y)}dxdy = -dxdy$$

so that

$$\begin{aligned} f*(g*h) &= \int_0^t \int_0^{t-x} f(y)g(t-x-y)h(x)dydx \\ &= \int_0^t h(x)\left[\int_0^{t-x} f(y)g(t-x-y)dy\right]dx \\ &= \int_0^t h(x)[f*g](t-x)dx = h*(f*g) \end{aligned}$$

and this is $(f*g)*h$ by part (a) which establishes the result.
(c) Taking the Laplace transform of the expression $f*f^{-1}=1$ gives

$$\mathcal{L}\{f\}\cdot\mathcal{L}\{f^{-1}\} = \frac{1}{s}$$

from which

$$\mathcal{L}\{f^{-1}\} = \frac{1}{s\bar{f}(s)}$$

using the usual notation ($\bar{f}(s)$ is the Laplace transform of $f(t)$). It must be the case that $\frac{1}{s\bar{f}(s)} \to 0$ as $s\to\infty$. The function f^{-1} is not uniquely defined. Using the properties of the Dirac-δ function, we can also write

$$\int_0^{t_+} f(\tau)\delta(t-\tau)d\tau = f(t)$$

from which

$$f^{-1}(t) = \frac{\delta(t-\tau)}{f(t)}.$$

Clearly, $f(t)\neq 0$.

2. Since $\mathcal{L}\{f\}=\bar{f}$ and $\mathcal{L}\{1\}=1/s$ we have

$$\mathcal{L}\{f*1\} = \frac{\bar{f}}{s}$$

so that, on inverting

$$\mathcal{L}^{-1}\left\{\frac{\bar{f}}{s}\right\} = f*1 = \int_0^t f(\tau)d\tau$$

as required.

3. These convolution integrals are straightforward to evaluate:

$$\text{(a)} \quad t * \cos t = \int_0^t (t-\tau)\cos\tau d\tau$$

this is, using integration by parts

$$1 - \cos t.$$

$$\text{(b)} \quad t * t = \int_0^t (t-\tau)\tau d\tau = \frac{t^3}{6}.$$

$$\text{(c)} \quad \sin t * \sin t = \int_0^t \sin(t-\tau)\sin\tau d\tau = \frac{1}{2}\int_0^t [\cos(2\tau - t) - \cos t] d\tau$$

this is now straightforwardly

$$\frac{1}{2}(\sin t - t\cos t).$$

$$\text{(d)} \quad e^t * t = \int_0^t e^{t-\tau}\tau d\tau$$

which on integration by parts gives

$$-1 - t + e^{-t}.$$

$$\text{(e)} \quad e^t * \cos t = \int_0^t e^{t-\tau}\cos\tau d\tau.$$

Integration by parts twice yields the following equation

$$\int_0^t e^{t-\tau}\cos\tau d\tau = \left[e^{-\tau}\sin\tau - e^{-\tau}\cos\tau\right]_0^t - \int_0^t e^{t-\tau}\cos\tau d\tau$$

from which

$$\int_0^t e^{t-\tau}\cos\tau d\tau = \frac{1}{2}(\sin t - \cos t + e^t).$$

4. (a) This is proved by using l'Hôpital's rule as follows

$$\lim_{x\to 0}\left\{\frac{\operatorname{erf}(x)}{x}\right\} = \lim_{x\to 0}\frac{1}{x}\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^x e^{-t^2}dt$$

and using Leibnitz' rule (or differentiation under the integral sign) this is

$$\lim_{x\to 0} \frac{2}{\sqrt{\pi}} e^{-x^2} = \frac{2}{\sqrt{\pi}}$$

as required.

(b) This part is tackled using power series expansions. First note that

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^{n+1}\frac{x^{2n}}{n!} + \cdots .$$

Integrating term by term (uniformly convergent for all x) gives

$$\int_0^{\sqrt{t}} e^{-x^2} dx = t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \cdots + (-1)^{n+1}\frac{t^{n+1/2}}{(2n+1)\cdot n!} + \cdots$$

from which

$$t^{-\frac{1}{2}}\mathrm{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\left(1 - \frac{t}{3} + \frac{t^2}{5.2!} - \frac{t^3}{7.3!} + \cdots \right.$$
$$\left. + (-1)^{n+1}\frac{t^n}{(2n+1)\cdot n!} + \cdots\right).$$

Taking the Laplace transform of this series term by term (again justified by the uniform convergence of the series for all t) gives

$$\mathcal{L}^{-1}\{t^{-\frac{1}{2}}\mathrm{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi}}\left(\frac{1}{s} - \frac{1}{3s^2} + \frac{1}{5s^3} - \frac{1}{7s^4} + \cdots \right.$$
$$\left. + \frac{(-1)^n}{(2n+1)s^{n+1}} + \cdots\right)$$

and taking out a factor $1/\sqrt{s}$ leaves the arctan series for $1/\sqrt{s}$. Hence we get the required result:

$$\mathcal{L}^{-1}\{t^{-1/2}\mathrm{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi s}}\tan^{-1}\left(\frac{1}{\sqrt{s}}\right).$$

5. All of these differential equations are solved by taking Laplace transforms. Only some of the more important steps are shown.
(a) The transformed equation is

$$s\bar{x}(s) - x(0) + 3\bar{x}(s) = \frac{1}{s-2}$$

from which, after partial fractions,

$$\bar{x}(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)} = \frac{4/5}{s+3} + \frac{1/5}{s-2}.$$

Inverting gives

$$x(t) = \frac{4}{5}e^{-3t} + \frac{1}{5}e^{2t}.$$

(b) This equation has Laplace transform

$$(s+3)\bar{x}(s) - x(0) = \frac{1}{s^2+1}$$

from which

$$\bar{x}(s) = \frac{x(0)}{s+3} - \frac{1/10}{s+3} + \frac{s/10 - 3/10}{s^2+1}.$$

The boundary condition $x(\pi) = 1$ is not natural for Laplace transforms, however inverting the above gives

$$x(t) = \left(x(0) - \frac{1}{10}\right)e^{-3t} - \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$$

and this is 1 when $x = \pi$, from which

$$x(0) - \frac{1}{10} = \frac{9}{10}e^{3\pi}$$

and the solution is

$$x(t) = \frac{9}{10}e^{3(\pi-t)} - \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t).$$

(c) This equation is second order; the principle is the same but the algebra is messier. The Laplace transform of the equation is

$$s^2\bar{x}(s) + 4s\bar{x}(s) + 5\bar{x}(s) = \frac{8}{s^2+1}$$

and rearranging using partial fractions gives

$$\bar{x}(s) = \frac{s+2}{(s+2)^2+1} + \frac{1}{(s+2)^2+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Taking the inverse then yields the result

$$x(t) = e^{-2t}(\cos t + \sin t) + \sin t - \cos t.$$

(d) The Laplace transform of the equation is

$$(s^2 - 3s - 2)\bar{x}(s) - s - 1 + 3 = \frac{6}{s}$$

from which, after rearranging and using partial fractions,

$$\bar{x}(s) = -\frac{3}{s} + \frac{4(s - \frac{3}{2})}{(s - \frac{3}{2})^2 - \frac{17}{4}} - \frac{5}{(s - \frac{3}{2})^2 - \frac{17}{4}}$$

which gives the solution

$$x(t) = -3 + 4e^{\frac{3}{2}t} \cosh\left(\frac{t}{2}\sqrt{17}\right) - \frac{10}{\sqrt{17}} e^{\frac{3}{2}t} \sinh\left(\frac{t}{2}\sqrt{17}\right).$$

(e) This equation is solved in a similar way. The transformed equation is

$$s^2\bar{y}(s) - 3s + \bar{y}(s) - 1 = \frac{6}{s^2 + 4}$$

from which

$$\bar{y}(s) = -\frac{2}{s^2 + 4} + \frac{3s + 3}{s^2 + 1}$$

and inverting, the solution

$$y(t) = -\sin(2t) + 3\cos t + 3\sin t$$

results.

6. Simultaneous ODEs are transformed into simultaneous algebraic equations and the algebra to solve them is often horrid. For parts (a) and (c) the algebra can be done by hand, for part (b) computer algebra is almost compulsory.
(a) The simultaneous equations in the transformed state after applying the boundary conditions are

$$(s - 2)\bar{x}(s) - (s + 1)\bar{y}(s) = \frac{6}{s - 3} + 3$$
$$(2s - 3)\bar{x}(s) + (s - 3)\bar{y}(s) = \frac{6}{s - 3} + 6$$

from which we solve and rearrange to obtain

$$\bar{x}(s) = \frac{4}{(s - 3)(s - 1)} + \frac{3s - 1}{(s - 1)^2}$$

so that, using partial fractions

$$\bar{x}(s) = \frac{2}{s-3} + \frac{1}{s-1} + \frac{2}{(s-1)^2}$$

giving, on inversion

$$x(t) = 2e^{3t} + e^t + 2te^t.$$

In order to find $y(t)$ we eliminate dy/dt from the original pair of simultaneous ODEs to give

$$y(t) = -3e^{3t} - \frac{5}{4}x(t) + \frac{3}{4}\frac{dx}{dt}.$$

Substituting for $x(t)$ then gives

$$y(t) = -e^{3t} + e^t - te^t.$$

(b) This equation is most easily tackled by substituting the derivative of

$$y = -4\frac{dx}{dt} - 6x + 2\sin(2t)$$

into the second equation to give

$$5\frac{d^2}{dx^2} + 6\frac{dx}{dt} + x = 4\cos(2t) + 3e^{-2t}.$$

The Laplace transform of this is then

$$5(s^2\bar{x}(s) - sx(0) - x'(0)) + 6(s\bar{x}(s) - x(0)) + \bar{x}(s) = \frac{4s}{s^2+4} + \frac{3}{s+2}.$$

After inserting the given boundary conditions and rearranging we are thus face with inverting

$$\bar{x}(s) = \frac{10s+2}{5s^2+6s+1} + \frac{4s}{(s^2+4)(5s^2+6s+1)} + \frac{3}{(s+2)(5s^2+6s+1)}.$$

Using a partial fractions package gives

$$\bar{x}(s) = \frac{29}{20(s+1)} + \frac{1}{3(s+2)} + \frac{2225}{1212(5s+1)} - \frac{4(19s-24)}{505(s^2+4)}$$

and inverting yields

$$x(t) = \frac{1}{3}e^{-2t} + \frac{29}{20}e^{-t} + \frac{445}{1212}e^{-\frac{1}{5}t} - \frac{76}{505}\cos(2t) + \frac{48}{505}\sin(2t).$$

Substituting back for $y(t)$ gives

$$y(t) = \frac{2}{3}e^{2t} - \frac{29}{10}e^{-t} - \frac{1157}{606}e^{-\frac{1}{5}t} + \frac{72}{505}\cos(2t) + \frac{118}{505}\sin(2t).$$

(c) This last problem is fully fourth order, but we do not change the line of approach. The Laplace transform gives the simultaneous equations

$$(s^2 - 1)\bar{x}(s) + 5s\bar{y}(s) - 5 = \frac{1}{s^2}$$
$$-2s\bar{x}(s) + (s^2 - 4)\bar{y}(s) - s = -\frac{2}{s}$$

in which the boundary conditions have already been applied. Solving for $\bar{y}(s)$ gives

$$\bar{y}(s) = \frac{s^4 + 7s^2 + 4}{s(s^2 + 4)(s^2 + 1)} = \frac{1}{s} - \frac{2}{3}\frac{s}{s^2 + 4} + \frac{2}{3}\frac{s}{s^2 + 1}$$

which inverts to the solution

$$y(t) = 1 - \frac{2}{3}\cos(2t) + \frac{2}{3}\cos t.$$

Substituting back into the second original equation gives

$$x(t) = -t - \frac{5}{3}\sin t + \frac{4}{3}\sin(2t).$$

7. Using Laplace transforms, the transform of x is given by

$$\bar{x}(s) = \frac{A}{(s^2 + 1)(s^2 + k^2)} + \frac{v_0}{(s^2 + k^2)} + \frac{sx(0)}{(s^2 + k^2)}.$$

If $k \neq 1$ this inverts to

$$x(t) = \frac{A}{k^2 - 1}\left(\sin t - \frac{\sin(kt)}{k}\right) + \frac{v_0}{k}\sin(kt) + x_0\cos(kt).$$

If $k = 1$ there is a term $(1 + s^2)^2$ in the denominator, and the inversion can be done using convolution. The result is

$$x(t) = \frac{A}{2}(\sin t - t\cos t) + v_0 \sin t + x(0)\cos t$$

and it can be seen that this tends to infinity as $t \to \infty$ due to the term $t \cos t$. This is called a *secular* term. It is not present in the solution for $k \neq 1$ which is purely oscillatory. The presence of a secular term denotes resonance.

8. Taking the Laplace transform of the equation, using the boundary conditions and rearranging gives

$$\bar{x}(s) = \frac{sv_0 + g}{s^2(s+a)}$$

which after partial fractions becomes

$$\bar{x}(s) = \frac{-\frac{1}{a^2}(av_0 - g)}{s+a} + \frac{-\frac{1}{a^2}(av_0 - g)s + \frac{g}{a}}{s^2}.$$

This inverts to the expression in the question. The speed

$$\frac{dx}{dt} = \frac{g}{a} - \frac{(av_0 - g)}{a}e^{-at}.$$

As $t \to \infty$ this tends to g/a which is the required terminal speed.

9. The set of equations in matrix form is determined by taking the Laplace transform of each. The resulting algebraic set is expressed in matrix form as follows:

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ R_1 & sL_2 & 0 & 0 \\ R_1 & 0 & R_3 & 1/C \\ 0 & 0 & 1 & -s \end{pmatrix} \begin{pmatrix} \bar{j}_1 \\ \bar{j}_2 \\ \bar{j}_3 \\ \bar{q}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ L_2 j_2(0) + E\omega/(\omega^2 + s^2) \\ E\omega/(\omega^2 + s^2) \\ -q_3(0) \end{pmatrix}.$$

10. The Laplace transform of this fourth order equation is

$$k(s^4\bar{y}(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0)) = \frac{\omega_0}{c}\left(\frac{c}{s} - \frac{1}{s^2} + \frac{e^{-as}}{s^2}\right)$$

Using the boundary conditions is easy for those given at $x = 0$, the others give

$$y''(0) - 2cy'''(0) + \frac{5}{6}\omega_0 c^2 = 0 \quad \text{and} \quad y'''(0) = \frac{1}{2}\omega_0 c.$$

So $y''(0) = \frac{1}{6}\omega_0 c^2$ and the full solution is, on inversion

$$y(x) = \frac{1}{12}\omega_0 c^2 x^2 - \frac{1}{12}\omega_0 c x^3 + \frac{\omega_0}{120}\left[5cx^4 - x^5 + (x-c)^5 H(x-c)\right]$$

where $0 \le x \le 2c$. Differentiating twice and putting $x = c/2$ gives $y''(c/2) = \frac{1}{48}\omega_0 c^2$.

11. Taking the Laplace transform and using the convolution theorem gives

$$\bar{\phi}(s) = \frac{2}{s^3} + \bar{\phi}(s)\frac{1}{s^2+1}$$

from which

$$\bar{\phi}(s) = \frac{2}{s^5} + \frac{2}{s^2}.$$

Inversion gives the solution

$$\phi(t) = t^2 + \frac{1}{12}t^4.$$

Exercises 4.7

1. The Riemann–Lebesgue lemma is stated as Theorem 4.2. As the constants b_n in a Fourier sine series for $g(t)$ in $[0, \pi]$ are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(t)\sin(nt)dt$$

and these sine functions form a basis for the linear space of piecewise continuous functions in $[0, \pi]$ (with the usual inner product) of which $g(t)$ is a member, the Riemann–Lebesgue lemma thus immediately gives the result. More directly, Parseval's Theorem:

$$\int_{-\pi}^{\pi}[g(t)]^2dt = \pi a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

yields the results

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} g(t)\cos(nt)dt = 0$$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} g(t)\sin(nt)dt = 0$$

as the nth term of the series on the right has to tend to zero as $n \to \infty$. As $g(t)$ is piecewise continuous over the half range $[0, \pi]$ and is free to be defined as odd over the full range $[-\pi, \pi]$, the result follows.

2. The Fourier series is found using the formulae in Sect. 4.2. The calculation is routine if lengthy and the answer is

$$\begin{aligned} f(t) = \frac{5\pi}{16} &- \frac{2}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right) \\ &- \frac{2}{\pi}\left(\frac{\cos(2t)}{2^2} + \frac{\cos(6t)}{6^2} + \frac{\cos(10t)}{10^2}\cdots\right) \\ &+ \frac{1}{\pi}\left(\sin(t) - \frac{\sin(3t)}{3^2} + \frac{\sin(5t)}{5^2} - \frac{\sin(7t)}{7^2}\cdots\right). \end{aligned}$$

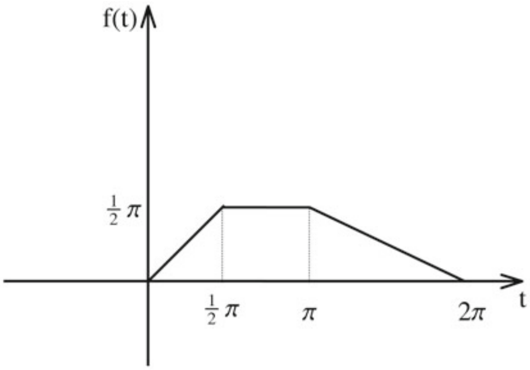


Fig. A.1 The original function composed of *straight line* segments

This function is displayed in Fig. A.1.

3. The Fourier series for $H(x)$ is found straightforwardly as

$$\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}.$$

Put $x = \pi/2$ and we get the series in the question and its sum:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

a series attributed to the Scottish mathematician James Gregory (1638–1675).

4. The Fourier series has the value

$$f(x) \sim -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n\sin(2nx)}{(2n+1)(2n-1)}.$$

5. This is another example where the Fourier series is found straightforwardly using integration by parts. The result is

$$1 - x^2 \sim \frac{1}{2}\left(\pi - \frac{\pi^3}{3}\right) - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

As the Fourier series is in fact continuous for this example there is no controversy, at $x = \pi$, $f(x) = 1 - \pi^2$.

6. Evaluating the integrals takes a little stamina this time.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)dx$$

and integrating twice by parts gives

$$b_n = \left[\frac{1}{a\pi}\sin(nx) - \frac{n}{a^2\pi}\cos(nx)\right]_{-\pi}^{\pi} - \frac{n^2}{a^2}b_n$$

from which

$$b_n = -\frac{2n\sinh\left(a\pi(-1)^n\right)}{a^2\pi}, \quad n = 1,\ 2, \ldots.$$

Similarly,

$$a_n = \frac{2a\sinh\left(a\pi(-1)^n\right)}{a^2\pi}, \quad n = 1,\ 2, \ldots,$$

and

$$a_0 = \frac{2\sinh(\pi a)}{\pi a}.$$

This gives the series in the question. Putting $x = 0$ gives the equation

$$e^0 = 1 = \frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{(-1)^n a}{n^2 + a^2}\right\}$$

from which

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + a^2} = \frac{1}{2a^2}(a\pi\text{cosech}\,(a\pi) - 1).$$

Also, since

$$\sum_{-\infty}^{\infty}\frac{(-1)^n}{n^2 + a^2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + a^2} + \frac{1}{a^2},$$

we get the result

$$\sum_{-\infty}^{\infty}\frac{(-1)^n}{n^2 + a^2} = \frac{\pi}{a}\text{cosech}\,(a\pi).$$

Putting $x = \pi$ and using Dirichlet's theorem (Theorem 4.3) we get

$$\frac{1}{2}(f(\pi) + f(-\pi)) = \cosh(a\pi) = \frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{a}{n^2 + a^2}\right\}$$

from which

$$\sum_{n=1}^{\infty}\frac{1}{n^2 + a^2} = \frac{1}{2a^2}(a\pi\coth\,(a\pi) - 1).$$

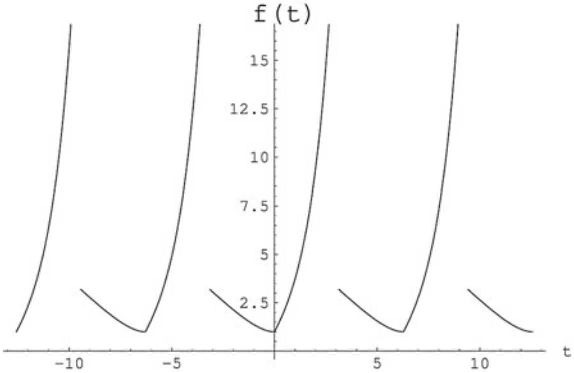


Fig. A.2 The function $f(t)$

Also, since

$$\sum_{-\infty}^{\infty} \frac{1}{n^2+a^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2}$$

we get the result

$$\sum_{-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(a\pi).$$

7. The graph is shown in Fig. A.2 and the Fourier series itself is given by

$$\begin{aligned} f(t) = \frac{1}{2}\pi + \frac{1}{\pi}\sinh(\pi) \\ + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2} + \frac{(-1)^n\sinh(\pi)}{n^2+1}\right]\cos(nt) \\ - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\sinh(\pi)\sin(nt). \end{aligned}$$

8. The Fourier series expansion over the range $[-\pi, \pi]$ is found by integration to be

$$f(t) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}\cos(nt) + \frac{(-1)^n}{n}\pi\sin(nt)\right] - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)^3}$$

and Fig. A.3 gives a picture of it. The required series are found by first putting $t = 0$ which gives

$$\pi^2 = \frac{2}{3}\pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

from which

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Putting $t = \pi$ gives, using Dirichlet's theorem (Theorem 4.3)

$$\frac{\pi^2}{2} = \frac{2}{3}\pi^2 - 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

from which

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

9. The sine series is given by the formula

$$a_n = 0 \quad b_n = \frac{2}{\pi}\int_0^{\pi}(t-\pi)^2\sin(nt)dt$$

with the result

$$f(t) \sim \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)t}{2k+1} + 2\pi\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}.$$

This is shown in Fig. A.5. The cosine series is given by

$$b_n = 0 \quad a_n = \frac{2}{\pi}\int_0^{\pi}(t-\pi)^2\cos(nt)dt$$

from which

$$f(t) \sim -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^2}$$

and this is pictured in Fig. A.4.

10. Parseval's theorem (Theorem 4.8) is

$$\int_{-\pi}^{\pi}[f(t)]^2dt = \pi a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

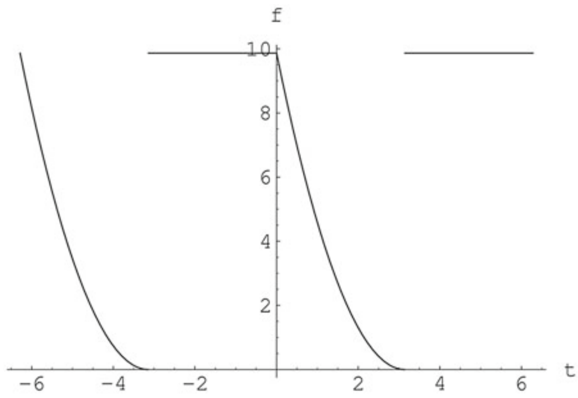


Fig. A.3 The function $f(t)$ as a full Fourier series

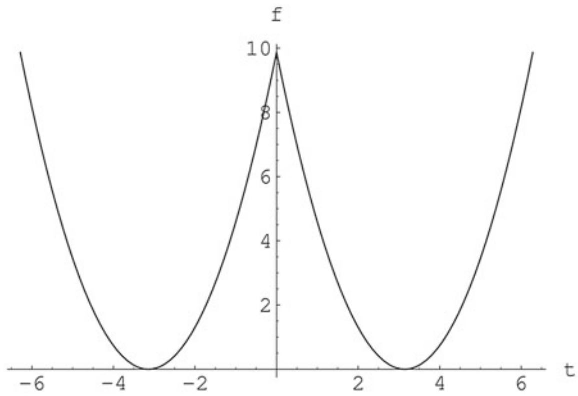


Fig. A.4 The function $f(t)$ as an even function

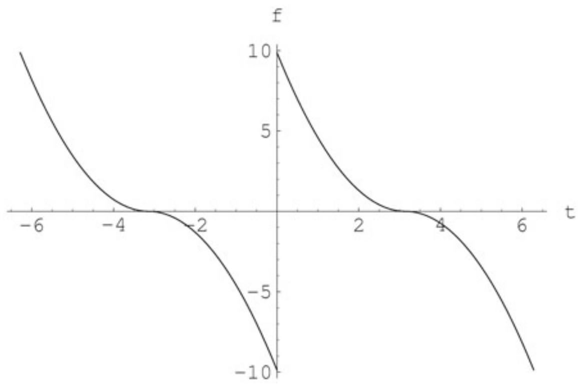


Fig. A.5 The function $f(t)$ as an odd function

Applying this to the Fourier series in the question is not straightforward, as we need the version for sine series. This is easily derived as

$$\int_0^{\pi} [f(t)]^2 dt = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2.$$

The left hand side is

$$\int_0^{\pi} t^2(\pi - t)^2 dt = \frac{\pi^5}{30}.$$

The right hand side is

$$\frac{32}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$$

whence the result

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

Noting that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \frac{1}{2^6} \sum_{n=1}^{\infty} \frac{1}{n^6}$$

gives the result

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{64}{63} \frac{\pi^6}{960} = \frac{\pi^6}{945}.$$

11. The Fourier series for the function x^4 is found as usual by evaluating the integrals

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \cos(nx) dx$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \sin(nx) dx.$$

However as x^4 is an even function, $b_n = 0$ and there are no sine terms. Evaluating the integral for the cosine terms gives the series in the question. With $x = 0$, the series becomes

$$0 = \frac{\pi^4}{5} + 8\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

and using the value of the second series $(\pi^2)/12$ gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{15}\frac{1}{48} = \frac{7\pi^4}{720}.$$

Differentiating term by term yields the Fourier series for x^3 for $-\pi < x < \pi$ as

$$x^3 \sim \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^3}(6 - \pi^2 n^2)\sin(nx).$$

12. Integrating the series

$$x \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$$

term by term gives

$$\frac{x^2}{2} \sim \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \cos(nx) + A$$

where A is a constant of integration. Integrating both sides with respect to x between the limits $-\pi$ and π gives

$$\frac{\pi^3}{3} = 2A\pi.$$

Hence $A = \pi^2/6$ and the Fourier series for x^2 is

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx),$$

where $-\pi < x < \pi$. Starting with the Fourier series for x^4 over the same range and integrating term by term we get

$$\frac{x^5}{5} \sim \frac{\pi^4 x}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^5}(\pi^2 n^2 - 6)\sin(nx) + B$$

where B is the constant of integration. This time setting $x = 0$ immediately gives $B = 0$, but there is an x on the right-hand side that has to be expressed in terms of a Fourier series. We thus use

$$x \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$$

to give the Fourier series for x^5 in $[-\pi, \pi]$ as

$$x^5 \sim \sum_{n=1}^{\infty}(-1)^n\left[\frac{40\pi^2}{n^3} - \frac{240}{n^5} - \frac{2\pi^4}{n}\right]\sin(nx).$$

13. Using the complex form of the Fourier series, we have that

$$V(t) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi it/5}.$$

The coefficients are given by the formula

$$c_n = \frac{1}{5}\int_0^5 V(t)e^{2in\pi t/5}dt.$$

By direct calculation they are

$$\begin{aligned}
c_{-4} &= \frac{5}{\pi}\left(i + e^{7\pi i/10}\right)\\
c_{-3} &= \frac{20}{3\pi}\left(i - e^{9\pi i/10}\right)\\
c_{-2} &= \frac{10}{\pi}\left(i - e^{\pi i/10}\right)\\
c_{-1} &= \frac{20}{\pi}\left(i + e^{3\pi i/10}\right)\\
c_0 &= 16\\
c_1 &= \frac{20}{\pi}\left(-i - e^{7\pi i/10}\right)\\
c_2 &= \frac{10}{\pi}\left(-i + e^{9\pi i/10}\right)\\
c_3 &= \frac{20}{3\pi}\left(-i + e^{\pi i/10}\right)\\
c_4 &= \frac{10}{\pi}\left(-i - e^{3\pi i/10}\right).
\end{aligned}$$

14. The differential equation can be written

$$\frac{d}{dt}\left[(1-t^2)\frac{dP_n}{dt}\right] = -n(n+1)P_n.$$

This means that the integral can be manipulated using integration by parts as follows:

$$\begin{aligned}\int_{-1}^{1} P_m P_n dt &= -\frac{1}{n(n+1)}\int_{-1}^{1}\frac{d}{dt}\left[(1-t^2)\frac{dP_n}{dt}\right]P_m dt\\ &= -\frac{1}{n(n+1)}\left[(1-t^2)\frac{dP_n}{dt}\frac{dP_m}{dt}\right]_{-1}^{1}\\ &\quad+\frac{1}{n(n+1)}\int_{-1}^{1}\frac{d}{dt}\left[(1-t^2)\frac{dP_m}{dt}\right]P_n dt\\ &=\frac{1}{n(n+1)}\left[(1-t^2)\frac{dP_m}{dt}\frac{dP_n}{dt}\right]_{-1}^{1}+\frac{m(m+1)}{n(n+1)}\int_{-1}^{1}P_m P_n dt\\ &=\frac{m(m+1)}{n(n+1)}\int_{-1}^{1}P_m P_n dt,\end{aligned}$$

all the integrated bits being zero. Therefore

$$\int_{-1}^{1} P_m P_n dt = 0$$

unless $m = n$ as required.

15. The first part is straightforward. Substituting $y = \sin(n\ln t)$ into the equation for y gives

$$\frac{dy}{dt}=\frac{n}{t}\cos(n\ln t),\quad \text{so}\quad t\frac{dy}{dt}=n\cos(n\ln t)$$

thus

$$t\frac{d}{dt}\left\{t\frac{dy}{dt}\right\}=-n^2\sin(n\ln t)=-n^2 y$$

as required. In order to do the next part we follow the last example in the chapter, or the previous exercise and integrate the ordinary differential equation. Write the ODE as:

$$\frac{d}{dt}\left\{t\frac{d\phi_n}{dt}\right\}+\frac{n^2}{t}\phi_n=0 \tag{A.1}$$

write it again substituting m for n:

$$\frac{d}{dt}\left\{t\frac{d\phi_m}{dt}\right\}+\frac{m^2}{t}\phi_m=0 \tag{A.2}$$

then form the combination $\phi_m\times$ Eq. (A.1) $-\ \phi_n\times$ Eq. (A.2) to obtain

$$\phi_m \frac{d}{dt}\left\{t\frac{d\phi_n}{dt}\right\} - \phi_n \frac{d}{dt}\left\{t\frac{d\phi_m}{dt}\right\} + \frac{n^2-m^2}{t}\phi_m\phi_n = 0$$

Integrating this between the two zeros (these are where $\ln t$ is 0 and π so $t = 1$ and e^{π}). The first pair of terms yield to integration by parts:

$$\left[\phi_m \cdot t \cdot \frac{d\phi_n}{dt} - \phi_n \cdot t \cdot \frac{d\phi_m}{dt}\right]_0^{e^{\pi}} - \int_0^{e^{\pi}} \frac{d\phi_m}{dt} \cdot t \cdot \frac{d\phi_n}{dt} - \frac{d\phi_n}{dt} \cdot t \cdot \frac{d\phi_m}{dt} dt$$

and both of these are zero, the first term as it vanishes at both limits and in the second the integrand is identically zero. Hence we have

$$\int_1^{e^{\pi}} \frac{n^2-m^2}{t}\phi_m\phi_n dt = (m^2-n^2)\int_1^{e^{\pi}} \frac{\phi_m\phi_n}{t} dt = 0$$

which implies orthogonality with weight function $1/t$.

16. The half range Fourier series for $g(x)$ is

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx = \frac{2}{n\pi}\left[-\cos(nx)\right]_0^{\pi} = \frac{2}{n\pi}[1-(-1)^n]$$

so $b_{2k} = 0$ and $b_{2k+1} = \dfrac{4}{(2k+1)\pi}$ So with $x = \ln t$

$$f(t) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{4k+1} \sin\{(2k+1)\ln t\}$$

Exercises 5.6

1. Using the separation of variable technique with

$$\phi(x,t) = \sum_k X_k(x)T_k(t)$$

gives the equations for $X_k(x)$ and $T_k(t)$ as

$$\frac{T_k'}{T_k} = \frac{\kappa X_k''}{X_k} = -\alpha^2$$

where $-\alpha^2$ is the separation constant. It is negative because $T_k(t)$ must not grow with t. In order to satisfy the boundary conditions we express $x(\pi/4 - x)$ as a Fourier sine series in the interval $0 \le x \le \pi/4$ as follows

$$x(x - \frac{\pi}{4}) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\sin[4(2k-1)x].$$

Now the function $X_k(x)$ is identified as

$$X_k(x) = \frac{1}{2\pi}\frac{1}{(2k-1)^3}\sin[4(2k-1)x]$$

so that the equation obeyed by $X_k(x)$ together with the boundary conditions $\phi(0, t) = \phi(\pi/4, t) = 0$ for all time are satisfied. Thus

$$\alpha = \frac{4(2k-1)\pi}{\sqrt{\kappa}}.$$

Putting this expression for $X_k(x)$ together with

$$T_k(t) = e^{-16(2k-1)^2\pi^2 t/\kappa}$$

gives the solution

$$\phi(x, t) = \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}e^{-16(2k-1)^2\pi^2 t/\kappa}\sin[4(2k-1)x].$$

2. The equation is

$$a\frac{\partial^2\phi}{\partial x^2} - b\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial t} = 0.$$

Taking the Laplace transform (in t) gives the ODE

$$a\bar{\phi}'' - b\bar{\phi}' - s\bar{\phi} = 0$$

after applying the boundary condition $\phi(x, 0) = 0$. Solving this and noting that

$$\phi(0, t) = 1 \Rightarrow \bar{\phi}(0, s) = \frac{1}{s} \quad \text{and} \quad \bar{\phi}(x, s) \to 0 \quad \text{as} \quad x \to \infty$$

gives the solution

$$\bar{\phi} = \frac{1}{s}\exp\left\{\frac{x}{2a}[b - \sqrt{b^2 + 4as}]\right\}.$$

3. Taking the Laplace transform of the heat conduction equation results in the ODE

$$\kappa \frac{d^2\bar{\phi}}{dx^2} - s\bar{\phi} = -x\left(\frac{\pi}{4} - x\right).$$

Although this is a second order non-homogeneous ODE, it is amenable to standard complimentary function, particular integral techniques. The general solution is

$$\bar{\phi}(x, s) = A\cosh\left(x\sqrt{\frac{s}{\kappa}}\right) + B\sinh\left(x\sqrt{\frac{s}{\kappa}}\right) - \frac{x^2}{s} + \frac{\pi x}{4s} - \frac{2\kappa}{s^2}.$$

The inverse of this gives the expression in the question. If this inverse is evaluated term by term using the series in Appendix A, the answer of Exercise 1 is not regained immediately. However, if the factor $2\kappa t$ is expressed as a Fourier series, then the series solution is the same as that in Exercise 1.

4. Taking the Laplace transform in y gives the ODE

$$\frac{d^2\bar{\phi}}{dx^2} = s\bar{\phi}.$$

Solving this, and applying the boundary condition $\phi(0, y) = 1$ which transforms to

$$\bar{\phi}(0, s) = \frac{1}{s}$$

gives the solution

$$\bar{\phi}(x, s) = \frac{1}{s}e^{-x\sqrt{s}}$$

which inverts to

$$\phi(x, y) = \text{erfc}\left\{\frac{x}{2\sqrt{y}}\right\}.$$

5. Using the transformation suggested in the question gives the equation obeyed by $\phi(x, t)$ as

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}.$$

This is the standard wave equation. To solve this using Laplace transform techniques, we transform in the variable t to obtain the equation

$$\frac{d^2\bar{\phi}}{dx^2} - \frac{s^2}{c^2}\bar{\phi} = -\frac{s}{c^2}\cos(mx).$$

The boundary conditions for $\phi(x, t)$ are

$$\phi(x, 0) = \cos(mx) \quad \text{and} \quad \phi'(x, 0) = -\frac{k}{2}\phi(x, 0) = -\frac{k}{2}\cos(mx).$$

This last condition arises since

$$u'(x,t) = \frac{k}{2}e^{kt/2}\phi(x,t) + e^{kt/2}\phi'(x,t).$$

Applying these conditions gives, after some algebra

$$\bar{\phi}(x,s) = \left[\frac{1}{s} - \frac{s-\frac{k}{2}}{s^2+m^2c^2}\cos(mx)\right]e^{-\frac{sx}{c}} + \frac{s-\frac{k}{2}}{s^2+m^2c^2}\cos(mx).$$

Using the second shift theorem (Theorem 2.4) we invert this to obtain

$$u = \begin{cases} e^{kt/2}(1-\cos(mct-mx)\cos(mx) + \frac{k}{2mc}\sin(mct-mx)\cos(mx) \\ \qquad +\cos(mct)\cos(mx) - \frac{k}{2mc}\sin(mct)\cos(mx)) \qquad t > x/c \\ \\ e^{kt/2}(\cos(mct)\cos(mx) - \frac{k}{2mc}\sin(mct)\cos(mx)) \qquad t < x/c. \end{cases}$$

6. Taking the Laplace transform of the one dimensional heat conduction equation gives

$$s\bar{u} = c^2\bar{u}_{xx}$$

as $u(x,0) = 0$. Solving this with the given boundary condition gives

$$\bar{u}(x,s) = \bar{f}(s)e^{-x\sqrt{s}/c}.$$

Using the standard form

$$\mathcal{L}^{-1}\{e^{-a\sqrt{s}}\} = \frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$$

gives, using the convolution theorem

$$u = \frac{x}{2}\int_0^t f(\tau)\sqrt{\frac{k}{\pi(t-\tau)^3}}e^{-x^2/4k(t-\tau)}d\tau.$$

When $f(t) = \delta(t)$, $u = \frac{x}{2}\sqrt{\frac{k}{\pi t^3}}e^{-x^2/4kt}$.

7. Assuming that the heat conduction equation applies gives that

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}$$

so that when transformed $\bar{T}(x,s)$ obeys the equation

$$s\bar{T}(x,s) - T(x,0) = \kappa \frac{d^2\bar{T}}{dx^2}(x,s).$$

Now, $T(x,0) = 0$ so this equation has solution

$$\bar{T}(x,s) = \bar{T}_0 e^{-x\sqrt{\frac{s}{\kappa}}}$$

and since

$$\frac{d\bar{T}}{dx}(s,0) = -\frac{\alpha}{s}$$

$$\bar{T}_0 = \alpha\sqrt{\frac{\kappa}{s}}$$

and the solution is using standard forms (or Chap. 3)

$$T(x,t) = \alpha\sqrt{\frac{\kappa}{\pi t}} e^{-x^2/4\kappa t}.$$

This gives, at $x = 0$ the desired form for $T(0,t)$. Note that for non-zero x the solution is not singular at $t = 0$.

8. Taking the Laplace transform of the wave equation given yields

$$s^2\bar{u} - \frac{\partial u}{\partial t}(x,0) = c^2\frac{d^2\bar{u}}{dx^2}$$

so that substituting the series

$$\bar{u} = \sum_{n=0}^{\infty} \frac{a_n(x)}{s^{n+k+1}}, \quad k \text{ integer}$$

as in Example 5.5 gives $k = 1$ all the odd powers are zero and

$$\frac{a_0}{s^2} + \frac{a_2}{s^4} + \frac{a_6}{s^6} + \cdots - \cos(x) = c^2\left(\frac{a_0''}{s^2} + \frac{a_2''}{s^4} + \frac{a_6''}{s^6} + \cdots\right)$$

so that

$$a_0 = \cos(x), \quad a_2 = c^2 a_0'' = -c^2\cos(x) \quad a_4 = c^4 \cos x \quad \text{etc.}$$

Hence

$$\bar{u}(x,s) = \cos(x)\left(\frac{1}{s^2} - \frac{c^2}{s^4} + \frac{c^4}{s^6} - \frac{c^6}{s^8} + \cdots\right)$$

which inverts term by term to

$$u(x,t) = \cos x \left(t - \frac{c^2 t^3}{3!} + \frac{c^4 t^5}{5!} + \cdots \right)$$

which in this case converges to the closed form solution

$$u = \frac{1}{c} \cos x \sin(ct).$$

9. For this problem we proceed similarly. Laplace transforms are taken and the equation to solve this time is

$$s^2 \bar{u} - s u(x,0) = c^2 \frac{d^2 \bar{u}}{dx^2}.$$

Once more substituting the series

$$\bar{u} = \sum_{n=0}^{\infty} \frac{a_n(x)}{s^{n+k+1}}, \quad k \text{ integer}$$

gives this time

$$a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots - \cos x = c^2 \left(a_0'' + \frac{a_1''}{s} + \frac{a_2''}{s^2} + \cdots \right)$$

so that

$$a_0 = \cos x, \quad a_1 = 0 \;\; a_2 = c^2 a_0'' = -c^2 \cos x \;\; a_3 = 0 \;\; a_4 = c^4 \cos x \;\; \text{etc.}$$

giving

$$\bar{u}(x,s) = \sum_{n=0}^{\infty} \frac{c^{2n}(-1)^n \cos x}{s^{2n+1}}.$$

Inverting term by term gives the answer

$$u(x,t) = \cos x \sum_{n=0}^{\infty} (-1)^n \frac{c^{2n} t^{2n}}{2n!}$$

which in fact in this instance converges to the result

$$u = \cos x \cos(ct).$$

Exercises 6.6

1. With the function $f(t)$ as defined, simple integration reveals that

$$
\begin{aligned}
F(\omega) &= \int_{-T}^{0} (t+T)e^{-i\omega t}dt + \int_{0}^{T} (T-t)e^{-i\omega t}dt \\
&= 2\int_{0}^{T} (T-t)\cos(\omega t)dt \\
&= \frac{2}{\omega}\left[(T-t)\frac{\sin(\omega t)}{\omega}\right]_0^T + \frac{2}{\omega}\int_0^T \sin(\omega t)dt \\
&= \frac{2}{\omega}\left[-\frac{\cos(\omega t)}{\omega}\right]_0^T \\
&= \frac{2(1-\cos(\omega T))}{\omega^2}
\end{aligned}
$$

2. With $f(t) = e^{-t^2}$ the Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} e^{-t^2}e^{-i\omega t}dt = \int_{-\infty}^{\infty} e^{-(t-\frac{1}{2}i\omega)^2}e^{-\frac{1}{4}\omega^2}dt.$$

Now although there is a complex number $(\frac{1}{2}i\omega)$ in the integrand, the change of variable $u = t - \frac{1}{2}i\omega$ can still be made. The limits are actually changed to $-\infty - \frac{1}{2}i\omega$ and $\infty - \frac{1}{2}i\omega$ but this does not change its value so we have that

$$\int_{-\infty}^{\infty} e^{-(t-\frac{1}{2}i\omega)^2}dt = \sqrt{\pi}.$$

Hence

$$F(\omega) = \sqrt{\pi}e^{-\frac{1}{4}\omega^2}.$$

3. Consider the Fourier transform of the square wave, Example 6.1. The inverse yields:

$$\frac{A}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega T)}{\omega}e^{i\omega t}d\omega = A$$

provided $|t| \leq T$. Let $t = 0$ and we get

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega T)}{\omega}d\omega = 1$$

Putting $T = 1$ and spotting that the integrand is even gives the result.

4. Using the definition of convolution given in the question, we have

$$\begin{aligned} f(at) * f(bt) &= \int_{-\infty}^{\infty} f(a(t-\tau)) f(b\tau) d\tau. \\ &= e^{-at} \int_0^{\infty} f(b\tau - a\tau) d\tau \\ &= -e^{-at} \frac{1}{b-a} [f(bt - at) - 1] \\ &= \frac{f(at) - f(bt)}{b-a}. \end{aligned}$$

As $b \to a$ we have

$$\begin{aligned} f(at) * f(at) &= -\lim_{b \to a} \frac{f(bt) - f(at)}{b-a} \\ &= -\frac{d}{da} f(at) = -tf'(at) = tf(at). \end{aligned}$$

5. With

$$g(x) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i x t} dt$$

let $u = t - 1/2x$, so that $du = dt$. This gives

$$e^{-2\pi i x t} = e^{-2\pi i x (u + 1/2x)} = -e^{-2\pi x u}.$$

Adding these two versions of $g(x)$ gives

$$\begin{aligned} |g(x)| &= \left| \frac{1}{2} \int_{-\infty}^{\infty} (f(u) - f(u + 1/2x)) e^{-2\pi i x u} du \right| \\ &\le \frac{1}{2} \int_{-\infty}^{\infty} |f(u) - f(u + 1/2x)| du \end{aligned}$$

and as $x \to \infty$, the right hand side $\to 0$. Hence

$$\int_{-\infty}^{\infty} f(t) \cos(2\pi x t) dt \to 0 \text{ and } \int_{-\infty}^{\infty} f(t) \sin(2\pi x t) dt \to 0$$

which illustrates the Riemann–Lebesgue lemma.

6. First of all we note that

$$G(it) = \int_{-\infty}^{\infty} g(\omega) e^{-i^2 \omega t} d\omega = \int_{-\infty}^{\infty} g(\omega) e^{\omega t} d\omega$$

therefore

$$\int_{-\infty}^{\infty} f(t)G(it)dt = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} g(\omega)e^{\omega t} d\omega dt.$$

Assuming that the integrals are uniformly convergent so that their order can be interchanged, the right hand side can be written

$$\int_{-\infty}^{\infty} g(\omega) \int_{-\infty}^{\infty} f(t)e^{\omega t} dt d\omega,$$

which is, straight away, in the required form

$$\int_{-\infty}^{\infty} g(\omega)F(i\omega)d\omega.$$

Changing the dummy variable from ω to t completes the proof.

7. Putting $f(t) = 1 - t^2$ where $f(t)$ is zero outside the range $-1 \leq t \leq 1$ and $g(t) = e^{-t}, \quad 0 \leq t < \infty$, we have

$$F(\omega) = \int_{-1}^{1} (1 - t^2)e^{-i\omega t} dt$$

and

$$G(\omega) = \int_{0}^{\infty} e^{-t} e^{i\omega t} dt.$$

Evaluating these integrals (the first involves integrating by parts twice) gives

$$F(\omega) = \frac{4}{\omega^3} (\omega \cos \omega - \sin \omega)$$

and

$$G(\omega) = \frac{1}{1 + i\omega}.$$

Thus, using Parseval's formula from the previous question, the imaginary unit disappears and we are left with

$$\int_{-1}^{1} (1 + t)dt = \int_{0}^{\infty} \frac{4e^{-t}}{t^3} (t \cosh t - \sinh t)\, dt$$

from which the desired integral is 2. Using Parseval's theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

we have that

$$\int_{-1}^{1}(1-t^2)^2dt = \frac{1}{2\pi}\int_0^\infty \frac{16\,(t\cos t - \sin t)^2}{t^6}dt.$$

Evaluating the integral on the left we get

$$\int_0^\infty \frac{(t\cos t - \sin t)^2}{t^6}dt = \frac{\pi}{15}.$$

8. The ordinary differential equation obeyed by the Laplace transform $\bar{u}(x, s)$ is

$$\frac{d^2\bar{u}(x,s)}{dx^2} - \frac{s}{k}\bar{u}(x,s) = -\frac{g(x)}{k}.$$

Taking the Fourier transform of this we obtain the solution

$$\bar{u}(x,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(\omega)}{s+\omega^2 k}e^{i\omega x}d\omega$$

where

$$G(\omega) = \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx$$

is the Fourier transform of $g(x)$. Now it is possible to write down the solution by inverting $\bar{u}(x, s)$ as the Laplace variable s only occurs in the denominator of a simple fraction. Inverting using standard forms thus gives

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega}e^{-\omega^2 kt}d\omega.$$

It is possible by completing the square and using the kind of "tricks" seen in Sect. 3.2 to convert this into the solution that can be obtained directly by Laplace transforms and convolution, namely

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-(x-\tau)^2/4t}g(\tau)d\tau.$$

9. To convert the partial differential equation into the integral form is a straightforward application of the theory of Sect. 6.4. Taking Fourier transforms in x and y using the notation

$$v(\lambda, y) = \int_{-\infty}^{\infty} u(x,y)e^{-i\lambda x}dx$$

and

$$w(\lambda, \mu) = \int_{-\infty}^{\infty} v(\lambda, y)e^{-i\mu y}dy$$

we obtain

$$-\lambda^2 w - \mu^2 w + k^2 w = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-\lambda\xi}e^{-\mu\eta}d\xi d\eta.$$

Using the inverse Fourier transform gives the answer in the question. The conditions are that for both $u(x, y)$ and $f(x, y)$ all first partial derivatives must vanish at $\pm\infty$.

10. The Fourier series written in complex form is

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Now it is straightforward to use the window function $W(x)$ to write

$$c_n = \frac{1}{2}\int_{-\infty}^{\infty} W\left(\frac{x-\pi}{2\pi}\right) f(x)e^{-inx}dx.$$

11. The easiest way to see how discrete Fourier transforms are derived is to consider a sampled time series as the original time series $f(t)$ multiplied by a function that picks out the discrete (sampled) values leaving all other values zero. This function is related to the Shah function (train of Dirac-δ functions) is not necessarily (but is usually) equally spaced. It is designed by using the window function $W(x)$ met in the last question. With such a function, taking the Fourier transform results in the finite sum of the kind seen in the question. The inverse is a similar evaluation, noting that because of the discrete nature of the function, there is a division by the total number of data points.
12. Inserting the values $\{1, 2, 1\}$ into the series of the previous question, $N = 3$ and $T = 1$ so we get the three values

$$F_0 = 1 + 2 + 1 = 4; \quad F_1 = 1 + 2e^{-2\pi i/3} + e^{-4\pi i/3} = e^{-2\pi i/3};$$

and

$$F_2 = 1 + 2e^{-4\pi i/3} + e^{-8\pi i/3} = e^{-4\pi i/3}.$$

13. Using the definition and essential property of the Dirac-δ function we have

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-i\omega t}dt = e^{-i\omega t_0}$$

inverting this gives the required answer

$$\delta(t - t_0) = \int_{-\infty}^{\infty} e^{-i\omega t_0} e^{i\omega t} d\omega = \int_{-\infty}^{\infty} e^{i\omega(t-t_0)} d\omega.$$

whence the second result is

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega.$$

14. Using the exponential forms of sine and cosine the Fourier transforms are immediately

$$\mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{2}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$
$$\mathcal{F}\{\sin(\omega_0 t)\} = \frac{1}{2i}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)).$$

Exercises 7.9

1. The wavelets being considered are $[1, 1, 1, 1]^T$, $[1, 1, -1, -1]^T$, $[1, -1, 0, 0]^T$ and $[0, 0, 1, -1]^T$ So we solve α times the first plus β times the second plus γ time the third plus δ times the fourth equals $[4, 8, 10, 14]^T$ which leads to

$$\alpha + \beta + \gamma = 4$$
$$\alpha + \beta - \gamma = 8$$
$$\alpha - \beta + \delta = 10$$
$$\alpha - \beta - \delta = 14.$$

Solving (easily by hand) gives $\alpha = 9$, $\beta = -3$, $\gamma = 6$ and $\delta = 12$. When the right hand side is $[a, b, c, d]^T$ the result is still easy to obtain by hand:

$$\alpha = \frac{1}{4}(a + b + c + d)$$
$$\beta = \frac{1}{4}(a + b - c - d)$$
$$\gamma = \frac{1}{2}(a + b)$$
$$\delta = \frac{1}{2}(c + d)$$

2. The 16 basis functions are the columns of the matrix:

$$\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

This layout is useful for the next part of the question. It is quickly apparent that each column is orthogonal to any other as the inner product is found by multiplying equivalent terms and adding up the whole 16. All of these sums are zero. The vectors are not orthonormal. Finally the vector $(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ can be put in terms of the basis functions as follows. Labelling the basis vectors a, b, up to p, then forming $\frac{1}{2}(a+b)$ we get a vector with 8 ones then below this, 8 zeros. Adding this to the third column c and diving by 2 gives a vector with 4 ones in the first four places and twelve zeros below that. The calculation is $\frac{1}{4}(a+b)+\frac{1}{2}c$. Now continuing in this fashion, add this vector to the vector represented by column e and divide by two to give two ones in the first two entries with 14 zeros below. The calculation is $\frac{1}{8}(a+b)+\frac{1}{4}c+\frac{1}{2}e$. Finally add this to the vector represented by i and divide by 2 and we get $(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ that achieves the required vector. The calculation is

$$\frac{1}{16}(a+b)+\frac{1}{8}c+\frac{1}{4}e+\frac{1}{2}i$$

3. Calculating the Fourier transform of $\psi_{j,k}(t) = 2^{j/2}\psi\left(2^j t - k\right)$ is a bit tricky. Note that

$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

so $\psi\left(2^j t - k\right)$ is always either 1 −1 or 0. In fact

$$\psi\left(2^j t - k\right) = \begin{cases} 1 & 2^j k \le t < (k+\frac{1}{2})2^j \\ -1 & (k+\frac{1}{2})2^j \le t < (k+1)2^j \\ 0 & \text{otherwise} \end{cases}$$

So, using the definition of Fourier transform:

$$\hat{\psi}_{j,k}(\omega) = \int_{-\infty}^{\infty} \psi_{j,k}(t) e^{-i\omega t} dt = 2^{j/2} \int_{-\infty}^{\infty} \psi\left(2^j t - k\right) e^{-i\omega t} dt$$

then using the above definition of $\psi\left(2^j t - k\right)$ we get

$$\begin{aligned}
\hat{\psi}_{j,k}(\omega) &= \int_{-\infty}^{\infty} 2^{j/2}\psi\left(2^j t - k\right) e^{-i\omega t} dt \\
&= 2^{j/2}\left[\int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} e^{-i\omega t} dt - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} e^{-i\omega t} dt\right] \\
&= 2^{j/2}\left[-\frac{e^{-i\omega t}}{i\omega}\bigg|_{k2^{j/2}}^{(k+\frac{1}{2})2^{j/2}} + \frac{e^{-i\omega t}}{i\omega}\bigg|_{(k+\frac{1}{2})2^{j/2}}^{(k+1)2^{j/2}}\right] \\
&= 2^{j/2}\frac{e^{-i\omega k 2^{-j}}}{i\omega}\left\{1 - 2e^{-i\omega(k+\frac{1}{2})2^{-j}} + e^{-i\omega(k+1)2^{-j}}\right\}
\end{aligned}$$

Some manipulation then gives

$$\hat{\psi}_{j,k}(\omega) = -\frac{2^{(j+4)/2}}{i\omega} e^{-i\omega(k+\frac{1}{2})2^{-j}} \sin^2\left(\frac{\omega}{4}2^{-j}\right)$$

which is the required Fourier transform.

4. Consider the general offspring (daughter) wavelet function

$$2^{j/2}\psi\left(2^j t - k\right)$$

So that

$$\int_{-\infty}^{\infty} |2^{j/2}\psi\left(2^j t - k\right)|^2 dt = 2^j \int_{k2^{-j}}^{(k+1)2^{-j}} 1 dt = 2^j[(k+1)2^{-j} - k2^{-j}] = 1.$$

This is independent of both j and k and so is the same as the mother wavelet $j = k = 0$ and for all daughter wavelets. This proves the result. It has the value 1.

5. With $f(t) = e^{-\frac{1}{2}t^2}$ before we calculate either the centre μ_f or the RMS value, we need to find the norm $||f(t)||$ defined by

$$||f(t)||^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

so $||f(t)|| = \pi^{1/4}$. The centre μ_f is defined by

$$\mu_f = \frac{1}{||f(t)||^2}\int_{-\infty}^{\infty} t|e^{-\frac{1}{2}t^2}|^2 dt = \pi^{-1/2}\left[-\frac{1}{2}e^{-t^2}\right]_{-\infty}^{\infty} = 0$$

so the centre is zero, this should not be a surprise. It could be deduced at once from the symmetry of $f(t)$.

$$\Delta_f = \frac{1}{||f(t)||}\left[\int_{-\infty}^{\infty} t^2 e^{-t^2} dt\right]^{1/2}$$

Using

$$\int_{-\infty}^{\infty} t^2 e^{-t^2} dt = \left[t \cdot \frac{1}{2}e^{-t^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2}e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}$$

gives

$$\Delta_f = \frac{1}{\pi^{1/4}}\sqrt{\frac{1}{2}\sqrt{\pi}} = \frac{1}{\sqrt{2}}$$

Given $\hat{f}(\omega) = \sqrt{2\pi}e^{-\frac{1}{2}\omega^2}$ the calculation of the centre $\mu_{\hat{f}}$ and $\Delta_{\hat{f}}$ follows along the same lines. This time $||\hat{f}(\omega)||^2 = 2\pi\sqrt{\pi}$, the centre $\mu_{\hat{f}}$ is of course once again zero. The RMS $\Delta_{\hat{f}}$ is once again $\frac{1}{\sqrt{2}}$ as the factor 2π cancels. Hence the Heisenberg inequality becomes

$$\Delta_f \cdot \Delta_{\hat{f}} = \frac{1}{2}$$

that is, equality.

6. Repeating the details of the above calculation is not necessary. With

$$f(t) = \frac{1}{2\sqrt{\alpha\pi}}e^{-\frac{t^2}{4\alpha}}$$

the norm is $||f(t)||$ where

$$||f(t)||^2 = \frac{1}{4\alpha\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\alpha}} dt = \frac{1}{2\sqrt{2\alpha\pi}}.$$

Thus the centre $\mu_f = 0$ as before but the RMS Δ_f is given by

$$\Delta_f^2 = \frac{1}{||f(t)||^2}\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\alpha}} dt = \alpha.$$

Similarly, $||\hat{f}(\omega)||$ is given by

$$||\hat{f}(\omega)||^2 = \int_{-\infty}^{\infty} e^{-2\alpha\omega^2} d\omega = \sqrt{\frac{\pi}{2\alpha}}$$

with $\mu_{\hat{f}} = 0$ and

$$\Delta_{\hat{f}}^2 = \frac{1}{||\hat{f}(\omega)||^2} \int_{-\infty}^{\infty} \omega^2 e^{-2\alpha\omega^2} d\omega = \frac{1}{4\alpha}.$$

Thus giving

$$\Delta_f \cdot \Delta_{\hat{f}} = \frac{1}{2}$$

as before, independent of α.

7. Linearity is straightforward to show. Form the linear sum

$$\alpha f_{1,G}(t_0, \omega) + \beta f_{2,G}(t_0, \omega) = \alpha \int_{-\infty}^{\infty} f_1(t)\overline{b_{t_0,\omega}(t)}dt + \beta \int_{-\infty}^{\infty} f_2(t)\overline{b_{t_0,\omega}(t)}dt$$

The right hand side can be written

$$\int_{-\infty}^{\infty} (\alpha f_1(t) + \beta f_2)\overline{b_{t_0,\omega}(t)}dt = f_G(t_0, \omega)$$

where

$$f(t) = \alpha f_1(t) + \beta f_2(t).$$

This establishes linearity.

(a) If $f_0 = f(t - t_1)$ then

$$f_{1G_b}(t_1, \omega) = \int_{-\infty}^{\infty} f(t - t_1)b(t - t_0)e^{-i\omega t} dt$$

Write $t - t_1 = \tau$ then $dt = d\tau$ and the right hand side is

$$= \int_{-\infty}^{\infty} f(\tau)b(\tau + t_1 - t_0)e^{-i\omega\tau - i\omega t_1} d\tau = e^{-i\omega t_1} f_{1G_b}(t_0 - t_1, \omega)$$

as required.

(b) With

$$f_2 = f(t)e^{i\omega_2 t}$$

we have

$$f_{2G_b}(t_0, \omega) = \int_{-\infty}^{\infty} f_2(t)e^{i\omega_2 t} b(t-t_0)e^{-i\omega t} dt = \int_{-\infty}^{\infty} f_2(t)b(t-t_0)e^{i(\omega-\omega_2)t} dt$$

and the right hand side is $f_{2G_b}(t_0, \omega - \omega_2)$ as required.

8. This follows example 7.9 in the text. The function $f_b(t) = f(t)b(t)$ and is given by

$$f_b(t) = \begin{cases} (1+t)\sin(\pi t) & -1 \le t < 0 \\ (1-t)\sin(\pi t) & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Thus

$$f_G(1,\omega) = \int_{-\infty}^{\infty} f_b(t)e^{-i\omega t}dt = \int_{-1}^{0}(1+t)\sin(\pi t)e^{-i\omega t}dt + \int_0^1 (1-t)\sin(\pi t)e^{-i\omega t}dt$$

The two integrals on the right are combined by letting $\tau = -t$ in the first to give

$$f_G(1,\omega) = 2i\int_0^1 (1-t)\sin(\pi t)\sin(\omega t)dt.$$

This yields to integration by parts and the answer is:

$$f_G(1,\omega) = 2i\left[\frac{1-\cos\{(\pi-\omega)t\}}{(\pi-\omega)^2} - \frac{1-\cos\{(\pi+\omega)t\}}{(\pi+\omega)^2}\right]$$

Exercises 8.7

1. In all of these examples, the location of the pole is obvious, and the residue is best found by use of the formula

$$\lim_{z\to a}(z-a)f(z)$$

where $z = a$ is the location of the simple pole. In these answers, the location of the pole is followed after the semicolon by its residue. Where there is more than one pole, the answers are sequential, poles first followed by the corresponding residues.

$$\begin{aligned}
&\text{(i) } z = -1; \quad 1,\\
&\text{(ii) } z = 1; \quad -1,\\
&\text{(iii) } z = 1, 3i, -3i; \quad \frac{1}{2}, \frac{5}{12}(3-i), \frac{5}{12}(3+i),\\
&\text{(iv) } z = 0, -2, -1; \quad \frac{3}{2}, -\frac{5}{2}, 1,\\
&\text{(v) } z = 0; \quad 1,\\
&\text{(vi) } z = n\pi \quad (-1)^n n\pi, \quad n \text{ integer},\\
&\text{(vii) } z = n\pi; \quad (-1)^n e^{n\pi}, \quad n \text{ integer}.
\end{aligned}$$

2. As in the first example, the location of the poles is straightforward. The methods vary. For parts (i), (ii) and (iii) the formula for finding the residue at a pole of order n is best, viz.

$$\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{(n-1)}}{dz^{(n-1)}}\{(z-a)^n f(z)\}.$$

For part (iv) expanding both numerator and denominator as power series and picking out the coefficient of $1/z$ works best. The answers are as follows

$$\begin{aligned}
&\text{(i)} \quad z = 1, \quad \text{order 2res} = 4\\
&\text{(ii)} \quad z = i, \quad \text{order 2res} = -\frac{1}{4}i\\
&\qquad z = -i, \text{ order 2res} = \frac{1}{4}i\\
&\text{(iii)} \quad z = 0, \quad \text{order 3res} = -\frac{1}{2}\\
&\text{(iv)} \quad z = 0, \quad \text{order 2res} = 1.
\end{aligned}$$

3. (i) Using the residue theorem, the integral is $2\pi i$ times the sum of the residues of the integrand at the three poles. The three residues are:

$$\frac{1}{3}(1-i)\ \ (\text{at } z=1),\quad \frac{4}{15}(-2-i)\ \ (\text{at } z=-2),\quad \frac{1}{5}(1+3i)\ \ (\text{at } z=-i).$$

The sum of these times $2\pi i$ gives the result

$$-\frac{2\pi}{15}.$$

(ii) This time the residue (calculated easily using the formula) is 6, whence the integral is $12\pi i$.
(iii) For this integral we use a semi circular contour on the upper half plane. By the estimation lemma, the integral around the curved portion tends to zero as the radius gets very large. Also the integral from $-\infty$ to 0 along the real axis is equal to the integral from 0 to ∞ since the integrand is even. Thus we have

$$2\int_0^\infty \frac{1}{x^6+1}dx = 2\pi i(\text{sum of residues at } z = e^{\pi i/6},\ \ i,\ \ e^{5\pi i/6})$$

from which we get the answer $\frac{\pi}{3}$.
(iv) This integral is evaluated using the same contour, and similar arguments tell us that

$$2\int_0^\infty \frac{\cos(2\pi x)}{x^4+x^2+1}dx = 2\pi i(\text{sum of residues at } z = e^{\pi i/3},\quad e^{2\pi i/3}).$$

(Note that the complex function considered is $\dfrac{e^{2\pi i z}}{z^4+z^2+1}$. Note also that the poles of the integrand are those of z^6-1 but excluding $z=\pm 1$.) The answer is, after a little algebra

$$-\frac{\pi}{2\sqrt{3}}e^{-\pi/\sqrt{3}}.$$

4. Problems (i) and (ii) are done by using the function

$$f(z) = \frac{(\ln(z))^2}{z^4+1}.$$

Integrated around the indented semi circular contour of Fig. 8.8, there are poles at $z = (\pm 1 \pm i)/\sqrt{2}$. Only those at $(\pm 1 + i)/\sqrt{2}$ or $z = e^{\pi i/4},\quad e^{3\pi i/4}$ are inside the contour. Evaluating

$$\int_{C'} f(z)dz$$

along all the parts of the contour gives the following contributions: those along the curved bits eventually contribute nothing (the denominator gets very large in absolute magnitude as the radius of the big semi-circle $\to\infty$, the integral around the small circle $\to 0$ as its radius $r\to 0$ since $r(\ln r)^2\to 0$.) The contributions along the real axis are

$$\int_0^\infty \frac{(\ln x)^2}{x^4+1}dx$$

along the positive real axis where $z = x$ and

$$\int_0^\infty \frac{(\ln x + i\pi)^2}{x^4+1}dx$$

along the negative real axis where $z = xe^{i\pi}$ so $\ln z = \ln x + i\pi$. The residue theorem thus gives

$$2\int_0^\infty \frac{(\ln x)^2}{x^4+1}dx + 2\pi i\int_0^\infty \frac{\ln x}{x^4+1}dx - \pi^2\int_0^\infty \frac{1}{x^4+1}dx \qquad \text{(A.3)}$$
$$= 2\pi i\{\text{sum of residues}\}.$$

The residue at $z = a$ is given by

$$\frac{(\ln a)^2}{4a^3}$$

using the formula for the residue of a simple pole. These sum to

$$-\frac{\pi^2}{64\sqrt{2}}(8-10i).$$

Equating real and imaginary parts of Eq. (A.3) gives the answers

$$\text{(i)}\quad \int_0^\infty \frac{(\ln x)^2}{x^4+1}dx = \frac{3\pi^3\sqrt{2}}{64};\quad \text{(ii)}\quad \int_0^\infty \frac{\ln x}{x^4+1}dx = -\frac{\pi^2}{16}\sqrt{2}$$

once the result

$$\int_0^\infty \frac{1}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}$$

from Example 8.1(ii) is used.
(iii) The third integral also uses the indented semi circular contour of Fig. 8.8. The contributions from the large and small semi circles are ultimately zero. There is a pole at $z=i$ which has residue $e^{\pi\lambda i/2}/2i$ and the straight parts contribute

$$\int_0^\infty \frac{x^\lambda}{1+x^2}dx$$

(positive real axis), and

$$-\int_\infty^0 \frac{x^\lambda e^{\lambda i\pi}}{1+x^2}dx$$

(negative real axis). Putting the contributions together yields

$$\int_0^\infty \frac{x^\lambda}{1+x^2}dx + e^{\lambda i\pi}\int_0^\infty \frac{x^\lambda}{1+x^2}dx = \pi e^{\lambda i\pi/2}$$

from which

$$\int_0^\infty \frac{x^\lambda}{1+x^2}dx = \frac{\pi}{2\cos(\frac{\lambda\pi}{2})}.$$

5. These inverse Laplace transforms are all evaluated form first principles using the Bromwich contour, although it is possible to deduce some of them by using previously derived results, for example if we assume that

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}$$

then we can carry on using the first shift theorem and convolution. However, we choose to use the Bromwich contour. The first two parts follow closely Example 8.5, though none of the branch points in these problems is in the expo-

nential. The principle is the same.
(i) This Bromwich contour has a cut along the negative real axis from -1 to $-\infty$. It is shown as Fig. A.6. Hence

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{1}{2\pi i}\int_{Br}\frac{e^{st}}{s\sqrt{s+1}}ds.$$

The integral is thus split into the following parts

$$\int_{C'} = \int_{Br} + \int_{\Gamma} + \int_{AB} + \int_{\gamma} + \int_{CD} = 2\pi i(\text{residue at } s=0)$$

where C' is the whole contour, Γ is the outer curved part, AB is the straight portion above the cut ($\Im s > 0$) γ is the small circle surrounding the branch point $s = -1$ and CD is the straight portion below the cut ($\Im s < 0$). The residue is one, the curved parts of the contour contribute nothing in the limit. The important contributions come from the integrals along AB and CD. On AB we can put $s = xe^{i\pi} - 1$. This leads to the integral

$$\int_{AB} = \int_{\infty}^{0}\frac{e^{t(-x-1)}}{(-x-1)i\sqrt{x}}dx$$

On CD we can put $s = xe^{-i\pi} - 1$. The integrals do not cancel because of the square root in the denominator (the reason the cut is there of course!). They in fact exactly reinforce. So the integral is

$$\int_{CD} = \int_{0}^{\infty}\frac{e^{t(-x-1)}}{(-x-1)(-i\sqrt{x})}dx.$$

Hence

$$\int_{C} = \int_{Br} -2\int_{\infty}^{0}\frac{e^{-t}e^{-xt}}{i(x+1)\sqrt{x}}dx = 2\pi i.$$

Using the integral

$$\int_{0}^{\infty}\frac{e^{-xt}}{(x+1)\sqrt{x}}dx = -\pi e^{t}[-1+\operatorname{erf}\sqrt{t}]$$

gives the answer that

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{1}{2\pi i}\int_{Br}\frac{e^{st}}{s\sqrt{s+1}}ds = \operatorname{erf}\sqrt{t}.$$

(ii) This second part is tackled in a similar way. The contour is identical (Fig. A.6). The important step is the correct parametrisation of the straight parts of the

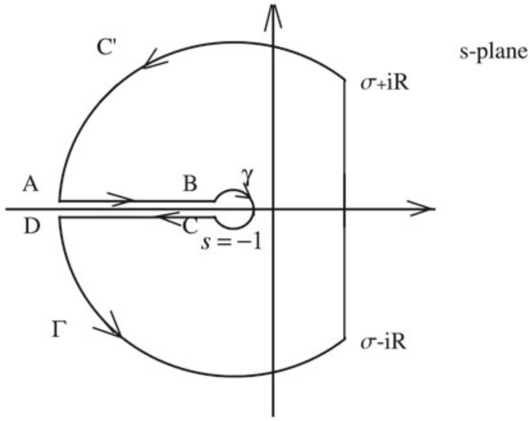


Fig. A.6 The cut Bromwich contour

contour just above and just below the branch cut. This time the two integrals along the cut are

$$\int_{AB} = -\int_{\infty}^{0} \frac{e^{(-x-1)t}}{1+i\sqrt{x}}dx$$

and

$$\int_{CD} = -\int_{0}^{\infty} \frac{e^{(-x-1)t}}{1-i\sqrt{x}}dx,$$

and the integrals combine to give

$$\mathcal{L}^{-1}\left\{\frac{1}{1+\sqrt{s+1}}\right\} = \frac{1}{2\pi i}\int_0^{\infty} \frac{e^{-t}e^{-xt}2i\sqrt{x}}{1+x}dx$$

which gives the result

$$\frac{e^{-t}}{\sqrt{\pi t}} - \text{erfc}\sqrt{t}.$$

(iii) This inverse can be obtained from part (ii) by using the first shift theorem (Theorem 1.2). The result is

$$\mathcal{L}^{-1}\left\{\frac{1}{1+\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} - e^{t}\text{erfc}\sqrt{t}.$$

(iv) Finally this last part can be deduced by using the formula

$$\frac{1}{\sqrt{s}+1} - \frac{1}{\sqrt{s}-1} = -2\frac{1}{s-1}.$$

The answer is

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}-1}\right\} = \frac{1}{\sqrt{\pi t}} + e^t(1+\mathrm{erf}\sqrt{t}).$$

6. This problem is best tackled by use of power series, especially so as there is no problem manipulating them as these exponential and related functions have series that are uniformly convergent for all finite values of t and s, excluding $s = 0$. The power series for the error function (obtained by integrating the exponential series term by term) yields:

$$\mathrm{erf}\left(\frac{1}{s}\right) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{s^{2n+1}(2n+1)n!}.$$

Taking the inverse Laplace transform using linearity and standard forms gives

$$\phi(t) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}\frac{t^{2n}}{(2n)!}.$$

After some tidying up, this implies that

$$\phi(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{(2n+1)!n!}.$$

Taking the Laplace transform of this series term by term gives

$$\mathcal{L}\left\{\phi(\sqrt{t})\right\} = \frac{2}{\sqrt{\pi s}}\sum_{n=0}^{\infty}(-1)^n\frac{(1/\sqrt{s})^{2n+1}}{(2n+1)!}$$

which is

$$\mathcal{L}\left\{\phi(\sqrt{t})\right\} = \frac{2}{\sqrt{\pi s}}\sin\left(\frac{1}{\sqrt{s}}\right)$$

as required.

7. This problem is tackled in a very similar way to the previous one. We simply integrate the series term by term and have to recognise

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{(k!)^2}\left(\frac{x}{s}\right)^{2k}$$

as the binomial series for

$$\left(1+\frac{x^2}{s^2}\right)^{-1/2}.$$

Again, the series are uniformly convergent except for $s = \pm ix$ which must be excluded alongside $s = 0$.

8. The integrand

$$\frac{\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}$$

has a singularity at the origin and wherever $\sqrt{s}$ is an odd multiple of $\pi/2$. The presence of the square roots leads one to expect branch points, but in fact there are only simple poles. There are however infinitely many of them at locations

$$s = -\left(n + \frac{1}{2}\right)^2 \pi^2$$

and at the origin. The (uncut) Bromwich contour can thus be used; all the singularities are certainly to the left of the line $s = \sigma$ in Fig. 8.9. The inverse is thus

$$\frac{1}{2\pi i}\int_{Br} e^{st}\frac{\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}ds = \text{sum of residues.}$$

The residue at $s = 0$ is straightforwardly

$$\lim_{s\to 0}(s-0)\left\{\frac{e^{st}\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}\right\} = 1.$$

The residue at the other poles is also calculated using the formula, but the calculation is messier, and the result is

$$\frac{4(-1)^n}{\pi(2n-1)}e^{-(n-1/2)^2\pi^2 t}\cos\left(n - \frac{1}{2}\right)\pi x.$$

Thus we have

$$\mathcal{L}^{-1}\left\{\frac{\cosh(x\sqrt{s})}{s\cosh(\sqrt{s})}\right\} = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(n-\frac{1}{2})^2\pi^2 t}\cos\left(n - \frac{1}{2}\right)\pi x.$$

9. The Bromwich contour for the function

$$e^{-s^{\frac{1}{3}}}$$

has a cut from the branch point at the origin. We can thus use the contour depicted in Fig. 8.10. As the origin has been excluded the integrand

$$e^{st-s^{\frac{1}{3}}}$$

has no singularities in the contour, so by Cauchy's theorem the integral around C' is zero. As is usual, the two parts of the integral that are curved give zero contribution as the outer radius of Γ gets larger and larger, and inner radius of the circle γ gets smaller. This is because $\cos\theta < 0$ on the left of the imaginary axis which means that the exponent in the integrand is negative on Γ, also on γ the ds contributes a zero as the radius of the circle γ decreases. The remaining contributions are

$$\int_{AB} = -\int_{\infty}^{0} e^{-xt-x^{\frac{1}{3}}e^{i\pi/3}}\,dx$$

and

$$\int_{CD} = -\int_{0}^{\infty} e^{-xt-x^{\frac{1}{3}}e^{-i\pi/3}}\,dx.$$

These combine to give

$$\int_{AB} + \int_{CD} = -\int_{0}^{\infty} e^{-xt-\frac{1}{2}x^{\frac{1}{3}}} \sin\left(\frac{x^{\frac{1}{3}}\sqrt{3}}{2}\right)dx.$$

Substituting $x = u^3$ gives the result

$$\mathcal{L}^{-1}\{e^{-s^{\frac{1}{3}}}\} = \frac{1}{2\pi i}\int_{Br} e^{st-s^{\frac{1}{3}}}\,ds = \frac{3}{\pi}\int_{0}^{\infty} u^2 e^{-u^3 t-\frac{1}{2}u} \sin\left(\frac{u\sqrt{3}}{2}\right)du.$$

10. Using Laplace transforms in t solving in the usual way gives the solution

$$\bar{\phi}(x, s) = \frac{1}{s^2}e^{-x\sqrt{s^2-1}}.$$

The singularity of $\bar{\phi}(x, s)$ with the largest real part is at $s = 1$. The others are at $s = -1, 0$. Expanding $\bar{\phi}(x, s)$ about $s = 1$ gives

$$\bar{\phi}(x, s) = 1 - x\sqrt{2}(s-1)^{\frac{1}{2}} + \cdots .$$

In terms of Theorem 8.2 this means that $k = 1/2$ and $a_0 = -x\sqrt{2}$. Hence the leading term in the asymptotic expansion for $\phi(x, t)$ for large t is

$$-\frac{1}{\pi}e^{t}\sin\left(\frac{1}{2}\pi\right)\left(-x\sqrt{2}\frac{\Gamma(3/2)}{t^{3/2}}\right)$$

whence

$$\phi(x, t) \sim \frac{xe^{t}}{\sqrt{2\pi t^3}}$$

as required.

Appendix B
Table of Laplace Transforms

In this table, t is a real variable, s is a complex variable and a and b are real constants. In a few entries, the real variable x also appears.

$\mathbf{f(s)}(=\int_0^\infty e^{-st}\mathbf{F(t)}dt)$	$\mathbf{F(t)}$
$\frac{1}{s}$	1
$\frac{1}{s^n}, n=1,2,\ldots$	$\frac{t^{n-1}}{(n-1)!},$
$\frac{1}{s^x}, x>0,$	$\frac{t^{x-1}}{\Gamma(x)},$
$\frac{1}{s-a}$	e^{at}
$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{a}{s^2+a^2}$	$\sin(at)$
$\frac{s}{s^2-a^2}$	$\cosh(at)$
$\frac{a}{s^2-a^2}$	$\sinh(at)$
$\frac{1}{(s-a)(s-b)} \quad a\neq b$	$\frac{e^{bt}-e^{at}}{b-a}$
$\frac{s}{(s-a)(s-b)} \quad a\neq b$	$\frac{be^{bt}-ae^{at}}{b-a}$
$\frac{1}{(s^2+a^2)^2}$	$\frac{\sin(at)-at\cos(at)}{2a^3}$
$\frac{s}{(s^2+a^2)^2}$	$\frac{t\sin(at)}{2a}$
$\frac{s^2}{(s^2+a^2)^2}$	$\frac{\sin(at)+at\cos(at)}{2a}$
$\frac{s^3}{(s^2+a^2)^2}$	$\cos(at)-\frac{1}{2}at\sin(at)$

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4,

$\mathbf{f(s)}(=\int_0^\infty e^{-st}\mathbf{F(t)}dt)$	$\mathbf{F(t)}$
$\frac{1}{\sqrt{s+a}+\sqrt{s+b}}$	$\frac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$
$\frac{e^{-a/s}}{\sqrt{s}}$	$\frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$
$\frac{e^{-a/s}}{s\sqrt{s}}$	$\frac{\sin 2\sqrt{at}}{\sqrt{\pi a}}$
$\frac{e^{-a/s}}{s^{n+1}}$	$\left(\frac{t}{a}\right)^{n/2} J_n(2\sqrt{at})$
1	$\delta(t)$
s^n	$\delta^{(n)}(t)$
$\frac{e^{-as}}{s}$	$H(t-a)$
$\frac{e^{-a\sqrt{s}}}{s}$	$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$
$e^{-a\sqrt{s}}$	$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$
$\frac{1}{s\sqrt{s+a}}$	$\frac{\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
$\frac{1}{\sqrt{s}(s-a)}$	$\frac{e^{at}\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
$\frac{1}{\sqrt{s-a}+b}$	$e^{at}\left\{\frac{1}{\sqrt{\pi t}} - be^{b^2t}\operatorname{erfc}(b\sqrt{t})\right\}$
$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
$\tan^{-1}(a/s)$	$\frac{\sin(at)}{t}$
$\frac{\sinh(sx)}{s\sinh(sa)}$	$\frac{x}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi t}{a}\right)$
$\frac{\sinh(sx)}{s\cosh(sa)}$	$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sin\left(\frac{(2n-1)\pi x}{2a}\right)\sin\left(\frac{(2n-1)\pi t}{2a}\right)$
$\frac{\cosh(sx)}{s\cosh(sa)}$	$1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left(\frac{(2n-1)\pi x}{2a}\right)\cos\left(\frac{(2n-1)\pi t}{2a}\right)$
$\frac{\cosh(sx)}{s\sinh(sa)}$	$\frac{t}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi t}{a}\right)$
$\frac{\sinh(sx)}{s^2\cosh(sa)}$	$x+\frac{8a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{2a}\right)\sin\left(\frac{(2n-1)\pi t}{2a}\right)$
$\frac{\sinh(x\sqrt{s})}{s\sinh(a\sqrt{s})}$	$\frac{x}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2t/a^2}\sin\left(\frac{n\pi x}{a}\right)$
$\frac{\cosh(x\sqrt{s})}{s\cosh(a\sqrt{s})}$	$1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2t/4a^2}\cos\left(\frac{(2n-1)\pi x}{2a}\right)$
$\frac{\sinh(x\sqrt{s})}{s^2\sinh(a\sqrt{s})}$	$\frac{xt}{a}+\frac{2a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}(1-e^{-n^2\pi^2t/a^2})\sin\left(\frac{n\pi x}{a}\right)$

In the last four entries $\gamma = 0.5772156\ldots$ is the Euler-Mascheroni constant. The next four Laplace transforms are of periodic functions that are given diagrammatically. The two column format is abandoned.

$\mathbf{f(s)}(=\int_0^\infty e^{-st}\mathbf{F(t)}dt)$	$\mathbf{F(t)}$
$\frac{1}{as^2}\tanh\left(\frac{as}{2}\right)$	$F(t)=\begin{cases} t/a & 0\le t\le a \\ 2-t/a & a<t\le 2a\end{cases}$ $F(t)=F(t+2a)$
$\frac{1}{s}\tanh\left(\frac{as}{2}\right)$	$F(t)=\begin{cases} 1 & 0\le t\le a \\ -1 & a<t\le 2a\end{cases}$ $F(t)=F(t+2a)$
$\frac{e^{-a/s}}{s^{n+1}}$	$\left(\frac{t}{a}\right)^{n/2} J_n(2\sqrt{at})$
$\frac{\pi a}{a^2+s^2}\coth\left(\frac{as}{2}\right)$	$\left\|\sin\left(\frac{\pi t}{a}\right)\right\|$
$\frac{\pi a}{(a^2s^2+\pi^2)(1-e^{-as})}$	$F(t)=\left\{\begin{matrix} \sin(\frac{\pi t}{a}) & 0\le t\le a \\ 0 & a<t\le 2a\end{matrix}\right\}$ $F(t)=F(t+2a)$
$\frac{1}{as^2}-\frac{e^{-as}}{s(1-e^{-as})}$	$F(t)=t/a,\quad 0\le t\le a\quad F(t)=F(t+a)$
$\ln\left(\frac{s+a}{s+b}\right)$	$\frac{e^{-bt}-e^{-at}}{t}$
$\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)$	$\frac{2(\cos bt-\cos at)}{t}$
$\frac{1}{s^3+a^3}$	$\frac{e^{at/2}}{3a^2}\left\{\sqrt{3}\sin\frac{\sqrt{3}at}{2}-\cos\frac{\sqrt{3}at}{2}+e^{-3at/2}\right\}$
$\frac{s}{s^3+a^3}$	$\frac{e^{at/2}}{3a^2}\left\{\sqrt{3}\sin\frac{\sqrt{3}at}{2}+\cos\frac{\sqrt{3}at}{2}-e^{-3at/2}\right\}$
$\frac{s^2}{s^3+a^3}$	$\frac{1}{3}\left(e^{-at}+2e^{at/2}\cos\frac{\sqrt{3}at}{2}\right)$
$\frac{1}{s^3-a^3}$	$\frac{e^{-at/2}}{3a^2}\left\{e^{3at/2}-\sqrt{3}\sin\frac{\sqrt{3}at}{2}-\cos\frac{\sqrt{3}at}{2}\right\}$
$\frac{s}{s^3-a^3}$	$\frac{e^{-at/2}}{3a^2}\left\{\sqrt{3}\sin\frac{\sqrt{3}at}{2}-\cos\frac{\sqrt{3}at}{2}+e^{3at/2}\right\}$
$\frac{s^2}{s^3-a^3}$	$\frac{1}{3}\left(e^{at}+2e^{-at/2}\cos\frac{\sqrt{3}at}{2}\right)$
$\frac{1}{s^4+4a^4}$	$\frac{1}{4a^3}(\sin at\cosh at-\cos at\sinh at)$
$\frac{s}{s^4+4a^4}$	$\frac{\sin at\sinh at}{2a^2}$
$-\frac{\gamma+\ln s}{s}$	$\ln t$
$\frac{\pi^2}{6s}+\frac{(\gamma+\ln s)^2}{s}$	$\ln^2 t$
$\frac{\ln s}{s}$	$-(\ln t+\gamma)$
$\frac{\ln^2 s}{s}$	$-(\ln t+\gamma)^2-\frac{1}{6}\pi^2$

Fig. B.1 The Laplace transform of the above function, the rectified sine wave $F(t) = |\sin\left(\frac{\pi t}{a}\right)|$ is given by $f(s) = \dfrac{\pi a}{a^2 s^2 + \pi^2} \coth\left(\dfrac{as}{2}\right)$

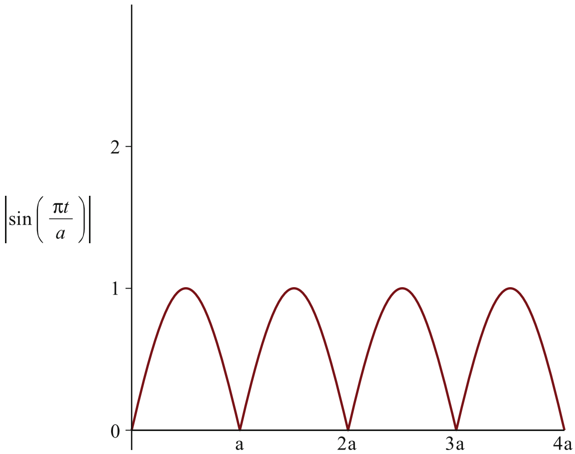


Fig. B.2 The Laplace transform of the above square wave function is given by $f(s) = \dfrac{1}{s} \tanh\left(\dfrac{1}{2}as\right)$

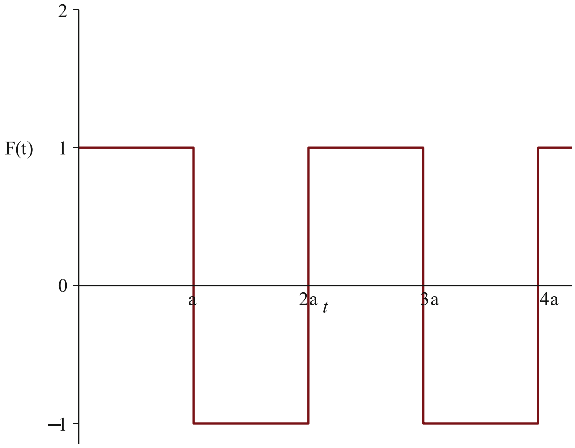
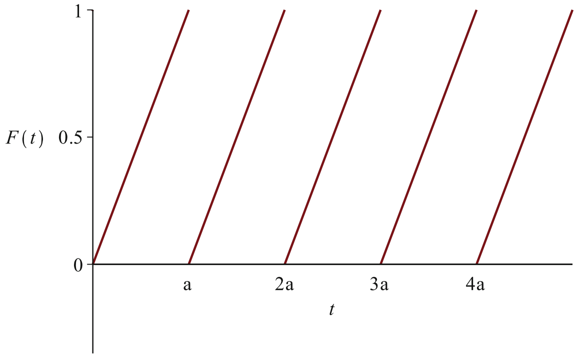


Fig. B.3 The Laplace transform of the above saw-tooth function is given by $f(s) = \dfrac{1}{as^2} - \dfrac{e^{-as}}{s(1 - e^{-as})}$



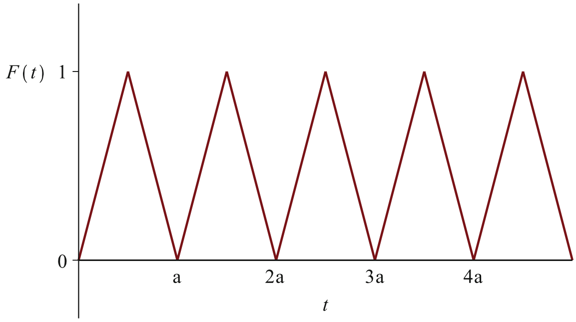


Fig. B.4 The Laplace transform of the above saw-tooth function is given by $f(s) = \frac{1}{as^2}\tanh\left(\frac{1}{2}as\right)$

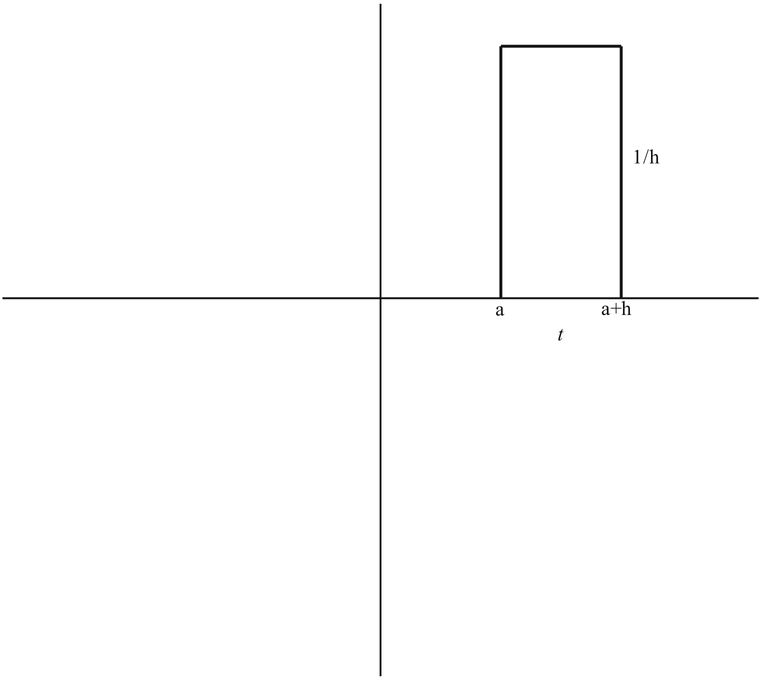


Fig. B.5 The Laplace transform of the above top hat function is given by $f(s) = \frac{e^{-as}}{sh}(1 - e^{-sh})$. Thus the Haar wavelet that corresponds to $a = 0$, $h = 1$ has the Laplace transform $\frac{1 - e^{-s}}{s}$

Appendix C
Linear Spaces

C.1 Linear Algebra

In this appendix, some fundamental concepts of linear algebra are given. The proofs are largely omitted; students are directed to textbooks on linear algebra for these. For this subject, we need to be precise in terms of the basic mathematical notions and notations we use. Therefore we uncharacteristically employ a formal mathematical style of prose. It is essential to be rigorous with the basic mathematics, but it is often the case that an over formal treatment can obscure rather than enlighten. That is why this material appears in an appendix rather than in the main body of the text.

A set of objects (called the elements of the set) is written as a sequence of (usually) lower case letters in between braces:-

$$A = \{a_1, a_2, a_3, \dots, a_n\}.$$

In discussing Fourier series, sets have infinitely many elements, so there is a row of dots after a_n too. The symbol $\in$ read as "belongs to" should be familiar to most. So $s \in S$ means that s is a member of the set S. Sometimes the alternative notation

$$S = \{x | f(x)\}$$

is used. The vertical line is read as "such that" so that $f(x)$ describes some property that x possess in order that $s \in S$. An example might be

$$S = \{x | x \in \mathbb{R}, |x| \le 2\}$$

so S is the set of real numbers that lie between -2 and $+2$.

The following notation is standard but is reiterated here for reference:
(a, b) denotes the open interval $\{x | a < x < b\}$,
$[a, b]$ denotes the closed interval $\{x | a \le x \le b\}$,
$[a, b)$ is the set $\{x | a \le x < b\}$,
and $(a, b]$ is the set $\{x | a < x \le b\}$.

P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4,

The last two are described as half closed or half open intervals and are reasonably obvious extensions of the first two definitions. In Fourier series, ∞ is often involved so the following intervals occur:-

$$(a, \infty) = \{x|a < x\} \quad [a, \infty) = \{x|a \le x\}$$
$$(-\infty, a) = \{x|x < a\} \quad (-\infty, a] = \{x|x \le a\}.$$

These are all obvious extensions. Where appropriate, use is also made of the following standard sets

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of integers

$\mathbb{Z}_+$ is the set of positive integers including zero: $\{0, 1, 2, 3, \ldots\}$

$\mathbb{R} = \{x|x \text{ is a real number}\} = (-\infty, \infty)$.

Sometimes (but very rarely) we might use the set of fractions or rationals $\mathbb{Q}$:

$$\mathbb{Q} = \left\{\frac{m}{n} \mid m \text{ and } n \text{ are integers }, n \neq 0\right\}.$$

$\mathbb{R}_+$ is the set of positive real numbers

$$\mathbb{R}_+ = \{x \mid x \in \mathbb{R}, x \ge 0\}.$$

Finally the set of complex numbers $\mathbb{C}$ is defined by

$$\mathbb{C} = \{z = x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}.$$

The standard notation

$$x = \Re\{z\}, \quad \text{the real part of } z$$
$$y = \Im\{z\}, \quad \text{the imaginary part of } z$$

has already been met in Chap. 1.

Hopefully, all of this is familiar to most of you. We will need these to define the particular normed spaces within which Fourier series operate. This we now proceed to do. A vector space V is an algebraic structure that consists of elements (called vectors) and two operations (called addition $+$ and multiplication $\times$). The following gives a list of properties obeyed by vectors **a,b,c** and scalars $\alpha, \beta \in F$ where F is a field (usually $\mathbb{R}$ or $\mathbb{C}$).

1. $\mathbf{a} + \mathbf{b}$ is also a vector (closure under addition).
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (associativity under addition).
3. There exists a zero vector denoted by $\mathbf{0}$ such that $\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a} \ \ \forall \mathbf{a} \in V$ (additive identity).

4. For every vector $\mathbf{a} \in V$ there is a vector $-\mathbf{a}$ (called "minus $\mathbf{a}$" such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.
5. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ for every $\mathbf{a}, \mathbf{b} \in V$ (additive commutativity).
6. $\alpha\mathbf{a} \in V$ for every $\alpha \in F$, $\mathbf{a} \in V$ (scalar multiplicity).
7. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$ for every $\alpha \in F$, $\mathbf{a}, \mathbf{b} \in V$ (first distributive law).
8. $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$ for every $\alpha, \beta \in F$, $\mathbf{a} \in V$ (second distributive law).
9. For the unit scalar 1 of the field F, and every $\mathbf{a} \in V$ $1 \cdot \mathbf{a} = \mathbf{a}$ (multiplicative identity).

The set V whose elements obey the above nine properties over a field F is called a vector space over F. The name linear space is also used in place of vector space and is useful as the name "vector" conjures up mechanics to many and gives a false impression in the present context. In the study of Fourier series, vectors are in fact functions. The name linear space emphasises the linearity property which is confirmed by the following definition and properties.

Definition C.1 *If* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n} \in V$ *where* V *is a linear space over a field* F *and if there exist scalars* $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ *such that*

$$\mathbf{b} = \alpha_1\mathbf{a_1} + \alpha_2\mathbf{a_2} + \cdots + \alpha_n\mathbf{a_n}$$

(called a linear combination *of the vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$*) then the collection of all such* $\mathbf{b}$ *which are a linear combination of the vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *is called the* span *of* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *denoted by span* $\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}\}$.

If this definition seems innocent, then the following one which depends on it is not. It is one of the most crucial properties possibly in the whole of mathematics.

Definition C.2 (linear independence) *If* V *is a linear (vector) space, the vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n} \in V$ *are said to be linearly independent if the equation*

$$\alpha_1\mathbf{a_1} + \alpha_2\mathbf{a_2} + \cdots + \alpha_n\mathbf{a_n} = \mathbf{0}$$

implies that all of the scalars are zero, i.e.

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0, (\alpha_1, \alpha_2, \ldots, \alpha_n \in F).$$

Otherwise, $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are said to be linearly dependent.*

Again, it is hoped that this is not a new concept. However, here is an example. Most texts take examples from geometry, true vectors indeed. This is not appropriate here so instead this example is algebraic.

Example C.1 *Is the set* $S = \{1, x, 2 + x, x^2\}$ *with* $F = \mathbf{R}$ *linearly independent?*

Solution The most general combination of $1, x, 2 + x, x^2$ is

$$y = \alpha_1 + \alpha_2 x + \alpha_3(2 + x) + \alpha_4 x^2$$

where x is a variable that can take any real value.

Now, $y = 0$ for all x does not imply $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, for if we choose $\alpha_1 + 2\alpha_2 = 0, \alpha_2 + \alpha_3 = 0$ and $\alpha_4 = 0$ then $y = 0$. The combination $\alpha_1 = 1, \alpha_3 = -\frac{1}{2}, \alpha_2 = \frac{1}{2}, \alpha_4 = 0$ will do. The set is therefore not linearly independent.

On the other hand, the set $\{1, x, x^2\}$ is most definitely linearly independent as

$$\alpha_1 + \alpha_2 x + \alpha_3 x^2 = 0 \text{ for all } x \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

It is possible to find many independent sets. One could choose $\{x, \sin x, \ln x\}$ for example: however sets like this are not very useful as they do not lead to any applications. The set $\{1, x, x^2\}$ spans all quadratic functions. Here is another definition that we hope is familiar.

Definition C.3 *A finite set of vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *is said to be a* basis *for the linear space* V *if the set of vectors* $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ *is linearly independent and* $V = span\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}\}$ *The natural number* n *is called the dimension of* V *and we write* $n = dim(V)$.

Example C.2 *Let* $[a, b]$ *(with* $a < b$*) denote the finite closed interval as already defined. Let* f *be a continuous real valued function whose value at the point* x *of* $[a, b]$ *is* $f(x)$. *Let* $C[a, b]$ *denote the set of all such functions. Now, if we define addition and scalar multiplication in the natural way, i.e.* $f_1 + f_2$ *is simply the value of* $f_1(x) + f_2(x)$ *and similarly* αf *is the value of* $\alpha f(x)$, *then it is clear that* $C[a, b]$ *is a real vector space. In this case, it is clear that the set* $x, x^2, x^3, \ldots, x^n$ *are all members of* $C[a, b]$ *for arbitrarily large* n. *It is therefore not possible for* $C[a, b]$ *to be finite dimensional.*

Perhaps it is now a little clearer as to why the set $\{1, x, x^2\}$ is useful as this is a basis for all quadratics, whereas $\{x, \sin x, \ln x\}$ does not form a basis for any well known space. Of course, there is usually an infinite choice of basis for any particular linear space. For the quadratic functions the sets $\{1-x, 1+x, x^2\}$ or $\{1, 1-x^2, 1+2x+x^2\}$ will do just as well. That we have these choices of bases is useful and will be exploited later.

Most books on elementary linear algebra are content to stop at this point and consolidate the above definitions through examples and exercises. However, we need a few more definitions and properties in order to meet the requirements of a Fourier series.

Definition C.4 *Let* V *be a real or complex linear space. (That is the field* F *over which the space is defined is either* $\mathbb{R}$ *or* $\mathbb{C}$.*) An inner product is an operation between two elements of* V *which results in a scalar. This scalar is denoted by* $\langle \mathbf{a_1}, \mathbf{a_2} \rangle$ *and has the following properties:-*

1. *For each* $\mathbf{a_1} \in V$, $\langle \mathbf{a_1}, \mathbf{a_1} \rangle$ *is a non-negative real number, i.e.*

$$\langle \mathbf{a_1}, \mathbf{a_1} \rangle \geq 0.$$

2. *For each* $\mathbf{a_1} \in V$, $\langle \mathbf{a_1}, \mathbf{a_1} \rangle = 0$ *if and only if* $\mathbf{a_1} = \mathbf{0}$.
3. *For each* $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3} \in V$ *and* $\alpha_1, \alpha_2 \in F$

$$\langle \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2, \mathbf{a}_3 \rangle = \alpha_1 \langle \mathbf{a}_1, \mathbf{a}_3 \rangle + \alpha_2 \langle \mathbf{a}_2, \mathbf{a}_3 \rangle.$$

4. *For each* $\mathbf{a_1}, \mathbf{a_2} \in V$, $\langle \mathbf{a_1}, \mathbf{a_2} \rangle = \overline{\langle \mathbf{a_2}, \mathbf{a_1} \rangle}$

where the overbar in the last property denotes the complex conjugate. If $F = \mathbb{R}$ α_1, α_2 *are real, and Property 4 becomes obvious.*

No doubt, students who are familiar with the geometry of vectors will be able to identify the inner product $\langle \mathbf{a_1}, \mathbf{a_2} \rangle$ with $\mathbf{a}_1.\mathbf{a}_2$ the scalar product of the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. This is one useful example, but it is by no means essential to the present text where most of the inner products take the form of integrals.

Inner products provide a rich source of properties that would be out of place to dwell on or prove here. For example:

$$\langle \mathbf{0}, \mathbf{a} \rangle = 0 \quad \forall\, \mathbf{a} \in V$$

and

$$\langle \alpha \mathbf{a}_1, \alpha \mathbf{a}_2 \rangle = |\alpha|^2 \langle \mathbf{a}_1, \mathbf{a}_2 \rangle.$$

Instead, we introduce two examples of inner product spaces.

1. If $\mathbb{C}^n$ is the vector space V, i.e. a typical element of V has the form $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ where $a_r = x_r + iy_r$, x_r, $y_r \in \mathbb{R}$. The inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 \overline{b_1} + a_2 \overline{b_2} + \cdots + a_n \overline{b_n},$$

   the overbar denoting complex conjugate.
2. Nearer to our applications of inner products is the choice $V = C[a, b]$ the linear space of all continuous functions f defined on the closed interval $[a, b]$. With the usual summation of functions and multiplication by scalars this can be verified to be a vector space over the field of complex numbers $\mathbb{C}^n$. Given a pair of continuous functions f, g we can define their inner product by

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

   It is left to the reader to verify that this is indeed an inner product space satisfying the correct properties in Definition C.4.

It is quite typical for linear spaces involving functions to be infinite dimensional. In fact it is very unusual for it to be otherwise.

What has been done so far is to define a linear space and an inner product on that space. It is nearly always true that we can define what is called a "norm" on a linear space. The norm is independent of the inner product in theory, but there is almost always a connection in practice. The norm is a generalisation of the notion of distance. If the linear space is simply two or three dimensional vectors, then the norm can indeed be distance. It is however, even in this case possible to define others. Here is the general definition of norm.

Definition C.5 *Let V be a linear space. A norm on V is a function from V to $\mathbb{R}_+$ (non-negative real numbers), denoted by being placed between two vertical lines $||\cdot||$ which satisfies the following four criteria:-*

1. *For each* $\mathbf{a}_1 \in V, \quad ||\mathbf{a}_1|| \geq 0.$
2. $||\mathbf{a}_1|| = 0$ *if and only if* $\mathbf{a}_1 = \mathbf{0}$.
3. *For each* $\mathbf{a}_1 \in V$ *and* $\alpha \in \mathbb{C}$

$$||\alpha\mathbf{a}_1|| = |\alpha|||\mathbf{a}_1||.$$

4. *For every* $\mathbf{a}_1, \mathbf{a}_2 \in V$

$$||\mathbf{a}_1 + \mathbf{a}_2|| \leq ||\mathbf{a}_1|| + ||\mathbf{a}_2||.$$

(4 is the triangle inequality.)

For the vector space comprising the elements $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ where $a_r = x_r + iy_r, x_r, y_r \in \mathbb{R}$, i.e. $\mathbb{C}^n$ met previously, the obvious norm is

$$\begin{aligned} ||\mathbf{a}|| &= [|a_1|^2 + |a_2|^2 + |a_3|^2 + \cdots + |a_n|^2]^{1/2} \\ &= [\langle \mathbf{a}, \mathbf{a} \rangle]^{1/2}. \end{aligned}$$

It is true in general that we can always define the norm $||\cdot||$ of a linear space equipped with an inner product $\langle ., . \rangle$ to be such that

$$||\mathbf{a}|| = [\langle \mathbf{a}, \mathbf{a} \rangle]^{1/2}.$$

This norm is used in the next example. A linear space equipped with an inner product is called an inner product space. The norm induced by the inner product, sometimes called the *natural* norm for the function space $C[a, b]$, is

$$||f|| = \left[\int_a^b |f|^2 dx\right]^{1/2}.$$

For applications to Fourier series we are able to make $||f|| = 1$ and we adjust elements of V, i.e. $C[a, b]$ so that this is achieved. This process is called normalisation. Linear spaces with special norms and other properties are the directions in which this subject now naturally moves. The interested reader is directed towards books on functional analysis.

We now establish an important inequality called the Cauchy–Schwarz inequality. We state it in the form of a theorem and prove it.

Theorem C.1 (Cauchy–Schwarz) *Let V be a linear space with inner product $\langle ., . \rangle$, then for each $\mathbf{a}, \mathbf{b} \in V$ we have:*

$$|\langle \mathbf{a}, \mathbf{b} \rangle|^2 \le ||\mathbf{a}|| \cdot ||\mathbf{b}||.$$

Proof If $\langle \mathbf{a}, \mathbf{b} \rangle = 0$ then the result is self evident. We therefore assume that $\langle \mathbf{a}, \mathbf{b} \rangle = \alpha \neq 0$, α may of course be complex. We start with the inequality

$$||\mathbf{a} - \lambda\alpha\mathbf{b}||^2 \ge 0$$

where λ is a real number. Now,

$$||\mathbf{a} - \lambda\alpha\mathbf{b}||^2 = \langle \mathbf{a} - \lambda\alpha\mathbf{b}, \mathbf{a} - \lambda\alpha\mathbf{b} \rangle.$$

We use the properties of the inner product to expand the right hand side as follows:-

$$\begin{aligned}
\langle \mathbf{a} - \lambda\alpha\mathbf{b}, \mathbf{a} - \lambda\alpha\mathbf{b} \rangle &= \langle \mathbf{a}, \mathbf{a} \rangle - \lambda\langle \alpha\mathbf{b}, \mathbf{a} \rangle - \lambda\langle \mathbf{a}, \alpha\mathbf{b} \rangle + \lambda^2|\alpha|^2\langle \mathbf{b}, \mathbf{b} \rangle \ge 0 \\
\text{so } &||\mathbf{a}||^2 - \lambda\alpha\langle \mathbf{b}, \mathbf{a} \rangle - \lambda\bar{\alpha}\langle \mathbf{a}, \mathbf{b} \rangle + \lambda^2|\alpha|^2||\mathbf{b}||^2 \ge 0 \\
\text{i.e. } &||\mathbf{a}||^2 - \lambda\alpha\bar{\alpha} - \lambda\bar{\alpha}\alpha + \lambda^2|\alpha|^2||\mathbf{b}||^2 \ge 0 \\
\text{so } &||\mathbf{a}||^2 - 2\lambda|\alpha|^2 + \lambda^2|\alpha|^2||\mathbf{b}||^2 \ge 0.
\end{aligned}$$

This last expression is a quadratic in the real parameter λ, and it has to be positive for all values of λ. The condition for the quadratic

$$a\lambda^2 + b\lambda + c$$

to be non-negative is that $b^2 \le 4ac$ and $a > 0$. With

$$a = |\alpha|^2||\mathbf{b}||^2, \quad b = -2|\alpha|^2, \quad c = ||\mathbf{a}||^2$$

the inequality $b^2 \le 4ac$ is

$$\begin{aligned}
4|\alpha|^4 &\le 4|\alpha|^2||\mathbf{a}||^2||\mathbf{b}||^2 \\
\text{or } |\alpha|^2 &\le ||\mathbf{a}||||\mathbf{b}||
\end{aligned}$$

and since $\alpha = \langle \mathbf{a}, \mathbf{b} \rangle$ the result follows. □

The following is an example that typifies the process of proving that something is a norm.

Example C.3 *Prove that $||\mathbf{a}|| = \sqrt{\langle \mathbf{a}.\mathbf{a} \rangle} \in V$ is indeed a norm for the vector space V with inner product $\langle , \rangle$.*

Proof The proof comprises showing that $\sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$ satisfies the four properties of a norm.

1. $||\mathbf{a}|| \geq 0$ follows immediately from the definition of square roots.
2. If $\mathbf{a} = \mathbf{0} \Longleftrightarrow \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = 0$.
3.

$$\begin{aligned} ||\alpha \mathbf{a}|| &= \sqrt{\langle \alpha \mathbf{a}, \alpha \mathbf{a} \rangle} = \\ \sqrt{\alpha \bar{\alpha} \langle \mathbf{a}, \mathbf{a} \rangle} &= \sqrt{|\alpha|^2 \langle \mathbf{a}, \mathbf{a} \rangle} = \\ |\alpha| \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} &= |\alpha| ||\mathbf{a}||. \end{aligned}$$

4. This fourth property is the only one that takes a little effort to prove. Consider $||\mathbf{a} + \mathbf{b}||^2$. This is equal to

$$\begin{aligned} \langle \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b} \rangle &= \langle \mathbf{a}, \mathbf{a} \rangle + \langle \mathbf{b}, \mathbf{a} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{b} \rangle \\ &= ||\mathbf{a}||^2 + \langle \mathbf{a}, \mathbf{b} \rangle + \overline{\langle \mathbf{a}, \mathbf{b} \rangle} + ||\mathbf{b}||^2. \end{aligned}$$

The expression $\langle \mathbf{a}, \mathbf{b} \rangle + \overline{\langle \mathbf{a}, \mathbf{b} \rangle}$, being the sum of a number and its complex conjugate, is real. In fact

$$\begin{aligned} |\langle \mathbf{a}, \mathbf{b} \rangle + \overline{\langle \mathbf{a}, \mathbf{b} \rangle}| &= |2\Re \langle \mathbf{a}, \mathbf{b} \rangle| \\ &\leq 2|\langle \mathbf{a}, \mathbf{b} \rangle| \\ &\leq 2||\mathbf{a}|| \cdot ||\mathbf{b}|| \end{aligned}$$

using the Cauchy–Schwarz inequality. Thus

$$\begin{aligned} ||\mathbf{a} + \mathbf{b}||^2 &= ||\mathbf{a}||^2 + \langle \mathbf{a}, \mathbf{b} \rangle + \overline{\langle \mathbf{a}, \mathbf{b} \rangle} + ||\mathbf{b}||^2 \\ &\leq ||\mathbf{a}||^2 + 2||\mathbf{a}||||\mathbf{b}|| + ||\mathbf{b}||^2 \\ &= (||\mathbf{a} + \mathbf{b}||)^2. \end{aligned}$$

Hence $||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}||+||\mathbf{b}||$ which establishes the triangle inequality, Property 4.

Hence $||\mathbf{a}|| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$ is a norm for V. □

An important property associated with linear spaces is orthogonality. It is a direct analogy/generalisation of the geometric result $\mathbf{a}.\mathbf{b} = 0, \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$ if $\mathbf{a}$ and $\mathbf{b}$ represent directions that are at right angles (i.e. are orthogonal) to each other. This idea leads to the following pair of definitions.

Definition C.6 *Let V be an inner product space, and let $\mathbf{a}, \mathbf{b} \in V$. If $\langle \mathbf{a}, \mathbf{b} \rangle = 0$ then vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal.*

Definition C.7 *Let V be a linear space, and let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a sequence of vectors, $\mathbf{a}_r \in V, \quad a_r \neq 0, \quad r = 1, 2, \ldots, n$, and let $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 0, \quad i \neq j, \quad 0 \leq i, j \leq n$. Then $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is called an orthogonal set of vectors.*

Further to these definitions, a vector $\mathbf{a} \in V$ for which $||\mathbf{a}|| = 1$ is called a unit vector and if an orthogonal set of vectors consists of all unit vectors, the set is called *orthonormal*.

It is also possible to let $n \to \infty$ and obtain an orthogonal set for an infinite dimensional inner product space. We make use of this later in this chapter, but for now let us look at an example.

Example C.4 *Determine an orthonormal set of vectors for the linear space that consists of all real linear functions:*

$$\{a + bx : a, b \in \mathbb{R} \;\; 0 \leq x \leq 1\}$$

using as inner product

$$\langle f, g \rangle = \int_0^1 fg dx.$$

Solution The set $\{1, x\}$ forms a basis, but it is not orthogonal. Let $a + bx$ and $c + dx$ be two vectors. In order to be orthogonal we must have

$$\langle a + bx, c + dx \rangle = \int_0^1 (a + bx)(c + dx)dx = 0.$$

Performing the elementary integration gives the following condition on the constants a, b, c, and d

$$ac + \frac{1}{2}(bc + ad) + \frac{1}{3}bd = 0.$$

In order to be orthonormal too we also need

$$||a + bx|| = 1 \text{ and } ||c + dx|| = 1$$

and these give, additionally,

$$a^2 + b^2 = 1, c^2 + d^2 = 1.$$

There are four unknowns and three equations here, so we can make a convenient choice. Let us set

$$a = -b = \frac{1}{\sqrt{2}}$$

which gives

$$\frac{1}{\sqrt{2}}(1 - x)$$

as one vector. The first equation now gives $3c = -d$ from which

$$c = \frac{1}{\sqrt{10}}, \quad d = -\frac{3}{\sqrt{10}}.$$

Hence the set $\{(1-x)/\sqrt{10}, (1-3x)/\sqrt{10}\}$ is a possible orthonormal one.

Of course there are infinitely many possible orthonormal sets, the above was one simple choice. The next definition follows naturally.

Definition C.8 *In an inner product space, an orthonormal set that is also a basis is called an orthonormal basis.*

This next example involves trigonometry which at last gets us close to discussing Fourier series.

Example C.5 *Show that* $\{\sin(x), \cos(x)\}$ *is an orthogonal basis for the inner product space* $V = \{a\sin(x) + b\cos(x) \mid a, b \in \mathbb{R},\ 0 \le x \le \pi\}$ *using as inner product*

$$\langle f, g\rangle = \int_0^1 fg dx,\ f, g \in V$$

and determine an orthonormal basis.

Solution V is two dimensional and the set $\{\sin(x), \cos(x)\}$ is obviously a basis. We merely need to check orthogonality. First of all,

$$\begin{aligned}\langle \sin(x), \cos(x)\rangle = \int_0^\pi \sin(x)\cos(x)dx &= \frac{1}{2}\int_0^\pi \sin(2x)dx \\ &= \left[-\frac{1}{4}\cos(2x)\right]_0^\pi \\ &= 0.\end{aligned}$$

Hence orthogonality is established. Also,

$$\langle \sin(x), \sin(x)\rangle = \int_0^\pi \sin^2(x)dx = \frac{\pi}{2}$$

and

$$\langle \cos(x), \cos(x)\rangle = \int_0^\pi \cos^2(x)dx = \frac{\pi}{2}.$$

Therefore

$$\left\{\sqrt{\frac{2}{\pi}}\sin(x), \sqrt{\frac{2}{\pi}}\cos(x)\right\}$$

is an orthonormal basis.

These two examples are reasonably simple, but for linear spaces of higher dimensions it is by no means obvious how to generate an orthonormal basis. One way of formally generating an orthonormal basis from an arbitrary basis is to use the Gramm–Schmidt orthonormalisation process, and this is given later in this appendix

There are some further points that need to be aired before we get to discussing Fourier series proper. These concern the properties of bases, especially regarding linear spaces of infinite dimension. If the basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ spans the linear space V, then *any* vector $\mathbf{v} \in V$ can be expressed as a linear combination of the basis vectors in the form

$$\mathbf{v} = \sum_{r=1}^{n} \alpha_r \mathbf{a}_r.$$

This result follows from the linear independence of the basis vectors, and that they span V.

If the basis is orthonormal, then a typical coefficient, α_k can be determined by taking the inner product of the vector $\mathbf{v}$ with the corresponding basis vector $\mathbf{a}_k$ as follows

$$\begin{aligned}\langle \mathbf{v}, \mathbf{a}_k \rangle &= \sum_{r=1}^{n} \alpha_r \langle \mathbf{a}_r, \mathbf{a}_k \rangle \\ &= \sum_{r=1}^{n} \alpha_r \delta_{kr} \\ &= a_k\end{aligned}$$

where δ_{kr} is the Kronecker delta:-

$$\delta_{kr} = \begin{cases} 1 & r = k \\ 0 & r \neq k \end{cases}.$$

If we try to generalise this to the case $n = \infty$ there are some difficulties. They are not insurmountable, but neither are they trivial. One extra need always arises when the case $n = \infty$ is considered and that is convergence. It is this that prevents the generalisation from being straightforward. The notion of *completeness* is also important. It has the following definition:

Definition C.9 *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\ldots\}$ *be an infinite orthonormal system in an inner product space* V. *The system is* complete *in* V *if only the zero vector*($\mathbf{u} = \mathbf{0}$) *satisfies the equation*

$$\langle \mathbf{u}, \mathbf{e}_n \rangle = \mathbf{0}, \quad n \in \mathbb{N}$$

A complete inner product space whose basis has infinitely many elements is called a Hilbert Space, the properties of which take us beyond the scope of this short appendix on linear algebra. The next step would be to move on to Bessel's Inequality which is stated but not proved in Chap. 4.

Here are a few more definitions that help when we have to deal with series of vectors rather than series of scalars

Definition C.10 *Let* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n, \ldots$ *be an infinite sequence of vectors in a normed linear space (e.g. an inner product space)* W*. We say that the sequence* converges in norm *to the vector* $\mathbf{w} \in W$ *if*

$$\lim_{n\to\infty} ||\mathbf{w} - \mathbf{w}_n|| = \mathbf{0}.$$

This means that for each $\epsilon > 0$*, there exists* $n > n(\epsilon)$ *such that* $||\mathbf{w}-\mathbf{w}_n|| < \epsilon, \forall n > n(\epsilon)$*.*

Definition C.11 *Let* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \ldots$ *be an infinite sequence of vectors in the normed linear space* V*. We say that the series*

$$\mathbf{w}_n = \sum_{r=1}^{n} \alpha_r \mathbf{a}_r$$

converges in norm *to the vector* $\mathbf{w}$ *if* $||\mathbf{w} - \mathbf{w}_n|| \to 0$ *as* $n \to \infty$*. We then write*

$$\mathbf{w} = \sum_{r=1}^{\infty} \alpha_r \mathbf{a}_r.$$

There is logic in this definition as $||\mathbf{w}_n - \mathbf{w}||$ measures the distance between the vectors $\mathbf{w}$ and $\mathbf{w}_n$ and if this gets smaller there is a sense in which $\mathbf{w}$ converges to $\mathbf{w}_n$.

Definition C.12 *If* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \ldots\}$ *is an infinite sequence of orthonormal vectors in a linear space* V *we say that the system is* closed in V *if, for every* $\mathbf{a} \in V$ *we have*

$$\lim_{n\to\infty} ||\mathbf{a} - \sum_{r=1}^{n} \langle \mathbf{a}, \mathbf{e}_r \rangle \mathbf{e}_r|| = 0.$$

There are many propositions that follow from these definitions, but for now we give one more definition that is useful in the context of Fourier series.

Definition C.13 *If* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \ldots\}$ *is an infinite sequence of orthonormal vectors in a linear space* V *of infinite dimension with an inner product, then we say that the system is* complete *in* V *if only the zero vector* $\mathbf{a} = \mathbf{0}$ *satisfies the equation*

$$\langle \mathbf{a}, \mathbf{e}_n \rangle = 0, \quad n \in \mathbb{N}.$$

There are many more general results and theorems on linear spaces that are useful to call on from within the study of Fourier series. However, in a book such as this a judgement has to be made as when to stop the theory. Enough theory of linear spaces has now been covered to enable Fourier series to be put in proper context. The reader who wishes to know more about the pure mathematics of particular spaces can enable their thirst to be quenched by many excellent texts on special spaces. (Banach Spaces and Sobolev Spaces both have a special place in applied mathematics, so inputting these names in the appropriate search engine should get results.)

C.2 Gramm–Schmidt Orthonormalisation Process

Even in an applied text such as this, it is important that we know formally how to construct an orthonormal set of basis vectors from a given basis. The Gramm–Schmidt process gives an infallible method of doing this. We state this process in the form of a theorem and prove it.

Theorem C.2 *Every finite dimensional inner product space has a basis consisting of orthonormal vectors.*

Proof Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be a basis for the inner product space V. A second equally valid basis can be constructed from this basis as follows

$$\begin{aligned}
u_1 &= v_1\\
u_2 &= v_2 - \frac{(v_2, u_1)}{||u_1||^2}u_1\\
u_3 &= v_3 - \frac{(v_3, u_2)}{||u_2||^2}u_2 - \frac{(v_3, u_1)}{||u_1||^2}u_1\\
&\vdots\\
u_n &= v_n - \frac{(v_n, u_{n-1})}{||u_{n-1}||^2}u_{n-1} - \cdots - \frac{(v_n, u_1)}{||u_1||^2}u_1
\end{aligned}$$

where $u_k \neq 0$ for all $k = 1, 2, \ldots, n$. If this has not been seen before, it may seem a cumbersome and rather odd construction; however, for every member of the new set $\{u_1, u_2, u_3, \ldots, u_n\}$ the terms consist of the corresponding member of the start basis $\{v_1, v_2, v_3, \ldots, v_n\}$ from which has been subtracted a series of terms. The coefficient of u_j in u_i $j < i$ is the inner product of v_i with respect to u_j divided by the length of u_j. The proof that the set $\{u_1, u_2, u_3, \ldots, u_n\}$ is orthogonal follows the standard induction method. It is so straightforward that it is left for the reader to complete. We now need two further steps. First, we show that $\{u_1, u_2, u_3, \ldots, u_n\}$ is a linearly independent set. Consider the linear combination

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$$

and take the inner product of this with the vector u_k to give the equation

$$\sum_{j=1}^{n} \alpha_j (u_j, u_k) = 0$$

from which we must have

$$\alpha_k (u_k, u_k) = 0$$

so that $\alpha_k = 0$ for all k. This establishes linear independence. Now the set $\{w_1, w_2, w_3, \ldots, w_n\}$ where

$$w_k = \frac{u_k}{||u_k||}$$

is at the same time, linearly independent, orthogonal and each element is of unit length. It is therefore the required orthonormal set of basis vectors for the inner product space. The proof is therefore complete. □

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P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4,

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P. Dyke, *An Introduction to Laplace Transforms and Fourier Series*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6395-4,